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Description of the Discs Structure of the Commuting Graph *C* (G, X) for Elements of Order Three in Conway Group Co_3

Osamah Hadi Mutlag
University of Baghdad, College of Science, RS & GIS Department,
osamahadimutlak@gmail.com

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Abstract

Suppose that X is a subset of a finite group G. The commuting graphis defines as a simple graph has X as a vertex set and two vertices $x,y \in$X are connected if $x \neq y$ and $xy = yx$. The commuting graph is denote by C(G,X). In this paper, we investigate the discs structure of the commuting graphwhen G is a Conway group $Co_3$and X is a conjugacy class for elements of order 3 in G .

Keywords: Exceptional groups; commuting graph; diameter, cliques.

1. Introduction

One of the current techniques of inspecting the shape of the group which has been recently confirmed to be as high quality is analyzing the action of a group on a graph.The elements of order 3 are critical in terms of algebraic features of finite groups, in regard to the central vital role in representing involution in finite groups.Many of the studies collected in this sense, for example, can be found here ([1,2]).If G is a finite group and X is a subset of G, the commuting graph C(G,X) has X as its own node set and two node$x$, $y\in$X are adjacent if x distinct and commute with y. In their seminal paper [3], Fowler and Brauer were the first to begin to consider the commuting graph. They were excellent for proving that for a given isomorphism form of an involution centralizer, there are finitely many non-abelian groups qualified for containing it.Everett and Rowley bring out several commuting graphs that have been discussed in [4].The commuting graph is called the commuting involution graph when G is a finite group and X is a conjugacy class of involution [5].Aubad [6] investigate the relation between the alternating group A_4 and the elements of order 3 in the exceptional group of lie type $^3D_4(2)$.

Now, during this paper we let $G\cong Co_3$ and $t\in G$ be an element of order 3, also let $X=t^G$ is now believed to be a G-conjugacy class of order 3 elements.The purpose of this study is to look at some of the features of the commuting graph $C(G,X)$ when G is the Conway group Co_3. Analyzing the disc structure and computing the diameters for such graphs was worthy of mention.

Assume that $x\in X$, the i^{th} disc of x, denoted by $\Delta_i(x)$, $(i\in\mathbb{N})$ is defined as

$$\Delta_i(x)=\{y\in X|\ d(x,y)=i\}$$

Upon on the graph $C(G,X)$, d(,) is established as the usual distance metric.Note that $\Delta_i(x)$ precisely splits up into a union of particular $C_G(t)$ –orbits (when G act by conjugation on X).As a consequence, we can determine the $C_G(t)$ -orbits of X in order to analyze the properties of the $C(G,X)$.We should note that G acting on X by conjugation embeds G in the group of graph automorphisms of $C(G,X)$, and G is transitive on the vertices of $C(G,X)$. We also denote by Diam C (G,X) the diameter of the commuting graph C(G,X) and is defined as

$$\mathrm{Diam}C(G,X) = \max_{x\in X}\{i|\ \Delta_i(x)\neq\emptyset \text{ and } \Delta_{i+1}(x)=\emptyset\}$$

Eventually, we should consult the Atlas [7] for the naming of G-conjugacy classes.

2.Discs Structure of *C* (G,X)

To analyze the discs structure of $C(G,X)$. In the next table, we include details about the classes of order 3 in Co_3, such as the class size, the $C_G(t)$ structure, and the permutation rank.

Group	$X=t^G$	$\|X\|$	Permutation Rank	$C_G(t)$ structures
Co_3	3A	1416800	24	$(C_3 \times C_3 \times C_3 \times C_3 \times C_3) : S_5$
	3B	17001600	719	$((C_3 \times ((C_3 \times C_3) : C_3)) : C_3) : (SL(2,9) : C_2)$
	3C	109296000	24282	$C_3 \times (PSL(2,8) : C_3)$

Table 2.1: The Classes Information

Dealing computationally with Co3, we used 276-point permutation representations see the OnLine Atlas [8].

Let C be a random G-Conjugacy classconsider the following set :

$$X_C=\{x\in X \mid tx\in X\}.$$

We note that if $X_C\neq\emptyset$, so it is a union of $C_G(t)$-orbits of X (such that$C_G(t)$ actsupon X by conjugation).So it would be valuable to assess which discs of $t$hold the vertices in X_C as the way of X_C splits into $C_G(t)$-orbits.It's also useful to know how large X_C is, which guides us to class structure constants.

