Coefficient Characterization for Some Subclasses of Generalized Rational Univalent Functions

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Abstract: This research work consists of two sections. Each section introduces a subclass of generalized rational functions and study of geometric properties like coefficient characterization, growth and distortion properties. First section introduces a starlike subclass $S_+^*(b_1, \alpha)$ of S_+. Second section introduces a convex subclass $C_+(b_1, \alpha)$ of S_+.

Key words: rational univalent, starlike, convex, coefficient characterization.

1. Introduction

A normalized function $f(z)$ analytic in the open unit disk around the origin and non-vanishing outside the origin of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ can be expressed in the form $\frac{z}{g(z)}$, where $g(z)$ has Taylor coefficients b_n's in U. Mitrinovic [2] obtained sufficient conditions for functions of the form $\frac{z}{1+b_1 z+\cdots+b_n z^n}$, $b_n \neq 0$ to be univalent in U.

Theorem[2]

The function $f(z) = \frac{z}{1+\sum_{n=1}^{\infty} b_n z^n}$ is in S if $\sum_{n=2}^{\infty}(n-1)|b_n| \leq 1$and $\sum_{n=1}^{\infty}|b_n| \leq 1$.

Reade et al.,[7] introduced different subclasses of univalent rational functions and obtained sufficient conditions for $f(z) \in S$ to be in those subclasses.

Obradovi´c. [5] studied on starlikeness of certain class of rational functions.

Ahuja and Pawan [1] studied properties of spiral-likeness of rational functions.

Obradovi´c and Ponnusamy [3] introduced a subclass of rational univalent functions S_+ as the subclass of functions of S which can be expressed in the form

$$\frac{z}{f(z)} = b_1 z + \sum_{n=1}^{\infty} \lambda_n \frac{z}{f_n(z)} \quad (1)$$

for some sequence $\{\lambda_n\}_{n=1}^{\infty}$ of non-negative real numbers with $\sum_{n=1}^{\infty}\lambda_n = 1$ and derived necessary and sufficient condition for functions of S to be in S_+.

Theorem [3**]** Let $f \in A$. Then $f \in S_+$ if and only if f has the form

$$\frac{z}{f(z)} = b_1 z + \sum_{n=1}^{\infty}\lambda_n \frac{z}{f_n(z)}$$

for some sequence $\{\lambda_n\}_{n=1}^{\infty}$ of non-negative real numbers with $\sum_{n=1}^{\infty}\lambda_n = 1$ and

$$\frac{z}{f_n(z)} = \begin{cases} 1, & \text{for } n = 1 \\ 1 + \frac{1}{n-1}z^n, & \text{for } n = 2,3,\dots \end{cases}$$

Now, this paper introduces different subclasses of S_+ by fixing b_1 and obtain coefficient characterization for these subclasses similar to that of [3] for S_+.

2. Starlike Subclass of Generalized Rational Univalent Functions

Reade et al.[6] obtained coefficient conditions on $\{b_n\}_{n=1}^{\infty}$ that ensure starlikeness of functions of the form $f(z) = \frac{z}{1+\sum_{n=1}^{\infty} b_n z^n}$.

Theorem [6**]**

Let $f(z) = \frac{z}{1+\sum_{n=1}^{\infty} b_n z^n}$, $z \in U$ and let α be a constant, $0 \le \alpha \le 1$. If the coefficients of $f(z)$ satisfy

$$\sum_{n=2}^{\infty}(n-1+\alpha)|b_n| \le \begin{cases} (1-\alpha) - (1-\alpha)|b_1|, & 0 \le \alpha \le \frac{1}{2} \\ (1-\alpha) - \alpha|b_1|, & \frac{1}{2} < \alpha \le 1 \end{cases}$$

then $f(z)$ is star-like of order α in the unit disk U.

Applying this condition, this section defines a subclass $S_+^*(b_1,\alpha)$ of class of starlike rational functions by fixing Taylor coefficient b_1 of $g(z)$. And obtains coefficient characterization, growth and distortion bounds for the subclass $S_+^*(b_1,\alpha)$.

Definition 2.1

Let $b_1 \in \mathbb{C}$, $|b_1| \le 1$ be fixed and $0 \le \alpha \le 1$.

