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On Certaın Common Coupled Fıxed Poınts of Ratıonal Contractıon Mappıngs ın Partıal $B-$ Metrıc Spaces and ıts Applıcatıons

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Abstract
In this paper, we prove common coupled fixed point results for Jungck-type maps by using rational contraction conditions in partial-b-metric space. Our results generalize and expand some well-known results. We also explore some of the applications to integral equations as well as Homotopy. Also, we proved an example to support our result.
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1. INTRODUCTION:

Fixed point theory is one of the most fruitful role in nonlinear analysis because of its wide applications in Homotopy theory, integral, integrodifferential, impulsive differential equations, and Approximation theory and has been studied in various metric spaces.

In 1993, Czerwik [1], [2] extended results related to the b-metric spaces. In 1994, Matthews [3] introduced the concept of partial metric space in which the self-distance of any point of space may not be zero. In 2013, Shukla [5] generalized both the concept of b-metric and partial metric spaces by introducing the partial b-metric spaces.

Bhaskar and Lakshmikantham [6] introduced the notion of a coupled fixed point and proved some coupled fixed point theorems for mixed monotone mappings in ordered metric spaces. After that many authors proved coupled fixed point theorems on various spaces for example ([7]-[12]).

In this paper, we prove a common coupled fixed point theorem for Jungck-type maps in partial $b-$metric spaces.

2. PRELIMINARIES:

In order to obtain our results we need to consider the followings.

Definition 2.1. ([1]) Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric if for all $x, y, z \in X$ the following conditions are satisfied:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) \leq s\,(d(x, z) + d(z, y))$.

The pair (X, d) is called a b-metric space. The number $s \geq 1$ is called the coefficient of (X, d).

Definition 2.2. ([3]) Let X be a nonempty set. A function $P : X \times X \to [0, \infty)$ is called a partial metric if if for all $x, y, z \in X$ the following conditions are satisfied:

(i) $x = y$ if and only if $P(x, x) = P(x, y) = P(y, y)$;

(ii) $P(x, x) \leq P(x, y)$;

(iii) $P(x, y) = P(y, x)$;

(iv) $P(x, y) \leq P(x, z) + P(z, y) - P(z, z)$.

The pair (X, P) is called a partial metric space.

Remark 2.3. It is clear that the partial metric space need not be a metric spaces, since in a b-metric space if $x = y$, then $d(x, x) = d(x, y) = d(y, y) = 0$. But in a partial metric space if $x = y$ then $P(x, x) = P(x, y) = P(y, y)$ may not be equal zero. Therefore the partial metric space may not be a b- metric space.

On the other hand, Shukla ([5]) introduced the notion of a partial b-metric space as follows:

Definition 2.4. ([5]) Let X be a nonempty set and $s \geq 1$ be a given real number. A function $P_b : X \times X \to [0, \infty)$ is called a partial b- metric if for all $x, y, z \in X$ the following conditions are satisfied:

(P_b1). $x = y$ if and only if $P_b(x, x) = P_b(x, y) = P_b(y, y)$;

(P_b2) $P_b(x, x) \leq P_b(x, y)$;

(P_b3) $P_b(x, y) = P_b(y, x)$;

(P_b4) $P_b(x, y) \leq s\,(P_b(x, z) + P_b(z, y) - P_b(z, z))$.

The pair (X, P_b) is called a partial b−metric space. The number $s \geq 1$ is called the coefficient of (X, P_b).

Remark 2.5. The class of partial b-metric space (X, P_b) is effectively larger than the class of partial metric space since a partial metric space is a special case of a partial b-metric space (X, P_b) when s = 1. Also, the class of partial b-metric space (X, P_b) is effectively larger than the class of b-metric space, since a b-metric space is a special case of a partial b-metric space (X, P_b) when the self-distance P (x; x) = 0. The following examples show that a partial b- metric on X need not be a partial metric, nor a b-metric on X see also ([4], [5]).

Example 2.6. ([5]) Let X = [0, 1) and the fumction $P_b : X \times X \to [0, \infty)$ be defined as $P_b(x; y) = [\max\{x, y\}]^2 + |x - y|^2$, for all $x, y \in X$. Then (X, P_b) is called a partial b-metric space with the coefficient $s \geq 1$. But, P_b is not a b-metric nor a partial metric on X.

