Coincidence and Fixed Point for Kannan Type Mappings

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Abstract

In this paper we have shown some coincidence and common fixed-point theorems for Kannan type mappings. We have used the additional conditions as complete metric space and asymptotic regularity and Kannan fixed point.

Keywords: Fixed Point, Coincidence Point complete metric space, asymptotic regularity, Kannan mapping.

Introduction:

The Banach contraction mapping is one of the significant results of functional analysis. It is generally source of metric fixed point theory. Also, it is used for many significant and applicability in number of branches of mathematics. In 1968 Kannan [6] proved a fixed-point theorem for map satisfying a contraction condition that did not require continuity at each point. We know as Kannan fixed point theorem.

The principal states that, if (X, d) is a complete metric space and $f: X \to X$ is a mapping such that there exists $k < \frac{1}{2}$ satisfying.

$d(fx, fy) \leq k[d(x, fx + d(y, fy)]$ for all $x, y \in X$

Then, f has a unique fixed point $a \in X$, and for any $x \in X$ the sequence of iterates $\{fx_n\}$ converges to v and $d(fx_{n+1}, u) \leq K.\left(\frac{K}{1-K}\right)^n .d(x, fx)$ where $n = 0,1,2,3 \ldots \ldots$

Kannan's theorem is important because Subrahmanyam [11] also proved that Kannan's theorem characterizes the metric completeness, i.e. A metric space X is complete if every mapping satisfying "Lipschitzian mappings" on X with constant $K < \frac{1}{2}$ has a fixed point.

The necessary existence of fixed points for Kannan type mappings implies that the corresponding metric space is complete but same is not true with the Banach contraction. Absolutely, there I an example of an incomplete metric space where every contraction has a fixed point [6]. Kannan type mappings, its generalizations and extensions in various spaces.

In this paper, we have proved coincidence point and common fixed point.

Definition 1.1

Let (X, d) be complete metric space and $f: X \to X$ is called Bannach contraction map that is $d(fx, fy) \leq d(x, y)$ for all $x, y \in X$ where $k \in (0,1)$ is constant. Then f has unique fixed point.

Definition 1.2 If (X, d) is complete metric space and f is self mapping is called Kannan type contraction$d(fx, fy) \leq \frac{1}{2}[d(x, fx) + d(y, fy)]$ for all $x \neq y \in X$

Definition 1.3 If (X, d) is metric space. A sequence $\{y_n\}$ is called conveyent for $\varepsilon > 0$ and $n \geq N$ we have $d(x_n, x) \in \varepsilon$ where x is called the limit point of the sequence $\{y_n\}$.

Definition 1.4 If (X, d) is metric space. A Sequence $\{y_n\}$ is called a Cauchy sequence for $\varepsilon > 0$ there exist a positive integer N such that $m, n \geq N$ we have $(x_m, x_n) < \varepsilon$

Definition 1.5A metric space (X, d) is said to be complete, if every Cauchy sequence is convergent.

Definition 1.6.[5].

A metric space (X, d) is said to be boundedly compact if every bounded sequence in X has a convergent subsequence. It is clear that every compact metric space is boundedly compact, but boundedly compact metric space need not be compact.

Definition 1.7.[6]. Asymptotic regularity: If (X, d) is a metric space. A mapping $f: S \to S$ satisfying the condition $\lim_{n\to\infty} d\,(fx_{n+1}, fx_n) = 0$ for all $x \in S$ is called asymptotic regular. In other words, suppose$S = \{0\} \cup [1,2]$ with S the standard metric. A mapping $f: S \to S$ defined by $f_0 = 1$ and $f_x = 0$ for $\leq x \leq 2$ for all $x \in S$ with $K = \frac{1}{2}$ and f is fixed point. The iterative sequence $\{x_n = fx_0\}$ is not conveyent Then. f is not asymptotically regular.

Example 2.1:[5]Suppose that $S = [0,1]$ with stander metric. If $f: [0,1] \to [0,1]$ is mapping defined by $f_0 = \frac{1}{2}$ and $f_x = \frac{x}{2}$ for $0 < x \leq 1$.

Then for $0 < x < y \leq 1$, $|f_x, f_y| = \frac{1}{2}(y - x) < \frac{1}{2}(x, y)$

$= |x - f_x| + |y - f_y|$ for $0 \leq x \leq 1$

$$|f_0 - f_x| \leq \frac{1}{2} - \frac{x}{2} < \frac{1}{2} + \frac{x}{2} = |0 - f_0| + |x - f_x|$$

Thus $|f_x - f_y| < |x - f_x| + |y - f_y|$ for all $x, y \in [0,1] x \neq y$

Hence, f is asymptotically regular and fixed point.

Definition 1.8. [5]. Fixed Function: Suppose C is any self-mapping defined on a family of a functions X. Then $g \in X$ is called fixed function of C. If $Cg = g$.

Example 2.2.[5]. Suppose $W = [1,2]$ and the mapping C is defined as $Cg(w) = g^2(w) - 2g(w) + 2$ for all $g \in X$ and $w \in W$. Then $g(w) = 2$ for all $w \in W$ and $g(w) = 1$ for all

$w \in W$ are two fixed functions of C.

Main Result:

Theorem3.1:

Suppose (X, d) is a complete metric space and $f, g: y \to X$ are asymptotically regular mappings such that $M < 1$ satisfying condition:

(i) $d(f_x, f_y) \leq M[d(g_x, f_x) + d(g_y, f_y) + d(g_x, g_y)]$ for all $x, y \in X$
(ii) $f(Y) \subseteq g(Y)$
(iii) Either $f(Y)$ or $g(Y)$ is complete.