Define $S_+^*(b_1,\alpha) =$

$$\left\{\begin{matrix} f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S : \frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n, \ z \in U \text{ and } b_n \ge 0, \text{ for } n \ge 2 \\ \sum_{n=2}^{\infty}(n-1+\alpha) b_n \le \begin{cases} (1-\alpha) - (1-\alpha)|b_1|, & 0 \le \alpha \le \frac{1}{2} \\ (1-\alpha) - \alpha|b_1|, & \frac{1}{2} < \alpha < 1 \end{cases} \end{matrix}\right\} \quad (2)$$

The following result shows coefficient characterization for the subclass $S_+^*(b_1,\alpha)$

Theorem 2.2

Let $f(z) \in S$ be of the form $f(z) = \frac{z}{1+\sum_{n=1}^{\infty} b_n z^n}$ for $z \in U$ and $b_1 \in \mathbb{C}$, $|b_1| \le 1$ be fixed.

Then $f(z) \in S_{+}^{*}(b_1, \alpha)$ if and only if $f(z)$ has the form

$$\frac{z}{f(z)} = b_1 z + \sum_{n=1}^{\infty} \lambda_n \frac{z}{f_n(z)}$$

for some sequence $\{\lambda_n\}_{n=1}^{\infty}$ of non-negative real numbers with $\sum_{n=1}^{\infty} \lambda_n = 1$ and

(i). for $0 \leq \alpha \leq \frac{1}{2}$, $\frac{z}{f_n(z)} = \begin{cases} 1, & \text{for } n = 1 \\ 1 + \frac{1-\alpha}{n-1+\alpha} z^n, & \text{for } n = 2,3,\ldots \end{cases}$

(ii). for $\frac{1}{2} < \alpha < 1$, $|b_1| \leq \frac{1-\alpha}{\alpha}$, $\frac{z}{f_n(z)} = \begin{cases} 1, & \text{for } n = 1 \\ 1 + \frac{1-\alpha}{n-1+\alpha} z^n, & \text{for } n = 2,3,\ldots \end{cases}$

Proof:

Case (i) for $0 \leq \alpha \leq \frac{1}{2}$

Suppose that $f(z) \in S$, $z \in U$ has the form (1) for some sequence of non-negative real numbers $\{\lambda_n\}_{n=1}^{\infty}$ with $\sum_{n=1}^{\infty} \lambda_n = 1$.

We need to prove that the function $f(z) \in S_{+}^{*}(b_1, \alpha)$.

for $z \in U$, rewrite $\frac{z}{f(z)}$ as

$$\frac{z}{f(z)} = b_1 z + \sum_{n=1}^{\infty} \lambda_n \frac{z}{f_n(z)}$$

$$= b_1 z + \lambda_1 + \sum_{n=2}^{\infty} \left[1 + \frac{1-\alpha}{n-1+\alpha} z^n\right] \lambda_n \qquad \text{(by the definition of } \frac{z}{f_n(z)}\text{)}$$

$$= 1 + b_1 z + \sum_{n=2}^{\infty} \lambda_n \frac{1-\alpha}{n-1+\alpha} z^n$$

$$= 1 + b_1 z + \sum_{n=2}^{\infty} b_n z^n \quad \text{where} \quad b_n = \lambda_n \frac{1-\alpha}{n-1+\alpha} \geq 0.$$

Choosing $\lambda_1 \in \mathbb{R}$ such that $|b_1| \leq \lambda_1 \leq 1$,

$$(1-\alpha)|b_1| + \sum_{n=2}^{\infty} (n-1+\alpha) b_n$$

$$\leq (1-\alpha)\lambda_1 + \sum_{n=2}^{\infty} \left[(n-1+\alpha)\lambda_n \frac{1-\alpha}{n-1+\alpha}\right]$$

$$= (1-\alpha)\lambda_1 + \sum_{n=2}^{\infty} (1-\alpha)\lambda_n$$

$$= (1-\alpha) \sum_{n=1}^{\infty} \lambda_n = (1-\alpha)$$

This shows that $f(z)$ satisfies (2).

Therefore $f(z) \in S_{+}^{*}(b_1, \alpha)$ for $0 \leq \alpha \leq \frac{1}{2}$.

Conversely, suppose $f(z) \in S_{+}^{*}(b_1, \alpha)$ for $0 \leq \alpha \leq \frac{1}{2}$.

Then $f(z)$ satisfies condition (2).