Definition 2.7. ([4]) Every partial b-metric P_b defines a b-metric d_{p_b}, where

$d_{P_b}(x, y) = 2P_b(x, y) - P_b(x, x) - P_b(y, y)$, for all $x, y \in X$.

Definition 2.8. ([4]) A sequence $\{x_n\}$ in a partial b-metric space (X, P_b) is said to be:

(i) P_b -convergent to a pointof $x \in X$ i f $\lim_{n\to\infty} p_b(x, x_n) = p_b(x, x)$

(ii) P_b -Cauchy sequence if $\lim_{n,m\to\infty} p_b(x_n, x_m)$ exist and finite

(iii) A partial b-metric space (X, P_b) is said to be P_b -complete if every P_b-Cauchy sequence $\{x_n\}$ in X is P_b converges to a point $x \in X$ such that

$\lim_{n,m\to\infty} P_b(x_n, x_m) = \lim_{n,\to\infty} P_b(x_n, x) = P_b(x, x)$.

Lemma 2.9. ([4]) A sequence $\{x_n\}$ is a P_b-Cauchy sequence in a partial b-metric space (X, P_b) if and only if it is a b-Cauchy sequence in the b-metric space (X, d_{P_b}).

Lemma 2.10. ([4]) A partial b-metric space (X, P_b) is P_b-complete if and only if the b-metric space (X, d_{P_b}) is b-complete.

Moreover $\lim_{n,m\to\infty} d_{p_b}(x_n, x_m) = 0$ iff $\lim_{m\to\infty} P_b(x_m, x) = \lim_{n\to\infty} P_b(x_n, x) = P_b(x, x)$.

Definition 2.11. ([6]) Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if x = F (x, y) and y = F (y, x).

Definition 2.12. ([8]) Let $F : X^2 \to X$ and $f : X \to X$ be two mappings . An element (x, y) is said to be a coupled coincident point of F and f if F (x, y) = fx, F (y, x) = fy.

Definition 2.13. ([8]) Let $F : X^2 \to X$ and $f : X \to X$ be two mappings. An element (x, y) is said to be a coupled common point of F and f if F (x, y) = fx = x, F (y, x) = fy = y.

Definition 2.14. ([8]) Let (X, P_b) be partial b-metric space. A pair (F, f) is called weakly compatible if

$f(F(x, y)) = F(fx, fy)$ whenever for all $x, y \in X$ such that $F(x, y) = fx$, $F(y, x) = fy$.

Now we prove our main result.

3. MAIN RESULTS :

Theorem 3.1. Let (X, P_b) be a Partial b- metric space, let $R: X \times X \to X$ and $A : X \to X$ be mappings satisfying the following

(3.1.1) $sP_b(R(x,y), R(u,v)) \le k \max\left\{ \begin{matrix} P_b(Ax,Au), P_b(Ay, Av), \\ \frac{P_b(Ax,R(x,y))P_b(Au,R(u,v)).}{1+P_b(Ax,Au)} \quad \frac{P_b(Ay,R(y,x))P_b(Av,R(v,u))}{1+P_b(Ay,Av)} \end{matrix} \right\}$

(3.1.2) $R(X \times X) \subseteq A(X)$ and $A(X)$ is complete subspace of X

(3.1.3) (R, A) is weakly compatible pair.

Then R and A have a unique common coupled fixed point in X.

Proof. Let x_0, y_0 be arbitrary points in X. From (3.1.2), There exist sequences $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ in X such that

$R(x_n, y_n) = Ax_{n+1} = u_n$ $R(y_n, x_n) = Ay_{n+1} = v_n$ for all $n \ge 0$.

For simplification we denote $R_n = \max\{P_b(u_n, u_{n+1}), P_b(v_n, v_{n+1})\}$.

