Modified Piezoconductivity Equation for Modeling Unsteady Filtration Processes in a Cavernous Medium

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Abstract
The numeric investigation of non-stationary filtration slightly-compressible viscous fluid in porous media with high-permeable vugs was carried out. Obtained evidence that dynamic of pressure distribution in such media could not be described by solution of classical piezoconductivity equation (analogue of the heat equation). Using multipole expansion for describing pressure field perturbation was taken into account. This approach allowed to describe the change of the liquid density in porous phase near vugs during filtration and, as a result, modified piezoconductivity equation was obtained. Comparison of numeric solution of modified piezoconductivity equation with results of direct numeric simulation of filtration in heterogeneous media shows that new equation could better describe this process.

Keywords: filtration, porous media, vugs, caves, heterogeneous media.

Inruduction

For describing the filtration process in porous media with inclusions or voids (vugs), specialist's usually use a classical conductivity equation, an analogue of the heat equation. This equation involves effective parameter that characterizes the mobility of fluid, which is a conductivity coefficient that equals to the ratio of permeability to fluid viscosity, effective compressibility of system fluid-rock and rock porosity (Basniev et al., 2006). James Maxwell was the first scientist, followed by others, to propose equations (Maxwell, 1892; McPhedran, McKenzie, 1978; McKenzie, McPhedran, Derrick, 1978). for calculation of effective media conductivity, which includes conductive spheres, forming a simple cubic, body-centered cubic, or face-centered cubic lattices. Also, expressions for effective media conductivity were obtained that equals to ratio of Darcy's coefficient to fluid viscosity, containing inclusions of arbitrary form (Fricke, 1924). However, in those works the effective transport coefficient was determined for problems, in which local electric potential (in problems of fluid filtration it equivalent to pressure) is constant in time. In this way Laplace's equation was considered in this approach. Specialists often use the obtained effective coefficients to solve non-stationary problems, though the legitimacy of such approach is not given, therefore, it is necessary to consider the possibility of modifying the non-stationary piezoconductivity equation. Then the obtained expressions were used to solve non-stationary equations of fluid transport in porous solid media. Therefore, in such approach only right side of conductivity equation modified. The rare example of the opposite approach is the derivation of an equation for describing of fluid in fractured-porous rock (Barenblat, Entov, Ryzhik, 1972). for dual-porosity model. In the paper was modified left side of equation, involving time derivative. In this work we try to modify the equation of conductivity for cavernous media, which leads to the appearance of a time derivative of a series of partial derivatives of pressure with respect to coordinates.

Theory

Considering filtration process in heterogeneous media, that contain high conductive inclusions (in the extreme case - vugs) of spherical form with radius *R*. In this case it could assume that the pressure *P* on inclusion's surface is constant, and this suppose does not diminish the generality of the conclusions presented below. Perturbation physical fields because of vug *P'* is presents as a sum of multipole fields. The multipole potential of order *l* is expressed in the next form:

$$\varphi^{(l)} = \frac{D^{\alpha_1 \dots \alpha_l} x_{\alpha_1 \dots \alpha_l}}{l! r^{2l+1}} \quad (1)$$

where α_i - the indices of the corresponding axes (α_i = 1 or 2 or 3), x – coordinates, r - radius vector module. The multipole potential coefficients are found by the formula $D^{\alpha_1 \dots \alpha_l} = R^{2l+1} \frac{\partial^l}{\partial x_{\alpha_1} \dots \partial x_{\alpha_l}}$. Choosing the degree r in the denominator, as in formula (1), leads to the fact that the Laplacian of the perturbation has the following form:

$$\Delta\varphi^{(l)} = \frac{\Delta\left(D^{\alpha_1 \dots \alpha_l} x_{\alpha_1 \dots \alpha_l}\right)}{l! r^{2l+1}} \quad (2)$$

At the each point in space around of inclusion, such a pressure perturbation leads, due to compressibility, to a change in the mass of the liquid. For each cavity, it could be find the "excess" pressure by integrating over space outside the cavity:

$$\Delta P' = \int_R^{R'} P' dV = \int_R^{R'} \sum_{i=1}^{+\infty} \varphi^{(l)} dV \quad (3)$$

