Mathematical Statistician and Engineering Applications
ISSN: 2094-0343
2326-9865

On recurrent Light like Hypersurfaces of Indefinite Kenmotsu Manifold

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Article Info
Page Number: 2144-2152
Publication Issue:
Vol. 71 No. 4 (2022)

Article History
Article Received: 25 March 2022
Revised: 30 April 2022
Accepted: 15 June 2022
Publication: 19 August 2022

Abstract
The object of present paper is to study the properties of recurrent lightlike hypersurfaces of indefinite Kenmotsu manifolds with (*l*, *m*) –type connection.

Keywords: Hypersurfaces, Kenmotsu manifold, Recurrent lightlike hypersurfaces.

1.Introduction

A linear connection $\bar{\nabla}$ on a semi-Riemannian manifold ($\bar{M}$, $\bar{g}$) is called an (*l*, *m*) –type connection [1] if $\bar{\nabla}$ and its torsion tensor $\bar{T}$ satisfy

(1.1)

$$(\bar{\nabla}_{\bar{X}}\bar{g})(\bar{Y},\bar{Z}) = l\{\theta(\bar{Y})\bar{g}(\bar{X},\bar{Z}) + \theta(\bar{Z})\bar{g}(\bar{X},\bar{Y})\} - m\{\theta(\bar{Y})\bar{g}(J\bar{X},\bar{Z}) + \theta(\bar{Z})\bar{g}(J\bar{X},\bar{Y})\}$$ and

(1.2) $$\bar{T}(\bar{X},\bar{Y}) = l\{\theta(\bar{Y})\bar{X} - \theta(\bar{X})Y)\} + m\{\theta(\bar{Y})J\bar{X} - \theta(\bar{X}),J\bar{Y})$$

where l and m are two smooth functions on $\bar{M}$, J is a tensor field of type (1,1) and θ is a 1-form associated with a smooth unit vector field ζ which is called the characteristic vector field of $\bar{M}$, given by $\theta(\bar{X}) = \bar{g}(\bar{X},\zeta)$.

By direct calculation it can be easily seen that a linear connection $\bar{\nabla}$ on M is an (*l*, *m*) –type connection if and only if $\bar{\nabla}$ satisfies

(1.3) $$\bar{\nabla}_{\bar{X}}\bar{Y} = \nabla_{\bar{X}}\bar{Y} + \theta(\bar{Y})\{l\bar{X} + mJ\bar{X}\},$$

where ∇ is the Levi-Civita connection of a semi-Riemannian manifold ($\overline{M}$, $\overline{g}$) with respect to $\overline{g}$.

In case (l, m) = (1, 0): The above connection $\overline{\nabla}$ turns into a semi-symmetric non-metric connection. The notion of semisymmetric non-metric connection on a Riemannian manifold was introduced by Ageshe-Chafle [2,3] and later, studied by several authors [4,5]. In case (l, m) = (0, 1): The above connection $\overline{\nabla}$ becomes a non-metric φ-symmetric connection such that $\varphi(\overline{X},\overline{Y}) = \overline{g}(J\overline{X},\overline{Y})$. The notion of the non-metric φsymmetric connection was introduced by Jin [6, 7, 8].

In case (l, m) = (1, 0) in (1.1) and (l, m) = (0, 1) in (1.2): The above connection $\overline{\nabla}$ reduces to a quarter-symmetric non-metric connection. The notion of quarter-symmetric non-metric connection was introduced by Golab [9] and then, studied by Sengupta-Biswas [10] and Ahmad-Haseeb [11]. In case (l, m) = (0, 0) in (1.1) and (l, m) = (0, 1) in (1.2): The above connection $\overline{\nabla}$ will be a quarter-symmetric metric connection. The notion of quarter-symmetric metric connection was introduced Yano-Imai [12]. In case (l, m) = (0, 0) in (1.1) and (l, m) = (1, 0) in (1.2): The above connection $\overline{\nabla}$ will be a semi-symmetric metric connection. The notion of semi-symmetric metric connection was introduced by Hayden [13].