Case(i): Suppose that $u_n = u_{n+1}$ and $v_n = v_{n+1}$ for some n

Claim: $u_{n+1} = u_{n+2}$ and $v_{n+1} = v_{n+2}$

Suppose $u_{n+1} \neq u_{n+2}$ and $v_{n+1} \neq v_{n+2}$. From (3.1.1) consider

$P_b(u_{n+1}, u_{n+2}) \le s.P_b(R(x_{n+1}, y_{n+1}), R(x_{n+2}, y_{n+2}))$

$$\le k\max\left\{ \begin{matrix} P_b(u_n,u_{n+1}), P_b(v_n,v_{n+1}), \\ \frac{P_b(u_n,u_{n+1}).P_b(u_{n+1},u_{n+2})}{1+P_b(u_n,u_{n+1})} \quad \frac{P_b(v_n,v_{n+1})P_b(v_{n+1},v_{n+2})}{1+P_b(v_n,v_{n+1})} \end{matrix} \right\}$$

$\le k.\max\{P_b(u_{n+1}, u_{n+2}), P_b(v_{n+1}, v_{n+2})\}$.

Similarly,

$P_b(v_{n+1}, v_{n+2}) \le k.\max\{P_b(u_{n+1}, u_{n+2}), P_b(v_{n+1}, v_{n+2})\}$

It follows that

$\max\{P_b(u_{n+1}, u_{n+2}), P_b(v_{n+1}, v_{n+2})\} \le k.\max\{P_b(u_{n+1}, u_{n+2}), P_b(v_{n+1}, v_{n+2})\}$

$$< \max\{P_b(u_{n+1}, u_{n+2}), P_b(v_{n+1}, v_{n+2})\},$$

is contradiction. Hence $u_{n+1} = u_{n+2}$ and $v_{n+1} = v_{n+2}$.

Continuing this way we can conclude that $u_n = u_{n+l}$ and $v_n = v_{n+l}$ for all l. It is clear that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequence in (X, P_b).

Case (ii): Suppose that $u_n \neq u_{n+1}$ and $v_n \neq v_{n+1}$

From (3.1.1), we have that

for all $n \geq 0$. $P_b(u_n, u_{n+1}) \leq s.P_b(R(x_n, y_n), R(x_{n+1}, y_{n+1}))$

$$\leq k\max\left\{\begin{matrix} P_b(u_{n-1}, u_n), P_b(v_{n-1}, v_n), \\ \frac{P_b(u_n, u_{n+1}).P_b(u_{n-1}, u_n)}{1+P_b(u_{n-1}, u_n)} \quad \frac{P_b(v_n, v_{n+1})P_b(v_{n-1}, v_n)}{1+P_b(v_n, v_{n+1})} \end{matrix}\right\}$$

$$\leq k.\max\{P_b(u_{n-1}, u_n), P_b(v_{n-1}, v_n), P_b(u_n, u_{n+1}), P_b(v_n, v_{n+1})\}.$$

$$\leq k.\max\{R_{n-1}, R_n\}$$

Similarly, $P_b(v_n, v_{n+1}) \leq k.\max\{R_{n-1}, R_n\}$.

Thus $R_n \leq k.\max\{R_{n-1}, R_n\}$.

If R_n is maximum, we have $R_n \leq k.R_n < R_n$. This is a contradiction. Hence R_{n-1} is maximum. Thus

$$\begin{aligned} R_n &\leq k.R_{n-1} \\ &\leq R_{n-1} \qquad (3.1) \end{aligned}$$

Therefore, $\{R_n\}$ is decreasing sequence and converges to $t \geq 0$.

Suppose $t > 0$, letting $n \to \infty$ in (3.1),

we have that $t \leq k.t < t$, is a contradiction . Hence $t = 0$.

Thus $\lim_{n\to\infty} R_n=0$. . It follows that $\lim_{n\to\infty} P_b(u_n, u_{n+1})=0=\lim_{n\to\infty} P_b(v_n, v_{n+1})$ (3.2)

From (3.2) and (P_b2), we have that $\lim_{n\to\infty} P_b(u_n, u_n)=0=\lim_{n\to\infty} P_b(v_n, v_n)$ (3.3)

From definition of d_{P_b}, (3.2) and (3.3), we have that

$$\lim_{n\to\infty} d_{p_b}(u_n, u_{n+1})=0=\lim_{n\to\infty} d_{p_b}(v_n, v_{n+1}) \qquad (3.4)$$

Now we prove that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequence in Partial b-metric space (X, P_b). If sufficient to prove that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequence in b-metric space (X, d_{P_b}).

On contrary suppose that either$\{u_n\}$ and$\{v_n\}$ are not Cauchy sequence. This gives that

$d_{P_b}(u_n, u_m) \nrightarrow 0$ and $d_{P_b}(v_n, v_m) \nrightarrow 0$ as $n, m \to \infty$.