It is easy to show that the result of integrating the potentials of multipoles of odd orders is equal to zero. Integration of the components of even multipoles containing any coordinate in an odd power will also give a zero result. Therefore, expression (2) will contain only even powers of the pressure derivative by the corresponding coordinates. However, integrating the quadrupole moment to infinity leads to a diverging integral, and it must be limited from physical considerations. It also follows from formula (2) that the disturbance from an external field, the Laplacian of which is not equal to zero, "creates" in each point around the cavity in accordance with the conductivity equation some "source of mass", which can also be found by integrating:

$$Q = \frac{\partial}{\partial t}\int_R^{R'} \sum_{i=1}^{+\infty} \rho\beta\varphi\varphi^{(l)}\, dV \ (4)$$

Here ρ - is the density of the liquid, β - is the compressibility, φ - is porosity of the medium. Similarly to the integral (3), in (4) only those terms of the multipole moments remain that contain the product of the squares of the two coordinates. These perturbations depend on the pressure Laplacian, which is zero for the stationary problem. Therefore, the perturbations are not zero only for the non-stationary case. Since the “additional” mass and “source of mass” are associated with the perturbation pressure of the cavity, they can be taken into account by applying the law of conservation of mass to a certain volume of the medium containing a sufficiently large number of cavities:

$$\rho\varphi\beta\begin{pmatrix} P + b_2(P_{11} + P_{22} + P_{33}) + b_4(P_{1111} + P_{2222} + P_{3333} + 2P_{1122} + 2P_{2233} + 2P_{1133}) + \\ b_6\left(\frac{5}{8}(P_{111111} + P_{222222} + P_{333333}) + \frac{12}{7}(P_{111122} + P_{111133} + P_{112222} + P_{113333} + P_{112233} + P_{222233})\right)\end{pmatrix}_t = \\ = \frac{k^*}{\mu}(P_{11} + P_{22} + P_{33})(5)$$

On the right side of the equation (5) k^* - is the effective permeability of the medium (Darcy coefficient), calculated, for example, using the approaches described in (Maxwell, 1892; McPhedran, McKenzie, 1978; McKenzie, McPhedran, Derrick, 1978; Fricke, 1924), μ - is the viscosity of the liquid, ρ - is the density of the liquid, β - is the compressibility, φ - is porosity of the medium. The subscripts at the pressure denote the corresponding derivatives by the corresponding coordinates, for example:

$$P_{1122} = \frac{\partial^4 P}{\partial^2 x_1^2 \partial^2 x_2^2} \ (6)$$

Consequently, the right part of the equation contains the pressure Laplacian emerging as the divergence from the vector of liquid filtration rate.

In the left part of the equation we see the derivative with time from the liquid mass in the volume unit of the porous rock containing vugs. It is also implicitly taken into account that if the vug concentration in the rock volume unit *n* is introduced, “the additional” mass in the rock volume unit is the product of *n* by “the additional mass” emerging due to the influence of this vug. There are 4 terms in large stocks. The first is the actual pressure of the liquid, the second is the additional mass caused by the quadrupole moment, the third and fourth are mulpole moments of orders 4 and 6.

The transformed equation containing partial derivatives of the 8^{th}, 10^{th} and higher even orders in the left part of the equation can be similarly considered, however, they are insignificant for this work. It should be pointed out that if the derivatives of n^{th} order are introduced into the equation, it should contain all lower order derivatives.

It should be specified that the proposed equation (5) contains the coefficients before the derivatives of different orders, which are calculated for the special case when the vugs are spherical. For example, there are factors 1 and 2 in brackets before the fourth derivatives, and factors 5/8 and 12/7 before the derivatives of the sixth order, obtained as a result of integrating the components of multipole moment of pressure disturbance around the spherical vugs. For vugs of different shape, the coefficients or relations between the coefficients can be different. However, irrespective of the vug shape, the pressure disturbance at quite long distance from them can be represented as the sum of multipole moments whose coefficients depend on the pressure derivatives of different order, and the integration of these disturbances around the vugs will give non-zero components only from even derivatives by the pressure coordinates, and then to the modified piezoconductivity equation of the same type as the equation (5).