Thus

$\sum_{n=2}^{\infty}(n-1+\alpha)b_n \leq (1-\alpha)-(1-\alpha)|b_1|$.

Now, set $b_n = \frac{(1-\alpha)}{(n-1+\alpha)}\lambda_n$ for $n = 2, 3, \ldots.$

so that $\lambda_n = \frac{(n-1+\alpha)}{(1-\alpha)} b_n$ for $n = 2, 3, \ldots.$ and $\lambda_1 = 1 - \sum_{n=2}^{\infty}\lambda_n$.

Then

$$\frac{z}{f(z)} = 1 + b_1 z + \sum_{n=2}^{\infty} b_n z^n$$

$$= b_1 z + \lambda_1 + \sum_{n=2}^{\infty}\left[1 + \frac{1-\alpha}{n-1+\alpha} z^n\right]\lambda_n$$

$$= b_1 z + \lambda_1 + \sum_{n=2}^{\infty}\lambda_n\left[1 + \frac{(1-\alpha)}{(n-1+\alpha)} z^n\right]$$

$$= b_1 z + \sum_{n=1}^{\infty}\lambda_n \frac{z}{f_n(z)}$$

Case (ii). for $\frac{1}{2} < \alpha < 1$

Suppose that $f(z) \in S$ has the form (1) for some sequence of non-negative real numbers $\{\lambda_n\}_{n=1}^{\infty}$ with $\sum_{n=1}^{\infty}\lambda_n = 1$.

We need to prove that the function $f(z) \in S_{+}^{*}(b_1, \alpha)$.

Now, rewrite $\frac{z}{f(z)}$ as

$$\frac{z}{f(z)} = b_1 z + \sum_{n=1}^{\infty}\lambda_n \frac{z}{f_n(z)}$$

$$= b_1 z + \lambda_1 + \sum_{n=2}^{\infty}\lambda_n\left[1 + \frac{1-\alpha}{n-1+\alpha} z^n\right] \qquad \text{(by the definition of } \frac{z}{f_n(z)}\text{)}$$

$$= 1 + b_1 z + \sum_{n=2}^{\infty}\lambda_n \frac{1-\alpha}{n-1+\alpha} z^n$$

$$= 1 + b_1 z + \sum_{n=2}^{\infty} b_n z^n \quad \text{where} \quad b_n = \lambda_n \frac{1-\alpha}{n-1+\alpha} \geq 0.$$

Choosing $\lambda_1 \in \mathbb{R}$ such that $|b_1| \leq \frac{(1-\alpha)}{\alpha}\lambda_1 \leq 1$,

$$\alpha|b_1| + \sum_{n=2}^{\infty}(n-1+\alpha)b_n$$

$$\leq \alpha\left[\frac{1-\alpha}{\alpha}\right]\lambda_1 + \sum_{n=2}^{\infty}\lambda_n\left[(n-1+\alpha)\frac{1-\alpha}{n-1+\alpha}\right]$$

$$= (1-\alpha)\lambda_1 + \sum_{n=2}^{\infty}(1-\alpha)\lambda_n$$

$$= (1-\alpha)\sum_{n=1}^{\infty}\lambda_n$$

$$= (1-\alpha)$$

This shows that (2) is satisfied.

Therefore $f(z) \in S_+^*(b_1,\alpha)$for $\frac{1}{2} < \alpha < 1$.

Conversely, suppose $f(z) \in S_+^*(b_1,\alpha)$ for $\frac{1}{2} < \alpha < 1$.

Then $f(z)$ satisfies condition (2).

Thus

$\alpha|b_1| + \sum_{n=2}^{\infty}(n-1+\alpha)b_n \leq (1-\alpha)$

Now, set $b_n = \lambda_n \frac{(1-\alpha)}{(n-1+\alpha)}$ for $n = 2, 3, ...$ and $\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n$

so that $\lambda_n = \frac{(n-1+\alpha)}{(1-\alpha)} b_n$ for $= 2, 3, ...$.