Class structure constants are define to be the sizes of the following set:

$$\{(h_1, h_2) \in C_1 \text{ x } C_2 \mid h_1h_2 = h\}$$

Such that C_1,C_2,C_3 are G-conjugacy classes and h is a random element in C_3. The above constants are now being measured accurately from G's complex character table, which is stored in the Atlas and available electronically in the computer algebra package's standard libraries of GAP[9]. If we take C_1 = C, C_2 = X = C_3 and $h = t$, then in this case

$$|X_C| = \frac{|G|}{|C_G(t)||C_G(s)|}\sum_{i=1}^{n}\frac{\chi_i(s)\chi_i(t)\overline{\chi_i(t)}}{\chi_i(1)}$$

Such that$s$ is a random element in the class C and $\chi_1, \chi_2,\ldots, \chi_n$ the complex irreducible characters of the group G.

To achieve the next results, computational style were used with the assist of Gap and the OnLine Atlas.The information we collate to investigate the discs structure of C(G,X)including theC_G(t)-orbits sizes in their action upon X_C , when $X_C\neq\emptyset$ for C is a G-conjugacy classes of $tx$with $x\in\Delta_i(t)$; $i\in\mathbb{N}$.In the tables that follow, exponential notation is used to denote the multiplicity of a given scale.

We give some detail on how to get the tables in this section. As previously mentioned, we use the class names from the Atlas.We further compact the letter part of the class name as we mean to union these classes and their characters are in alphabetical order to make it simpler.For instant, in **Table 2.3**, when$G\cong Co_3$ and X = 3B, 20AB is short-hand for 20A$\cup$ 20B.

2.1 **Discs Structure of *C*(Co_3,3A)**

The distance between *t* and *x* in *C*(G,X) is mostly specified by the G-conjugacy classC to which $tx \in$C, as seen in the following table.The exclusion is15A(58320^2) $\in \Delta_2$(t) and 15A(69984,349920) $\in \Delta_3$(t).

Δ_1(t)	Δ_2(t)	Δ_3(t)
1A(1), 3A(360), 3B(360^2)	4B(3645,7290),6A(3645), 6B(7290),6C(29160), 10A(29160,58320), 12B(29160),15A(58320^2), 24B(58320^2),30A(58320)	5B(69984),7A(58320), 9A(58320),15A(69984,349920), 15B(349920)

Table 2.2: Discs structure of *C*(Co_3,3A)

2.2 **Discs Structure of *C*(Co_3,3B)**

The distance between *t* and *x* in *C*(G,X) is classified by the G-conjugacy classC to which $tx \in$C, only in the case C $\in${1A,2AB,3A,3C,4B,5A,6DE}.

Δ_1(t)	Δ_2(t)	Δ_3(t)	Δ_4(t)
1A(1), 3A(20, 30), 3B(20, 60,540)	2A(1215), 3A (540, 1080, 1215, 9720), 3C (16202), 5B(4860, 9720), 6A (97202) 6C (97202), 9A (16202), 9B (32404,	3B(7290^2),4B(1215, 2430, 3645, 7290^2, 14580^4, 29160^4), 5A(1458^2),5B(14580^7, 29160^{23}), 6A(2430),6C(1215, 2430, 3645, 14580^8, 29160^6),7A(14580^{17}, 29160^{49}),8A(14580^8), 8C(14580^{16},29160^{26}), 9A(4860^4),9B(29160^{26}), 10A(2430, 4860, 14580, 29160^{22}),10B(29160^2),11AB(14 580,29160^{41}),12B(2430, 7290, 14580^{10}),12C($14580^{4,}29160^{12}$), 14A(14580^6,29160^{16}),15A (4860^2,14580^2,29160^{26}), 15B(29160^{20}),18A(14580^7, 29160^{26}),20AB(7290^2,14580^4, 29160^{10}),21A(29160^6), 22AB(29160^2),23AB(29160^{14}), 24B(4860^2,29160^2),	2B(243^2),3B(243^2,4860^2), 6A(14580^4),6D(4860^4), 6E(14580^2),7A(4860^6,29160^{18}), 8A(29160^2),8C(7290^4,14580^4, 29160^8),9B(29160^4),10A (7290^5,29160^{14}),11AB(14580, 29160^8),12B(14580^2, 29160^4), 12C(7290^4),14A(29160^8), 15A(14580^2, 29160^6), 15B(29160^4), 18A(29160^2), 20AB(7290^2,29160^2), 24B(14580^2, 29160^6), 30A(29160^2)

	97203), 15B (97203)	30A(4860,14580^2, 29160^2)	

Table 2.3: Discs structure of $C(\mathrm{Co}_3,3\mathrm{B})$

2.3 Discs Structure of $C(\mathrm{Co}_3,3\mathrm{B})$

The distance between t and x in C(G,X) is classified by the G-conjugacy classC to which $tx \in$C, only in the case C $\in$\{1A,2AB\}.