If the vug shape is unknown, we can write down a little bit different expression with a large number of unknown variables:

$$\rho\varphi\beta\big(P + (b_{21}P_{11} + b_{22}P_{22} + b_{23}P_{33}) + (b_{41111}P_{1111} + b_{42222}P_{2222} + b_{43333}P_{3333} + b_{41122}P_{1122} + b_{42233}P_{2233} + b_{41133}P_{1133})\big)_t = \frac{k^*}{\mu}(P_{11} + P_{22} + P_{33}) \quad (7)$$

It is taken into account in this expression that for the vugs of the arbitrary shape coefficients $D^{\alpha_1 \dots \alpha_l}$ also depend on the corresponding derivatives but with different coefficients $C_{\alpha_1 \dots \alpha_l}$:

$$D^{\alpha_1 \dots \alpha_l} = C_{\alpha_1 \dots \alpha_l}\frac{\partial^l P}{\partial x_{\alpha_1} \dots \partial x_{\alpha_l}} \quad (8)$$

The expression of these coefficients for spherical vugs was given before. This results in the fact that the coefficients at "similar" derivatives in the modified piezoconductivity equation are different and the medium can demonstrate anisotropic properties. It should be taken into account that in the formula (7) the components containing only the derivatives up to the fourth order are given, and in the formula (5) – also the components with the derivatives up to the sixth order.

From the formula (7) it is seen that the number of coefficients describing the filtration properties of the cavernous medium significantly increases. The obtained equation (7) can describe anisotropic filtration properties of the rock.

Results and Discussions

To check the obtained equation, one can use the classical problem of fluid filtration in porous media in a half-space. In this case, only the derivatives by one coordinate and time will be left in the equation (5). Consider a semi-infinite region containing cavities. At the boundary of the region the pressure rises sharply and then is maintained constant:

$$\frac{\partial p}{\partial t} + b_2\frac{\partial}{\partial t}\frac{\partial^2 p}{\partial x^2} + b_4\frac{\partial}{\partial t}\frac{\partial^4 p}{\partial x^4} = a_2\frac{\partial^2 p}{\partial x^2}$$

$$p(0,t) = p_0$$

$$p(x,t) \to 0; \frac{\partial p}{\partial x} \to 0; x \to +\infty \quad (9)$$

For verification of the equation (9) was simulated filtration of viscous slightly-compressible fluid in long bar in three cases: homogeneous bar without vugs, bar with lattice of spherical vugs (Fig. 1 A), and bar of irregular arrays of vugs (Fig. 1 B).

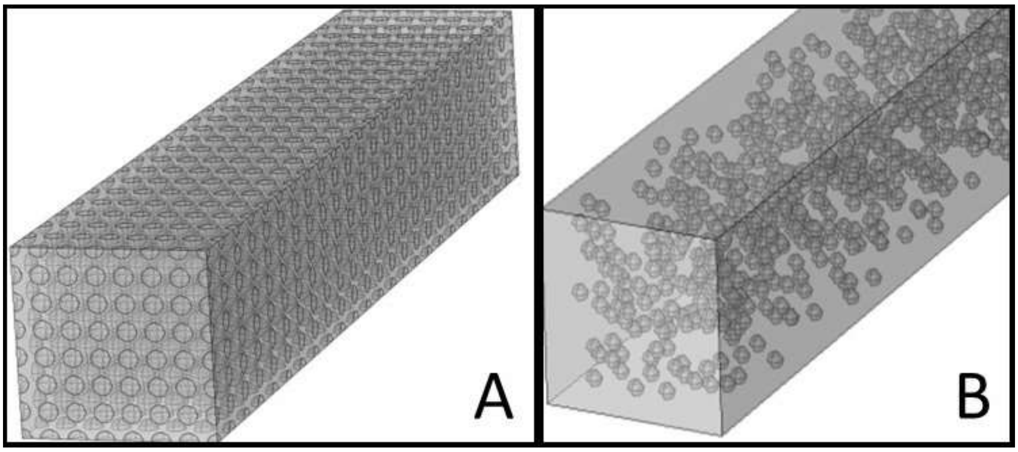

Figure 1 View of computational domains with a periodic arrangement of cavities (A) and with a uniformly random arrangement (B)