Then $\frac{z}{f(z)}$ has the form

$$\frac{z}{f(z)} = 1 + b_1 z + \sum_{n=2}^{\infty} b_n z^n$$

$$= b_1 z + \lambda_1 + \sum_{n=2}^{\infty}\left[1 + \frac{1-\alpha}{n-1+\alpha} z^n\right]\lambda_n$$

$$= b_1 z + \lambda_1 + \sum_{n=2}^{\infty} \lambda_n \left[1 + \frac{(1-\alpha)}{(n-1+\alpha)} z^n\right]$$

$$= b_1 z + \sum_{n=1}^{\infty} \lambda_n \frac{z}{f_n(z)}$$

This completes the proof.

Next results discuss growth and distortion bounds for $S_+^*(b_1,\alpha)$

Theorem 2.3

If $f \in S_+^*(b_1,\alpha)$, $z \in U$, for $0 \leq \alpha < 1$, then $|z| = r < 1$, then

$$\max\left\{0,\ 1 - |b_1|r - \frac{1-\alpha}{1+\alpha} r^2\right\} \leq \left|\frac{z}{f(z)}\right| \leq 1 + |b_1|r + \frac{1-\alpha}{1+\alpha} r^2 \qquad (3)$$

Proof: Since $f(z) \in S_+^*(b_1,\alpha)$,

by Theorem 1.2, $\frac{z}{f(z)}$ has the form

$$\frac{z}{f(z)} = b_1 z + \sum_{n=1}^{\infty} \lambda_n \frac{z}{f_n(z)}$$

$$= b_1 z + \lambda_1 \frac{z}{f_1(z)} + \sum_{n=2}^{\infty} \lambda_n \frac{z}{f_n(z)}$$

$$= b_1 z + \lambda_1 + \sum_{n=2}^{\infty} \lambda_n \left[1 + \frac{(1-\alpha)}{(n-1+\alpha)} z^n\right]$$

$$= 1 + b_1 z + \sum_{n=2}^{\infty} \lambda_n \frac{(1-\alpha)}{(n-1+\alpha)} z^n \tag{4}$$

Then

$$\left|\frac{z}{f(z)}\right| \leq 1 + |b_1 z| + \left|\sum_{n=2}^{\infty} \lambda_n \frac{(1-\alpha)}{(n-1+\alpha)} z^n\right|$$

$$\leq 1 + |b_1||z| + |z|^2 \left|\sum_{n=2}^{\infty} \lambda_n \frac{(1-\alpha)}{(n-1+\alpha)}\right|$$

$$\leq 1 + |b_1| r + \frac{1-\alpha}{1+\alpha} r^2 \quad \text{for } |z| \leq r < 1 \qquad \left(\text{since } \frac{(1-\alpha)}{(n-1+\alpha)} \text{is decreasing}\right)$$

And also from (4),

$$\left|\frac{z}{f(z)}\right| \geq 1 - |b_1 z| - \left|\sum_{n=2}^{\infty} \lambda_n \frac{(1-\alpha)}{(n-1+\alpha)} z^n\right|$$

$$\geq 1 - |b_1||z| - |z|^2 \left|\sum_{n=2}^{\infty} \lambda_n \frac{(1-\alpha)}{(n-1+\alpha)}\right|$$

$$\geq 1 - |b_1| r - \frac{1-\alpha}{1+\alpha} r^2 \quad \text{for } |z| \leq r < 1$$

Therefore

$$\max\left\{0,\ 1 - |b_1| r - \frac{1-\alpha}{1+\alpha} r^2\right\} \leq \left|\frac{z}{f(z)}\right| \leq 1 + |b_1| r + \frac{1-\alpha}{1+\alpha} r^2$$

Theorem 2.4

If $f \in S_+^*(b_1, \alpha)$, $z \in U$ and $0 \leq \alpha < 1$, then

$$max\left\{0,\ |b_1| - \frac{2(1-\alpha)}{1+\alpha} r\right\} \leq \left|\left\{\frac{z}{f(z)}\right\}'\right| \leq |b_1| + \frac{2(1-\alpha)}{1+\alpha} r, \quad \text{for } |z| = r < 1$$

Proof: Since $f(z) \in S_+^*(b_1, \alpha)$,

using (4) $\frac{z}{f(z)}$ can be written as

$$\frac{z}{f(z)} = 1 + b_1 z + \sum_{n=2}^{\infty} \lambda_n \frac{(1-\alpha)}{(n-1+\alpha)} z^n$$