Consequently, $\max\{ d_{P_b}(u_n, u_m), d_{P_b}(v_n, v_m)\} \nrightarrow 0$ as $n, m \to \infty$.

Then there exist an $\epsilon > 0$ and monotonically increases sequences of natural numbers $\{m_k\}$, $\{n_k\}$ such that $n_k > m_k > k$

$$\max\{d_{P_b}(u_{n_k}, u_{m_k}), d_{P_b}(v_{n_k}, v_{m_k})\} \geq \epsilon \qquad (3.5)$$

and $$\max\{d_{P_b}(u_{n_{k-1}}, u_{m_k}), d_{P_b}(v_{n_{k-1}}, v_{m_k})\} < \epsilon. \quad (3.6)$$

From (3.5) and (3.6), we have that

$\epsilon \leq \max\{d_{P_b}(u_{n_k}, u_{m_k}), d_{P_b}(v_{n_k}, v_{m_k})\}$

$\leq s.\max\{d_{P_b}(u_{m_k}, u_{n_{k-1}}), d_{P_b}(v_{m_k}, v_{n_k})\} + s.\max\{d_{P_b}(u_{n_{k-1}}, u_{n_k}), d_{P_b}(v_{n_{k-1}}, v_{n_k})\}$

$< s.\epsilon + s.\max\{d_{P_b}(u_{n_{k-1}}, u_{n_k}), d_{P_b}(v_{n_{k-1}}, v_{n_k})\}$

Taking upper limit as $k \to \infty$ and from (3.2) , we have that

$$\epsilon \leq \limsup_{k\to\infty} \max\{d_{P_b}(u_{n_k}, u_{m_k}), d_{P_b}(v_{n_k}, v_{m_k})\} \leq s.\epsilon. \quad (3.7)$$

Also

$\epsilon \leq \max\{d_{P_b}(u_{n_k}, u_{m_k}), d_{P_b}(v_{n_k}, v_{m_k})\}$

$\leq s.\max\{d_{P_b}(u_{m_k}, u_{n_{k+1}}), d_{P_b}(v_{m_k}, v_{n_{k+1}})\}$

$+ s.\max\{d_{P_b}(u_{n_{k+1}}, u_{n_k}), d_{P_b}(v_{n_{k+1}}, v_{n_k})\}$

Taking upper limit as $k \to \infty$ and from (3.4), we have that

$\frac{\epsilon}{s} \leq \limsup \max\{d_{P_b}(u_{m_k}, u_{n_{k+1}}), d_{P_b}(v_{m_k}, v_{n_{k+1}})\}$. (3.8)

On the other hand

$\max\{d_{P_b}(u_{m_k}, u_{n_{k+1}}), d_{P_b}(v_{m_k}, v_{n_{k+1}})\}$

$\leq s.\max\{d_{P_b}(u_{m_k}, u_{n_k}), d_{P_b}(v_{m_{k+1}}, v_{n_k})\}$

$+ s.\max\{d_{P_b}(u_{n_k}, u_{n_{k+1}}), d_{P_b}(v_{n_k}, v_{n_{k+1}})\}$

Taking upper limit as $k \to \infty$ and from (3.4), we have that

$$\limsup \max\{d_P(u_m, u_{n+1}), d_P(v_m, v_{n+1})\} \leq \epsilon s^2 \quad (3.9)$$

also, from (3.5), we have that

$\epsilon \leq \max\{d_{P_b}(u_{n_k}, u_{m_k}), d_{P_b}(v_{n_k}, v_{m_k})\}$

$\leq s.\max\{d_{P_b}(u_{m_k}, u_{m\,k+1}), d_{P_b}(v_{m_k}, v_{m\,k+1})\}$

$+ s.\max\{d_{P_b}(u_{n\,k+1}, u_{n_k}), d_{P_b}(v_{n\,k+1}, v_{n_k})\}$

$$\leq \left\{\begin{array}{c} s.\max\{d_{p_b}(u_{m_k}, u_{m_{k+1}}),\ d_{p_b}(v_{m_k}, v_{m_{k+1}})\} \\ +s^2.\max\{d_{p_b}(u_{m_{k+1}}, u_{n_{k+2}}),\ d_{p_b}(v_{m_{k+1}}, v_{n_{k+2}})\} \\ +s^2.\max\{d_{p_b}(u_{n_{k+2}}, u_{n_k}),\ d_{p_b}(v_{n_{k+2}}, v_{n_k})\} \end{array}\right\}$$