The permeability of the vugs material was set 4 orders of magnitude higher than the material of the bars. Initially, the pressure field was set to be uniform; however, at the initial moment of the calculation, at one of its ends, the pressure rose sharply and then was kept constant. On the lateral boundary, the "non-leakage" condition was set.

For the homogeneous medium, in which the vugs are missing, in the equations (9) coefficients b_2, b_4 and a_4 equal zero. The solution of the problem set is well-known (Basniev et al., 2006). and is represented as follows:

$$P = P_0 + \frac{P(0)-P_0}{\sqrt{\pi}} \int_{x/\sqrt{4a_2 t}}^{+\infty} exp\left(-\frac{\xi^2}{2}\right) d\xi (10)$$

Here P_0 – initial pressure value, P(0) – pressure in the beginning of coordinates at all time moments t>0, P(0)-P_0 – instant pressure change in the beginning of coordinates.

From the solution it is seen that it depends on dimensionless coordinate $x/\sqrt{t}$, which will be further called as "dimensionless coordinate".

In Fig. 2 shown the results of calculations in "self-similar" coordinates. For a homogeneous bar (Fig. 2. A) all points form one curve, however for bars with vugs this is not the case (see Fig. 2 B). This means that the classical piezoconductivity conduction equation could give only an approximate solution. Also in Figure 2B we can see that the curves for different time moments at the value of dimensionless coordinate natural logarithm equal to 1 differ by 8%. It is assumed that if specialists carry out the calculations using a classical piezoconductivity equation, the error in determining the parameters can be such value, therefore, for more accurate calculations the improved system of equations similar to the equations (5) and (9) can be required.

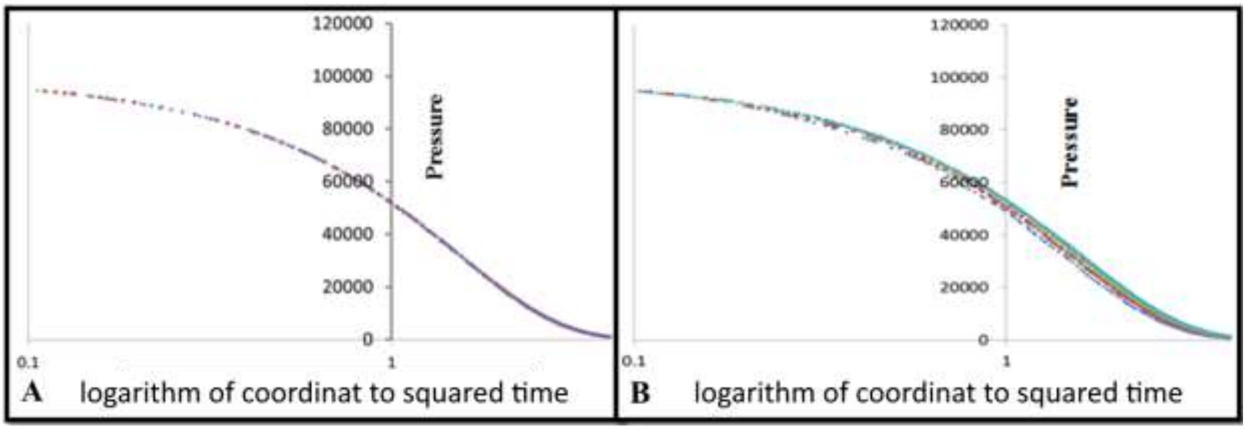

Figure 2 Pressure distribution at different times in a homogeneous computational domain (A) and in a cavernous computational domain (B).