2. Preliminaries

Let M be an almost contact manifold equipped with an almost contact metric structure $\{J,\zeta,\theta,\overline{g}\}$ consisting of a (1,1) tensor field J, a vector field ζ, a 1-form θ and a compatible Riemannian metric $\overline{g}$ satisfying

(2.1) $J^2\overline{X} = -\overline{X} + \theta(\overline{X})\zeta,\ \ \overline{g}(J\overline{X}, J\overline{Y}) = \overline{g}(\overline{X},\overline{Y}) - \theta(\overline{X}), \theta(\overline{Y}), \theta(\zeta) = 1,$

From this, we also have $J\zeta = 0,\ \theta o J = 0, \overline{g}(J\overline{X},\overline{Y}) = -\overline{g}(\overline{X}, J\overline{Y}), \theta(\overline{X}) = \overline{g}(\overline{X},\zeta).$

for all $X, Y \in \chi(M)$.

An almost contact metric manifold M is a Kenmotsu manifold [14] if and only if it satisfies

$(\overline{\nabla}_{\overline{X}}\, J)\overline{Y} = \overline{g}(J\overline{X},\overline{Y})\zeta - \theta(\overline{Y})J\overline{X},\ \ X, Y \in \chi(M),$

where ∇ is the Levi-Civita connection of the Riemannian metric g.

With the above equation and (1.3), (2.1) and $\theta(JY) = 0$, it follows that

(2.2) $(\overline{\nabla}_{\overline{X}}\, J)\overline{Y} = \overline{g}(J\overline{X},\overline{Y})\zeta - \theta(\overline{Y})J\overline{X} - \theta(\overline{Y})\{lJ\overline{X} - m\overline{X} + m\theta(\overline{X})\zeta\}.$

Taking $\overline{Y} = \zeta$ and using $J\zeta = 0$ with $\theta(\overline{\nabla}_X \zeta) = l\theta(X)$, we have

(2.3) $\overline{\nabla}_{\overline{X}}\zeta = mJ\overline{X} + (l+1)\overline{X} - \theta(\overline{X})\zeta.$

Let (M, g) be a lightlike hypersurface of $\overline{M}$. The normal bundle $TM^{\perp}$ of M is a subbundle of the tangent bundle TM of M, of rank 1, and coincides with the radical distribution Rad(TM) = TM $\cap TM^{\perp}$. Denote by F (M) the algebra of smooth functions on M and by T(E) the F (M) module of smooth sections of any vector bundle E over M.

A complementary vector bundle S(TM) of Rad(TM) in TM is non-degenerate distribution on M, which is called a screen distribution on M, such that

TM = Rad(TM) $\oplus_{orth}$ S(TM),

where $\oplus_{orth}$ denotes the orthogonal direct sum. For any null section ξ of Rad(TM), there exists a unique null section N of a unique lightlike vector bundle tr(TM) in the orthogonal complement $S(TM)^{\perp}$ of S(TM) satisfying

$\bar{g}(\xi, N) = 1$, $\bar{g}(N; N) = \bar{g}(N; X) = 0$; $\forall 8\ X \in T(S(TM))$:

We call tr(TM) and N the transversal vector bundle and the null transversal vector field of M with respect to the screen distribution S(TM), respectively.

The tangent bundle T $\overline{M}$ of $\overline{M}$ is decomposed as follow:

T $\overline{M}$ = TM $\oplus$ tr(TM) = {Rad(TM) $\oplus$ tr(TM)} $\oplus_{orth}$ S(TM):

In the sequel, let X, Y, Z and W be the vector fields on M, unless otherwise specified. Let P be the projection morphism of TM on S(TM). Then the local Gauss and Weingartan formulas of M and S(TM) are given respectively by

(2.4) $\bar{\nabla}_X Y = \nabla_X Y + B(X.Y)N,$

(2.5) $\bar{\nabla}_X N = -A_N X + \tau(X)N,$

(2.6) $\nabla_X PY = \nabla^*_X PY + C(X.PY)\xi,$

(2.7) $\nabla_X \xi = -A^*_{\xi} X - \sigma(X)\xi.$

where ∇ and ∇^* are the induced linear connections on TM and S(TM) respectively, B and C are the local second fundamental forms on TM and S(TM) respectively, A_N and A^*_{ξ} are the shape operators on TM and S(TM) respectively, and τ and σ are 1-forms on M.