$$\leq \left\{\begin{array}{c} s.\max\{d_{p_b}(u_{m_k}, u_{m_{k+1}}),\ d_{p_b}(v_{m_k}, v_{m_{k+1}})\} \\ +s^2.\max\{d_{p_b}(u_{n_{k+1}}, u_{n_{k+2}}), d_{p_b}(v_{n_{k+1}}, v_{n_{k+2}})\} \\ +s^3.\max\{d_{p_b}(u_{n_{k+1}}, u_{n_{k+2}}), d_{p_b}(v_{n_{k+1}}, v_{n_{k+2}})\} \\ +s^3.\max\{d_{p_b}(u_{n_{k+1}}, u_{n_k}), d_{p_b}(v_{n_{k+1}}, v_{n_k})\} \end{array}\right\}$$

Taking upper limit as $k \to \infty$ and from (3.4), we have that

$\frac{\epsilon}{s^3} \leq \limsup \max\{d_{P_b}(u_{m\,k+1}, u_{n\,k+2}), d_{P_b}(v_{m\,k+1}, v_{n\,k+2})\}$

On other hand

$\max\{d_{P_b}(u_{m\,k+1}, u_{n\,k+2}), d_{P_b}(v_{m\,k+1}, v_{n\,k+2})\}$

$\leq s.\max\{d_{P_b}(u_{m\,k+1}, u_{m_k}), d_{P_b}(v_{m\,k+1}, v_{m_k})\}$

$+ s.\max\{d_{P_b}(u_{m_k}, u_{n\,k+2}), d_{P_b}(v_{m_k}, v_{n\,k+2})\}$

$$\leq \left\{\begin{array}{c} s.\max\{d_{p_b}(u_{m_{k+1}}, u_{m_k}), d_{p_b}(v_{m_{k+1}}, v_{m_k})\} \\ +s^2.\max\{d_{p_b}(u_{m_k}, u_{n_k}), d_{p_b}(v_{m_k}, v_{n_k}))\} \\ +s^3.\max\{d_{p_b}(u_{n_k}, u_{n_{k+1}}), d_{p_b}(v_{n_k}, v_{n_{k+1}})\} \\ +s^3.\max\{d_{p_b}(u_{n_{k+1}}, u_{n_{k+2}}), d_{p_b}(v_{n_{k+1}}, v_{n_{k+2}})\} \end{array}\right\}$$

Taking upper limit as $k \to \infty$ and from (3.4), (3.7) we have that

$$\limsup \max\{d_P(u_{m+1}, u_{n+2}), d_P(v_{m+1}, v_{n+2})\} \leq \epsilon.s^3. \quad (3.10)$$

Now

$p_b(u_{m_{k+1}}, u_{n_{k+2}}) \leq s.p_b(u_{m_{k+1}}, u_{n_{k+2}})$

$\leq s.P_b(R(x_{m_{k+1}}, y_{m_{k+1}}), R(x_{n_{k+2}}, y_{n_{k+2}}))$

$$\leq k\max\left\{\begin{array}{c} p_b(u_{m_k}, u_{m_{k+1}})p_b(v_{m_k}, v_{m_{k+1}}) \\ \frac{p_b(u_{m_k}, u_{m_{k+1}}).Pb(u_{n_{k+1}}, u_{n_{k+2}})}{1+Pb(u_{m_k}, u_{n_{k+1}})} \\ \frac{Pb(vmk, vmk+1).Pb(vnk+1, vnk+2)}{1+p_b(v_{m_k}, v_{n_{k+1}})} \end{array}\right\}.$$

Similarly, we can prove

$$p_b\,(v_{m_{k+1}}, v_{n_{k+2}}) \leq k \max\left\{\begin{array}{c} p_b\,(u_{m_k}, u_{m_{k+1}})p_b\,(v_{m_k}, v_{m_{k+1}}) \\ \frac{p_b\,(u_{m_k}, u_{m_{k+1}}).Pb(u_{n_{k+1}}, u_{n_{k+2}})}{1+Pb(u_{m_k}, u_{n_{k+1}})} \\ \frac{Pb(vmk, vmk+1).Pb(vnk+1, vnk+2)}{1+p_b\,(v_{m_k}, v_{n_{k+1}})} \end{array}\right\}.$$