Using expression (9), it is possible to numerically find such constants b_2, b_4, a_2 at which the curves obtained in the numerical calculation will be satisfactorily described by the solution of equation (5).

To solve the equation (9) with the constant value of the required function on the boundary, the program in the Python software was written, which numerically solves the problem at the specified coefficients b_2, b_4, a_2, based on the implicit numerical scheme of the second order of convergence by coordinate and fifth order of convergence by time allowing to find the solution by sweep method in each time layer. Varying the step by time and step by coordinate, it was found that the convergence order by time was 1.98, and the convergence order by coordinate – 4.186.

Then, a series of calculations at different values of coefficients were made with the help of this program to select the best. As a result, the values of coefficients were found, at which the pressure distribution obtained in the process of 3D numerical modeling coincided with the solution results in the equation (9): b_2=- *0.000265*, b_4= *0.0000000000021*, a_2=*120*.

As you can see in Fig. 3, the solution of the derived modified equation describes the points of the numerical experiment much better than the solution of the classical equation. The demonstrated dependencies describe the calculation results shown in Fig. 2B, however, in Fig. 3 the coordinates are not dimensionless for better visual comparison with the solution results in the equation (9). Therefore, it can be argued that the obtained modified equation describes the pressure field in vug media with a higher accuracy than the classical equation of piezoconductivity. In a similar way, a modified equation can be derived based on expressions for multipoles of the third and higher order, where partial derivatives of higher orders are present, and describing the dependences of the pressure fields even more accurately.

It also seems complicated now to predict coefficients for a general case of location and shape of vugs, but it can be recommended, if all the necessary data are known, to carry out the numerical modeling of 3D case in the heterogeneous medium of some problem and then choose the coefficients in the modified piezoconductivity equation in such a way that the solutions will coincide.