Then f, g have a coincidence point.

Proof:

Let x_0, x_1 be point of Y such that fx_0, gx_1 (This is possible for $f(Y) \subseteq g(Y)$). Since $f(Y) \subseteq g(Y)$ and $fx_0 = gx_1$, hence we can construct a sequence $\{x_n\}$ such that $fx_n = gx_{n+1}$

Now $d(fx_n, fx_{n+1}) = d(z_n, z_{n+1})$

$$d(fx_n, fx_{n+1}) \leq M[d(gx_n, fx_n) + d(gx_{n+1}, fx_{n+1}) + d(gx_n, gx_{n+1})]$$

$$\begin{aligned} d(z_n, z_{n+1}) &\leq M[d(z_{n-1}, z_n) + d(z_n, z_{n+1}) + d(z_{n-1}, z_n)] \\ &\leq 2Md(z_{n-1}, z_n) + Md(z_n, z_{n+1}) \end{aligned}$$

$$d(z_n, z_{n+1}) \leq \frac{2M}{1-M}(d(z_{n-1}, z_n) + d(z_n, z_{n+1})] \to 0$$

Using Lemma (4.1), $z_n = \{fx_n\}$ is Cauchy sequence. Hence $g(Y)$ is complete. Then there exists $p \in g(Y)$.

Such that $\lim_{n\to\infty} z_n \to p$. Then $\exists\ z \in y$ such that $g_z = p$

Putting $x = x_n, y = z$ we have

$d(fx_n, fz) \leq M[d(gx_n, fx_n) + d(gz, fz) + d(gx_n, gz)$

$d(p, fz) \leq M[d(p,p) + d(gz, fz) + d(p, fz)]$

$d(p, fz) \leq M(p, fz)$

Which is contradiction when $1 \leq M$. Hence it is possible only if $d(p, fz) = 0$, So

$$p = f(z) = g(z).$$

Hence p is coincidence point of $f$and$g$.

Again if $f(Y)$ is complete then $z_n \to p \in f(Y) \subseteq g(Y)$

Hence z is coincidence point of f and g.

Theorem 3.2

Suppose (X, d) is a complete metric space and $f, g: x \to x$ is an asymptotically regular mapping such that exists $M < 1$ satisfying condition.

(i) $d(fx, fy) \leq M[d(gx, fx) + d(gy, fy) + d(gx, gy)$ for all $x, y \in X$.
(ii) $f(X) \subseteq g(X)$ is complete.
(iii) f and g are commuting at their coincidence point, f and g have unique common fixed point.

Proof:

If we take $Y = X$ in theorem (3.1) then we get $z_n = fx_n$ such that $\{z_n\}$ is cauchy sequence. Suppose $g(X)$ is complete. Then $z_n \to p(X) \in g(X)$, hence there exists $z \in X$ such that $f(z) = p$. Putting $x = x_n; y = f(z)$in (i) we have $f(z) = g(z) = p$.

Since $f$and$g$ are commuting at their coincidence point So, $fgz = gfz = g_p = f_p$.

Putting $x = p$ and $y = p' \, in \, (i) \, we \, get$

$$d(fz, ffz) \leq M[d(gz, fz) + d(gfz, ffz) + d(gz, gfz)$$

$$d(p, fp) \leq M[d(p, fp) + d(gp, fp) + d(p, gp)$$

Again, we have $g_p = p$

$$d(p, fp) \leq M[d(p, fp) + d(p, fp) + d(p, p)$$

$d(p, fp) \leq 2M[d(p, fp)$ which is not possible when $d(p, fp) \neq 0$ therefore $p = f_p = g_p$ that is p is a common fixed point of $f$and$g$.

For uniqueness, letp'beanother common fixed point then put $x = p, y = p'$ such that $p = p'$

$$d(fp, fp') \leq M[d(gp, fp) + d(gp', fp') + d(gp, fp')]$$

$$d(p, p') \leq M[d(p, p) + d(p', p') + d(p, p')] \leq Md(p, p')$$

A contradiction! which is mean such that $p = p'$. Thus p is unique common fixed point of f and g.Hence, we prove the uniqueness.

Theorem 3.3:

Let $(X, d)$is a complete metric space and $f: X \to X$ is asymptotically regular mapping such that exists $M < 1$ satisfying

$$d(fx, fy) \leq M[d(x, fx) + d(y, fy) + d(x, y)]$$

For all $x, y \in X$.

Then f has a unique fixed point $u \in X$.

Lemma 4.1:

If C is a non-empty closed sub-set of a complete metric space. (x, d) and let $f: C \to C$ is a mapping such that there exists $K < 1$ satisfying

$d(fx, fy) \leq k[d(x, fx) + d(y, fy)]$ for all $x, y \in X$

We suppose that there exists constant $x, y \in \mathbb{R}$ such that $0 \leq x \leq 1$ and $y > 0$.

If for arbitrary $a \in C$ there exists $x \in C$ such that $d(v, fv) \leq x. d(a, fx)$ and $d(v, a) \leq y. d(a, fa)$ then f has at least one fixed point.

Conclusion:

In our paper, we have used two mappings for Kannan type mappings. If we take $X = Y$ and

$g = I$ (Identity map) our theorem (3.1) then results of "Jaroslow Gomichi" [6] is became a particular case of our result. In this sense, our result is generalized form "JaroslowGomichi".

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