Δ_1(t)	Δ_2(t)	Δ_3(t)	Δ_4(t)	Δ_5(t)
1A(1), 3A(56), 3B (84^2) 3C (56, 84^2)	2B(756^2), 3B(756^2, 1512^2), 3C(756^2, 1512^2), 6D (1512^{12}), 6E(756^2, 1512^{12}), 7A (1512^7), 9AB (1512^9), 21A (1512^7)	3C(1512^2, 2268), 4B(1512^2, 2268^3, 4536^7),5A(4536^2), 5B(1512^2, 2268^6, 4536^{30}),6A(4536), 6B(1512^2, 4536^8), 6C(1512^2, 2268^6, 4536^{23}),6D(4536^{20}), 6E(4536^{54}),7A(4536^{101}), 8A(4536^{32}),8B(4536^4), 8C(4536^{104}),9A(4536^{26}), 9B(4536^{52}),10A(1512^2, 4536^{112}),10B(4536^{256}), 11AB(2268^4, 4536^{144}), 12A(4536^{16}),12B(4536^{76}), 12C(4536^{92}),14A(4536^{264}),15A(4536^{203}), 15B(1512^2, 4536^{276}), 18A(4536^{178}), 20AB(4536^{264}), 21A(4536^{223}), 22AB(4536^{162}), 23AB(4536^{153}), 24A(4536^{212}), 24B(4536^{136}), 30A(1512^2, 4536^{128})	2A(378),3A(504), 3B(252^2, 504, 1512^4), 3C(378, 2268),4B(756^2, 1512^3,2268,4536^{10}), 5A(4536^8), 5B(378,1512^3,2268^2, 4536^{79}),6A(1512^5,4536^{11}), 6B(1512^{14},4536^{14}), 6C(756^2,1512^{11},2268^4, 4536^{110}),6D(4536^{148}), 6E(2268^4,4536^{310}), 7A(4536^{523}),8A(2268^8, 4536^{108}),8B(4536^{76}), 8C(2268^8,4536^{660}), 9A(1512^4, 4536^{163}) 9B(1512^{13}, 4536^{209}), 10A(1512^{10}, 4536^{384}), 10B(2268^4,4536^{1040}), 11AB(2268^4,4536^{919}), 12A(4536^{146}), 12B(1512^{12}, 4536^{442}), 12C(4536^{556}), 14A(4536^{1320}), 15A(4536^{797}), 15B(1512^{10}, 4536^{1531}), 18A(4536^{1118}), 20AB(4536^{996}), 21A(4536^{869}),	3B(168^3), 3C(168^3)

			22AB(4536^{836}), 23AB(4536^{809}), 24(4536^{700}), 24B(4536^{840}), 30A(1512^{22}, 4536^{624})	

Table 2.4: Discs structure of $C(Co_3,3B)$

3. Main Result

In the next theorem we give a full information about the diameters , the discs sizes of the commuting graph C(G,X) when G≅ Co_3 and X is a G-conjugacy class of elements of order 3 in G. The main theorem describe as follows:

Theorem 3.1: Let G be isomorphic to Co_3 . Then

1−The sizes of the discs $\Delta_i(t)$ are listed in **Table 3.1**.
2-If (G,X)=(Co_3,3A) then Dim C(G,X)=3.

3-If (G,X)=(Co_3,3B) then Dim C(G,X)=4.

4-If (G,X)=(Co_3,3C) then Dim C(G,X)=5.

G	$X=t^G$	$\|\Delta_1(t)\|$	$\|\Delta_2(t)\|$	$\|\Delta_3(t)\|$	$\|\Delta_4(t)\|$	$\|\Delta_5(t)\|$
Co_3	3A	1081	459270	956448		
	3B	671	144990	13435956	3419982	
	3C	449	96768	18594576	90603198	1008

Table 3.1: The Discs for C (G,X)., G≅Co_3.

Proof: The proof of the above theorem can be hold form **Table 2.2, Table 2.3** and **Table 3.3**.

4.Conclusions:.

The aim of this work is to investigate the algebraic structure of G-conjugacy classes of elements of order 3 in Conway groups Co_3.For this, we used the commuting graph on these G-conjugacy classes. The sizes of $C_G(t)$-orbits, discs, and diameters of the graphs, for example, have all yielded very interesting results.

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