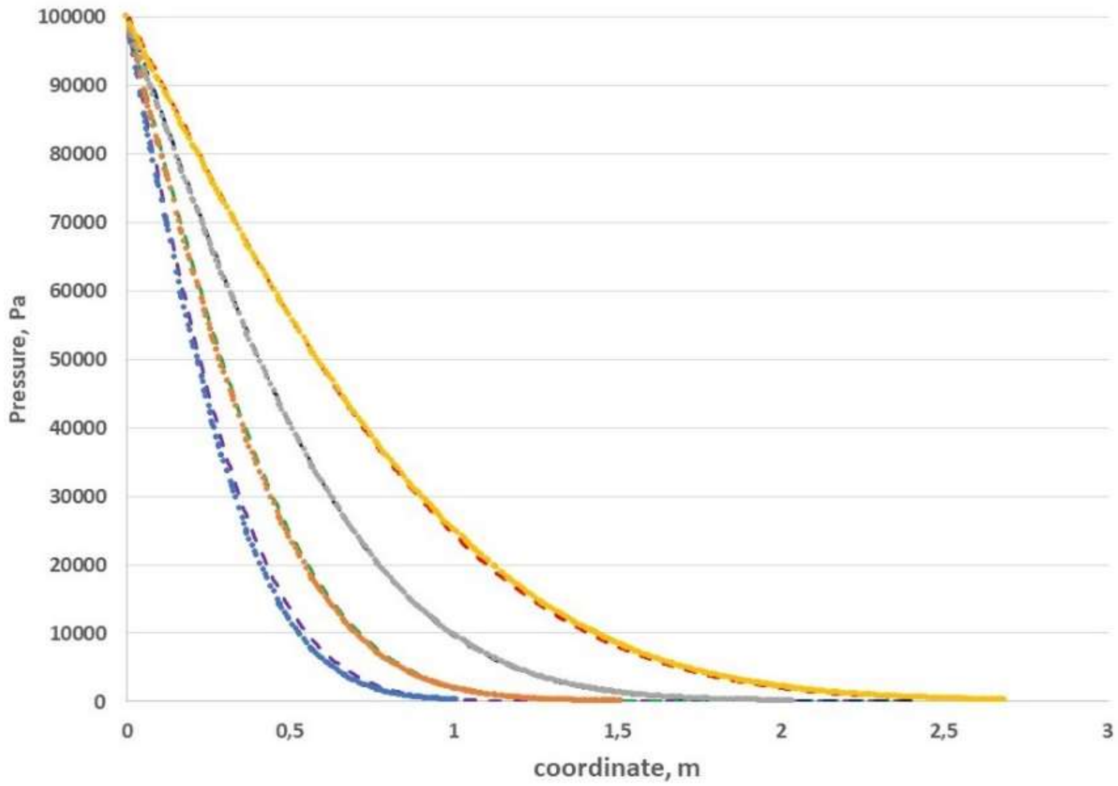

Figure 3 Comparison of the results of direct modeling of the pressure distribution along the cavernous bar at different points in time (dotted line) with the solution of the modified piezoconductivity equation (dashed lines).

To derive the modified system of equations, we also did not consider the issue of setting the boundary conditions, as well as the influence of vugs near the boundary of the calculation region on the liquid filtration rate. However, if the constant value of the required function is set on the boundary, only the pressure distribution in the calculation region is of interest, out of which the liquid flows can be determined. At the same time, the influence of vugs on the liquid flow from the wall into the calculation region is not considered in the paper. The calculation of the flow expression applying the multipole approximation of the pressure disturbance around the vugs will contain not only the pressure gradient components but also the derivatives of a higher order.

The case when the vugs consist of highly-permeable material in comparison with the permeability of the material into which they are impregnated was considered in the paper. However, if there is a vug with any different permeability from the surrounding rock, the pressure disturbance around it can be also described with the help of multipole approximation, the coefficients of which will depend on the relation of these permeabilities.

The issue of correctness of setting boundary problems for the derived type of equation, existence and uniqueness of the solution was also not considered in this paper.

Conclusion

In numeric simulation of viscous and compressible fluid filtration in porous media with high-permeable inclusions was get evidences that pressure dynamic is not exactly descries by solution of classical piezoconductivity equation. Using multipole expansion for describing the pressure field perturbation because of vugs was obtained modification of piezoconductivity equation. This equation has more complicated structure and includes a time derivative of a series of partial derivatives of pressure with respect to coordinates. Using numerical solution of modified piezoconductivity equation was shown that the modified equation could more batter describe the proses of fluid filtration in heterogeneous media with high-permeable inclusions.

The results obtained can also be applied to solve problems of heat transfer in solid media containing inclusions, since in each point the temperature change is described by the heat conductivity equation and heat flow vector – by Fourier formula, which are analogous to piezoconductivity equation and Darcy's law.

References

[1]. Basniev K. S., Dmitriev N. M., Kanevskaya R. D., Maksimov V. M. (2006). Underground hydromechanics: student's book for HEIs. 2006, Moscow; Izhevsk: Institute of Computer Research. 495 p

[2]. Maxwell J. C. (1892). Electricity and Magnetism. V. 1. Oxford: Claredonpress.

[3]. McPhedran R. C., McKenzie D. R. and Brown Robert (1978). The conductivity of lattices of spheres I. The simple cubic lattice, Proc. R. Soc. Lond. A, 359 http://doi.org/10.1098/rspa.1978.0031

[4]. McKenzie D. R., McPhedran R. C., Derrick G. H. (1978). The Conductivity of Lattices of Spheres. II The Body Centred and Face Centred Cubic Lattices. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 362(1709), 211–232. doi:10.1098/rspa.1978.0129

[5]. Fricke H. (1924). The Electric Conductivity of a Suspension of Homogeneous Spheroids, Phys. Rev. 24, 575. DOI:10.1103/PhysRev.24.575.

[6]. Barenblat G. I., Entov V. M., Ryzhik V. M. (1972). Theory of non-stationary filtration of liquid and gas. M.: "Nedra", p. 288.