So

$$\left\{\frac{z}{f(z)}\right\}' = b_1 + \sum_{n=2}^{\infty} n\lambda_n \frac{(1-\alpha)}{(n-1+\alpha)} z^{n-1}$$

$$\left|\left\{\frac{z}{f(z)}\right\}'\right| \leq |b_1| + \left|\sum_{n=2}^{\infty} n\lambda_n \frac{(1-\alpha)}{(n-1+\alpha)} z^{n-1}\right|$$

$$\leq |b_1| + |z| \left|\sum_{n=2}^{\infty} n\lambda_n \frac{(1-\alpha)}{(n-1+\alpha)}\right|$$

$$\leq |b_1| + \frac{2(1-\alpha)}{1+\alpha} r \quad \text{for } |z| = r \qquad \left(\text{since } \frac{(1-\alpha)}{(n-1+\alpha)} \text{ is decreasing}\right)$$

and also

$$\left|\left\{\frac{z}{f(z)}\right\}'\right| \geq |b_1| - \left|\sum_{n=2}^{\infty} n\lambda_n \frac{(1-\alpha)}{(n-1+\alpha)} z^{n-1}\right|$$

$$\geq |b_1| - |z| \left|\sum_{n=2}^{\infty} n\lambda_n \frac{(1-\alpha)}{(n-1+\alpha)}\right|$$

$$\geq |b_1| - \frac{2(1-\alpha)}{1+\alpha} r \quad \text{for } |z| = r < 1$$

Therefore

$$\max\left\{0, \; |b_1| - \frac{2(1-\alpha)}{1+\alpha} r\right\} \leq \left|\left\{\frac{z}{f(z)}\right\}'\right| \leq |b_1| + \frac{2(1-\alpha)}{1+\alpha} r$$

3. Convex subclass of Generalized Rational Univalent Functions

Ahuja and Pawan [1] obtained sufficient condition for convexity of generalized rational functions. Also proved the following condition:

The function $f(z) = \frac{z}{1+\sum_{n=1}^{\infty} b_n z^n}$ is convex of order α in U if

$$\frac{4-\alpha}{1-\alpha}|b_1| + \sum_{n=1}^{\infty} \frac{(n-1)(3n+1-\alpha)}{1-\alpha}|b_n| \leq 1 \tag{5}$$

Imposing this condition, now this section defines a subclass $C_+(b_1, \alpha)$ of S_+.

Definition 3.1

Let $b_1 \in \mathbb{C}$, $0 \leq |b_1| \leq 1/4$ be fixed and $0 \leq \alpha < 1$.

$C_+(b_1,\alpha) =\{f(z) \in S : \frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n, \; z \in U, \; b_n \geq 0$ for $n \geq 2$,

$(4-\alpha)|b_1| + \sum_{n=1}^{\infty}(n-1)(3n+1-\alpha)\, b_n \leq 1-\alpha\}$.
(6)

Now, the next result shows coefficient characterization for the subclass $C_+(b_1, \alpha)$

Theorem 3.2

Let $f(z) \in S$ for $z \in U$ be of the form $f(z) = \frac{z}{1+\sum_{n=1}^{\infty} b_n z^n}$ and $b_1 \in \mathbb{C}$, $|b_1| \leq 1/4$ be fixed. Then $f \in C_+(b_1, \alpha)$ for $0 \leq \alpha < 1$ if and only if $f(z)$ has the form

$$\frac{z}{f(z)} = b_1 z + \sum_{n=1}^{\infty} \lambda_n \frac{z}{f_n(z)}$$

For some sequence $\{\lambda_n\}_{n=1}^{\infty}$ of non-negative real numbers with $\sum_{n=1}^{\infty} \lambda_n = 1$ and

$$\frac{z}{f_n(z)} = \begin{cases} 1, & \text{for } n = 1 \\ 1 + \frac{1-\alpha}{(n-1)(3n+1-\alpha)} z^n, & \text{for } n = 2,3,\ldots \end{cases} \tag{7}$$

Proof:

Suppose that $f(z) \in S$ has the form (1) for some sequence $\{\lambda_n\}_{n=1}^{\infty}$of non-negative real numbers with $\sum_{n=1}^{\infty}\lambda_n = 1$. To prove that the function $f(z) \in C_+(b_1,\alpha)$.