For a lightlike hypersurface M of indefinite Kenmotsu manifold ($\overline{M}, \bar{g}$), it is known [11] that J(Rad(TM)) and J(tr(TM)) are subbundles of S(TM), of rank 1 such that J(Rad(TM)) $\cap$J(tr(TM)) = 0. Thus there exist two non-degenerate almost complex distributions D_o and D on M with respect to J, i.e., J(D_o) = D_o and J(D) = D, such that

S(TM) = J(Rad(TM)) $\oplus$ J(tr(TM)) $\oplus_{orth}$ D_o;

$D = \{Rad(TM) \oplus_{orth} J(Rad(TM))\} \oplus_{orth} D_o$;

$TM = D \oplus J(tr(TM))$.

Consider two null vector fields U and V , and two 1-forms u and v such that

(2.8) $U = -JN$, $V = -J\xi$, $u(X) = g(X, V)$; $v(X) = g(X, U)$.

Denote by S the projection morphism of TM on D. Any vector field X of M is expressed as X = SX + u(X)U. Applying J to this form, we have

(2.9) $JX = FX + u(X)N$,

where F is a tensor field of type (1, 1) globally defined on M by F = JoS. Applying J to (2.9) and using (1.2) , (1.3) and (2.8), we have

(2.10) $F^2X = -X + u(X)U + \theta(X)\zeta$.

As u(U) = 1 and FU = 0, the set (F, u, U) defines an indefinite almost contact structure on M and F is called the structure tensor field of M.

3. (*l*, *m*)-type connections

Let $(\overline{M}, \overline{g}, J)$ be a Kenmotsu manifold with a semi- symmetric metric connection $\overline{\nabla}$. Using (1.1), (2.1) and (2.7), we obtain

$$
(3.1) \quad \begin{aligned}(\nabla_X g)(Y, Z) &= B(X,Y)\eta(Z) + B(X, Z)\eta(Y) \\ &-l\{\theta(Y)g(X,Z) + \theta(Z)g(X,Y)\} \\ &-m\{\theta(Y)g(JX,Z) + \theta(Z)g(JX,Y)\},\end{aligned}
$$

$$
(3.2) \quad \begin{aligned}T(X,Y) = &\, l\{\theta(Y)X - \theta(X)Y\} \\ &+ m\{\theta(Y)FX - \theta(X)FY\},\end{aligned}
$$

$$
(3.3) \quad B(X,Y) - B(Y, X) = m\{\theta(Y)u(X) - \theta(X)u(Y)\},
$$

where T is the torsion tensor with respect to ∇ and η is a 1-form such that

$\eta(X) = \overline{g}(X, N)$.

Proposition 1: Let M be a lightlike hypersurface of indefinite Kenmotsu manifold $\overline{M}$ with an (*l*,*m*)-type connection such that ζ is tangent to M.Then if m = 0, then B is symmetric and conversely if B is symmetric then m = 0.

Proof: If m = 0, then B is symmetric by (3.3). Conversely, if B is symmetric,then replacing X by ζ and Y by U,we get m = 0.

As $B(X,Y) = \overline{g}(\overline{\nabla}_X Y, \xi)$,so B is independent of the choice of S(TM) and satisfies

$$(3.4) \qquad B(X,\xi) = 0, \qquad B(\xi, X) = 0.$$

Local second fundamental forms are related to their shape operators by

(3.5) $B(X,Y) = g(A^*_\xi X, Y) + m u(X)\theta(Y)$,

(3.6) $C(X, PY) = g(A_N X, PY) + \{l\eta(X) + m u(X)\}\theta(PY)$,

(3.7) $\bar{g}(A^*_\xi X, N) = 0$, $\bar{g}(A_N X, N) = 0$, $\sigma = \tau$.