Now, write $\frac{z}{f(z)}$ as

$$\frac{z}{f(z)} = b_1 z + \sum_{n=1}^{\infty}\lambda_n \frac{z}{f_n(z)}$$

$$= b_1 z + \lambda_1 + \sum_{n=2}^{\infty}\lambda_n \left[1 + \frac{1-\alpha}{(n-1)(3n+1-\alpha)} z^n\right] \quad \text{(by the definition of } \frac{z}{f_n(z)}\text{)}$$

$$= b_1 z + \lambda_1 + \sum_{n=2}^{\infty}\lambda_n + \sum_{n=2}^{\infty}\lambda_n \frac{1-\alpha}{(n-1)(3n+1-\alpha)} z^n$$

$$= 1 + b_1 z + \sum_{n=2}^{\infty}\lambda_n \frac{1-\alpha}{(n-1)(3n+1-\alpha)} z^n$$

$$= 1 + b_1 z + \sum_{n=2}^{\infty} b_n z^n \quad \text{where} \quad b_n = \lambda_n \frac{1-\alpha}{(n-1)(3n+1-\alpha)} \geq 0,\ n \geq 2.$$

Taking $\lambda_1 \in \mathbb{R}$ such that $|b_1| \leq \frac{1-\alpha}{4-\alpha}\lambda_1$,

$$(4-\alpha)|b_1| + \sum_{n=2}^{\infty}(n-1)(3n+1-\alpha)\, b_n$$

$$\leq (1-\alpha)\lambda_1 + \sum_{n=2}^{\infty}(n-1)(3n+1-\alpha)\lambda_n \frac{1-\alpha}{(n-1)(3n+1-\alpha)}$$

$$= (1-\alpha)\lambda_1 + \sum_{n=2}^{\infty}(1-\alpha)\lambda_n$$

$$= (1-\alpha)\sum_{n=1}^{\infty}\lambda_n = (1-\alpha)$$

This shows that con**d**ition (6) is satisfied.

Hence $f(z) \in C_+(b_1,\alpha)$ for $0 \leq \alpha < 1$.

Conversely, suppose $f(z) \in C_+(b_1,\alpha)$ for $z \in U$.

Therefore, by (6)

$$(4-\alpha)|b_1| + \sum_{n=2}^{\infty}(n-1)(3n+1-\alpha)\, b_n \leq (1-\alpha)$$

Now, set $\lambda_n = \frac{(n-1)(3n+1-\alpha)}{(1-\alpha)} b_n$ for $n \geq 2$ and $\lambda_1 = 1 - \sum_{n=2}^{\infty}\lambda_n$

Therefore $\frac{z}{f(z)} = 1 + b_1 z + \sum_{n=2}^{\infty} b_n z^n$

$$= \lambda_1 + \sum_{n=2}^{\infty}\lambda_n + b_1 z + \sum_{n=2}^{\infty}\lambda_n \frac{1-\alpha}{(n-1)(3n+1-\alpha)} z^n$$

$$= b_1 z + \lambda_1 + \sum_{n=2}^{\infty}\lambda_n \left[1 + \frac{1-\alpha}{(n-1)(3n+1-\alpha)} z^n\right]$$

$$= b_1 z + \sum_{n=1}^{\infty}\lambda_n \frac{z}{f_n(z)}$$

This completes the proof.

The following results discuss growth and distortion bounds for the subclass $C_+(b_1, \alpha)$.

Theorem 3.3

If $f \in C_+(b_1, \alpha)$ for $z \in U$ and $0 \le \alpha < 1$, $0 \le |b_1| \le 1/4$, then for $|z| = r < 1$, then

$$\max\left\{0,\ 1 - |b_1|r - \frac{1-\alpha}{7-\alpha} r^2\right\} \le \left|\frac{z}{f(z)}\right| \le 1 + |b_1|r + \frac{1-\alpha}{7-\alpha} r^2$$

Proof: Since $f(z) \in C_+(b_1, \alpha)$, by Theorem 2.2

$$\frac{z}{f(z)} = b_1 z + \sum_{n=1}^{\infty} \lambda_n \frac{z}{f_n(z)}$$

$$= b_1 z + \lambda_1 + \sum_{n=2}^{\infty} \lambda_n \left[1 + \frac{1-\alpha}{(n-1)(3n+1-\alpha)} z^n\right]$$

$$= 1 + b_1 z + \sum_{n=2}^{\infty} \lambda_n \frac{1-\alpha}{(n-1)(3n+1-\alpha)} z^n \qquad (8)$$