S(TM) is non-degenerate,so using $(3.4)_2$, (3.5), we have

$$(3.8) \qquad A^*_\xi \xi = 0, \qquad \bar{\nabla}_X \xi = -A^*_\xi X - \tau(X)\xi..$$

Taking $\bar{\nabla}_X$ to $\bar{g}(\zeta,\xi)=0$ and $\bar{g}(\zeta, N)=0$ and using (1.1),(2.3),(2.5),(3.5),(3.6) and (3.8), we have

$$(3.9) \qquad g(A^*_\xi X, \zeta) = 0, \qquad B(X,\zeta) = m u(X).$$

$$(3.10) \qquad g\left(A_N X, \zeta\right) = \eta\left(X\right), \; C(X,\zeta) = (l+1)\eta(X) + mv(X)\}.$$

By (2.9),(2.3) and (2.4), we have

$$(3.11) \qquad \nabla_X \zeta = mFX + (l+1)X - \theta(X)\zeta.$$

Applying $\bar{\nabla}_X$ to (2.8) and (2.9) and using (2.2), (2.4), (2.5), (2.9), (2.10), (3.1), (3.6), (3.8) with $\theta(U) = \theta(V) = 0$, we have

(3.12) $B(X,U) = C(X,V)$,

(3.13) $\nabla_X U = F(A_N X) + \tau\left(X\right)U - V(X)\zeta$,

(3.14) $\nabla_X V = F(A^*_\xi X) - \tau\left(X\right)V - u(X)\zeta$,

$$(3.15) \qquad \begin{aligned} (\nabla_X F)Y &= u\left(Y\right)A_N X - B\left(X,Y\right)U \\ &+ \{\bar{g}(JX,Y) - m\theta(X)\theta(Y)\}\zeta \\ &+ m\theta(Y)X - (l+1)\theta(Y)FX, \end{aligned}$$

(3.16) $(\nabla_X u)Y = -u\left(Y\right)\tau(X) - B\left(X, FY\right) - (l+1)\theta(Y)u(X)$,

$$(3.17) \qquad \begin{aligned} (\nabla_X u)Y &= v\left(Y\right)\tau(X) - g\left(A_N X, FY\right) \\ &- (l+1)\theta(Y)v(X) + m\,\theta(Y)\eta(X). \end{aligned}$$

4. Recurrent hypersurfaces

Structure tensor field F of M is said to be recurrent [15] if there exists a 1-form ω on TM such that

$$(\nabla_X F)Y = w(X)FY$$

A lightlike hypersurface M of indefinite Kenmotsu manifold $\overline{M}$ is called recurrent if it admits a recurrent structure tensor field F .

Theorem 1: There exist no recurrent lightlike hypersurface of indefinite Kenmotsu manifold with an (*l,m*)-type connection such that ζ is tangent to M and F is recurrent .

Proof: As M is recurrent so by definition and (3.15), we have

$$\begin{aligned} w(X)FY &= u(Y)A_N X - B(X,Y)U \\ &+\{\overline{g}(JX,Y) - m\theta(X)\theta(Y)\}\zeta \\ &+ m\theta(Y)X - (l+1)\theta(Y)FX\}. \end{aligned} \tag{4.1}$$

Taking $Y = \xi$ and using (3.4) with $F\xi = -V$, we have

$$w(X)V + u(X)\zeta = 0.$$

Taking scalar product with U, we get $w = 0$.

Hence F is parallel to ∇ .

Replacing Y by ξ and using (3.9) , we get $m\{\mathrm{X} - \mathrm{u(X)}\,\mathrm{U} - \theta(X)\zeta\} = lFX$. Replacing X by V, we get $mVC = l\xi$, which implies $m = 0$ and $l = 0$.

Taking scalar product with ζ to (4.1) and using (3.10), we get

$$\mathrm{u(X)}v(Y) - u(Y)v(X) = 0.$$

Hence $m = 0$, which is a contradiction that (*l,m*) $\neq$ (0,0). Hence the theorem follows.

Corollary: There exist no recurrent lightlike hypersurface of Indefinite Kenmotsu manifold with an (*l,m*)-type connection such that ζ is tangent to M and F is parallel with respect to connection ∇ of M .

5. Lie Recurrent hypersurfaces

Structure tensor field F of M is said to be Lie recurrent [15] if there exists a 1-form ν on TM such that

$$(L_X \mathrm{F})\mathrm{Y} = \nu(\mathrm{X})\mathrm{FY}.$$

where L_X denote the Lie derivative on M with respect to X .

Structure tensor field F is called Lie parallel if L_X F = 0. A lightlike hypersurface M of Indefinite Kenmotsu manifold $\overline{M}$ is called Lie recurrent if it admits a Lie recurrent structure tensor field F .

Theorem 2: Let M be a Lie recurrent lightlike hypersurface of indefinite Kenmotsu manifold $\overline{M}$ with an (l,m)-type connection such that ζ is tangent to M and F is Lie recurrent. Then

(1) F is Lie parallel,

(2) 1-form τ satisfies $\tau = 0$ and

(3) Shape operator A_ξ^* satisfies $A_\xi^* U = A_\xi^* V = 0$.

Proof: (1) By definition of Lie recurrent, (2.9) , (2.10), (3.2) and (3.15), we have

(4.2) $$\begin{aligned} \vartheta(X)FY &= -\nabla_{FY}X + F\nabla_Y X + u(Y)A_N X \\ &\quad -\{B(X,Y) - \mathrm{m}\,\theta(Y)u(X)\}U \\ &\quad -\theta(Y)(FX) + \overline{g}(JX,Y)\zeta. \end{aligned}$$

Replacing Y by ξ and using (3.4), we have

(4.3) $$-\upsilon(X)V = \nabla_V X + F\nabla_\xi X + u(X)\zeta.$$

Taking the scalar product with V and ξ , we get

(4.4) $$u(\nabla_V X) = 0, \qquad \theta(\nabla_V X) + u(X) = 0.$$

Taking Y = V in (4.2) and using $\theta(V) = 0$, we get

(4.5) $$-\upsilon(X)\xi = -\nabla_\xi X + F\nabla_V X - B(X,V)U..$$

Applying F and using (2.10) and (4.4), we have

$$\upsilon(X)V = \nabla_V X + F\nabla_\xi X + u(X)\zeta.$$

Comparing this with (4.3), we have $\upsilon = 0$.Hence F is Lie parallel.

(2) Taking scalar product with N to (4.2) and using (3.70, we have

(4.6) $$-\overline{g}(\nabla_{FY}X, N) + \overline{g}(\nabla_Y X, U) = 0.$$

Taking X = ξ and using (2.7) with(3.5), we get

(4.7) $$B(X,U) = \tau(FX)$$

Taking X = U and using (3.12) with FU = 0, we have

(4.8) $$C(U,V) = B(U,U) = 0.$$

Taking X = V in (4.6) and using (3.5) with (3.14), we have

$$B(FY,U) = -\tau(Y).$$

Taking Y = U and Y = ζ with the fact $FU = F\zeta = 0$, we get

(4.9) $\tau(U) = 0, \qquad \tau(\xi) = 0.$

Taking X = U to (4.2) and using (3.3), (3.10), (3.12), (3.13), we get

$u(y)A_N U - F(A_N FY) - A_N Y - \tau(FY)U + \eta(Y)\zeta = 0..$

Taking scalar product with V and using (3.6),(3.12) and (4.8) , we get

$B(X,U) = -\tau(\mathrm{FX}).$

Comparing with (4.70, we have $\tau(\mathrm{FX}) = 0.$

Replacing X by FY and using (2.10) with (4.9), we have

$$\tau = 0\,.$$

(3)Taking X= U in (3.3) and using (4.7) with $\tau = 0$, we have

$$B(U, X) = m\theta(X).$$

Taking X= U in (3.5)) and using (4.10), we have($g(A^*_\xi U, X) = 0$ .Hence $A^*_\xi U = 0$

Replacing X by ξ in (4.3) and using (3.8) with $\tau = 0$, we have $A_N V = 0$.

Acknowledgment

This work is financially supported by Minor research Project grant of VBS Purvanchal University Jaunpur Letter No.: 133/ P. U. /IQAC/2022 Dated: 23/03/2022.

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