So

$$\left|\frac{z}{f(z)}\right| \le 1 + |b_1 z| + \left|\sum_{n=2}^{\infty} \lambda_n \frac{1-\alpha}{(n-1)(3n+1-\alpha)} z^n\right|$$

$$\le 1 + |b_1||z| + |z|^2 \left|\sum_{n=2}^{\infty} \lambda_n \frac{1-\alpha}{(n-1)(3n+1-\alpha)}\right|$$

$$\le 1 + |b_1|r + \frac{1-\alpha}{(7-\alpha)} r^2 \quad \text{for } |z| = r < 1$$

$$\left(\text{since } \frac{1-\alpha}{(n-1)(3n+1-\alpha)} \text{ is decreasing}\right)$$

From (8), write

$$\left|\frac{z}{f(z)}\right| \ge 1 - |b_1 z| - \left|\sum_{n=2}^{\infty} \lambda_n \frac{1-\alpha}{(n-1)(3n+1-\alpha)} z^n\right|$$

$$\ge 1 - |b_1||z| - |z|^2 \left|\sum_{n=2}^{\infty} \lambda_n \frac{1-\alpha}{(n-1)(3n+1-\alpha)}\right|$$

$$\ge 1 - |b_1|r - \frac{1-\alpha}{(7-\alpha)} r^2 \quad \text{for } |z| = r < 1$$

Therefore

$$\max\left\{0,\ 1 - |b_1|r - \frac{1-\alpha}{(7-\alpha)} r^2\right\} \le \left|\frac{z}{f(z)}\right| \le 1 + |b_1|r + \frac{1-\alpha}{(7-\alpha)} r^2$$

Theorem 3.4

If $f \in C_+(b_1, \alpha)$, $z \in U$ for $0 \le \alpha < 1$, $0 \le |b_1| \le 1/4$, $|z| = r < 1$, then

$$\max\left\{0,\ |b_1| - \frac{2(1-\alpha)}{7-\alpha} r\right\} \le \left|\left\{\frac{z}{f(z)}\right\}'\right| \le |b_1| + \frac{2(1-\alpha)}{7-\alpha} r.$$

Proof: Let $f(z) \in C_+(b_1, \alpha)$, then from (8),

$$\frac{z}{f(z)} = 1 + b_1 z + \sum_{n=2}^{\infty} \lambda_n \frac{1-\alpha}{(n-1)(3n+1-\alpha)} z^n$$

$$\left\{\frac{z}{f(z)}\right\}' = b_1 + \sum_{n=2}^{\infty} n\,\lambda_n \frac{1-\alpha}{(n-1)(3n+1-\alpha)} z^{n-1}$$

And

$$\left|\left\{\frac{z}{f(z)}\right\}'\right| \le |b_1| + \left|\sum_{n=2}^{\infty} n\,\lambda_n \frac{1-\alpha}{(n-1)(3n+1-\alpha)} z^{n-1}\right|$$

$$\le |b_1| + |z| \left|\sum_{n=2}^{\infty} n\,\lambda_n \frac{1-\alpha}{(n-1)(3n+1-\alpha)}\right|$$

$$\le |b_1| + \frac{2(1-\alpha)}{7-\alpha} r \quad \text{for } |z| = r < 1$$

(since $\frac{1-\alpha}{(n-1)(3n+1-\alpha)}$ is decreasing)

also

$$\left|\left\{\frac{z}{f(z)}\right\}'\right| \ge |b_1| - \left|\sum_{n=2}^{\infty} n\,\lambda_n \frac{1-\alpha}{(n-1)(3n+1-\alpha)} z^{n-1}\right|$$

$$\ge |b_1| - |z| \left|\sum_{n=2}^{\infty} n\,\lambda_n \frac{1-\alpha}{(n-1)(3n+1-\alpha)}\right|$$

$$\ge |b_1| - \frac{2(1-\alpha)}{7-\alpha} r \quad \text{for } |z| = r < 1$$

Therefore

$$\max\left\{0,\ |b_1| - \frac{2(1-\alpha)}{7-\alpha} r\right\} \le \left|\left\{\frac{z}{f(z)}\right\}'\right| \le |b_1| + \frac{2(1-\alpha)}{7-\alpha} r$$

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