Thus

$$\max\left\{\begin{array}{c} p_b\,(u_{m_{k+1}}, u_{n_{k+2}}) \\ p_b\,(v_{m_{k+1}}, v_{n_{k+2}}) \end{array}\right\}) \leq k \max\left\{\begin{array}{c} p_b\,(u_{m_k}, u_{m_{k+1}})p_b\,(v_{m_k}, v_{m_{k+1}}) \\ \frac{p_b\,(u_{m_k}, u_{m_{k+1}}).Pb(u_{n_{k+1}}, u_{n_{k+2}})}{1+Pb(u_{m_k}, u_{n_{k+1}})} \\ \frac{Pb(vmk, vmk+1).Pb(vnk+1, vnk+2)}{1+p_b\,(v_{m_k}, v_{n_{k+1}})} \end{array}\right\}.$$

Taking upper limit as $k \to \infty$ and from (3.2), (3.9) and (3.10) we have that

$\epsilon.s^3 \leq k\epsilon s^2$

Sub Case (i): If $s = 1, \epsilon \leq k.\epsilon < \epsilon$, is contradiction.

Sub Case (ii): If $s > 1, \epsilon.s^3 \leq k\epsilon s^2$.

It follows that $s \leq k < 1$, is contradiction. Hence, $\{u_n\}$ and $\{v_n\}$ are Cauchy sequence in (X, d_b).

Suppose A(x) is complete subspace of X. Then $\{Ax_{n+1}\}$ and $\{Ay_{n+1}\}$ are converges to a and b in $(A(X), d_{P_b})$,

thus $dP_b\,(Ax_{n+1}, a) = 0 = d_{P_b}\,(Ay_{n+1}, b)$ for some $a = Ax$ and $b = Ay$. we have that

$$P_b(a, a) = \lim_{n,m\to\infty} P_b\,(Ax_n, Ax_m) = \lim_{n\to\infty} P_b\,(Ax_n, a)\,) = \lim_{n\to\infty} P_b\,(Ax_{n+1}, a) = 0 \qquad (3.11)$$

$$P_b(b, b) = \lim_{n,m\to\infty} P_b\,(Ay_n, Ay_m) = \lim_{n\to\infty} P_b\,(Ay_n, b)\,) = \lim_{n\to\infty} P_b\,(Ay_{n+1}, b) = 0 \qquad (3.12)$$

Now we claim that $R(x, y) = a$ and $R(y, x) = b$ From (3.1.1), we have that

$P_b\,(R(x, y), R(x_n, y_n)) \leq\ s.P_b\,(R(x, y), R(x_n, y_n))$

$$\leq \max\left\{\begin{array}{c} p_b\,(a,\ Ax_n), p_b\,(b,\ Ay_n) \\ \frac{p_b(a,\ R(x,y)).p_b(Ax_n,\ Ax_{n+1})}{1+p_b(a,\ Ax_n)} \\ \frac{Pb(b,\ R(y,\ x)).Pb(Ay_n,\ Ay_{n+1})}{1+Pb(b,\ Ay_n)} \end{array}\right\} \to 0 \text{ as } n \to \infty$$

It follows that R(x, y)=a=Ax and similarly, R(y, x)=b=Ay. Since (R, A) is weakly compatible pair,

we have R(a, b)=Aa and R(b, a)=Ab.

From (3.1.1) we have that

P_b (R(a, b), R(x_n, y_n))≤sP_b (R(a, b), R(x_n, y_n))

$$\leq k\max\left\{\begin{matrix} P_b\,(Aa, Ax_n), P_b\,(Ab, Ay_n) \\ \frac{P_b(Aa,\ R(a,b)).\ P_b(Ax_n,\ Ax_{n+1})}{1+P_b(a,\ Ax_n)}, \end{matrix} \frac{P_b(Ab,\ R(b,a)).\ P_b(Ay_n,\ Ay_{n+1})}{1+P_b(b,\ Ay_n)}\right\}$$

Letting $n \to \infty$, we have that

P_b (R(a, b), a) ≤ k. max {P_b(R(a, b), a), P_b(R(b, a), b)}.

Similarly we can also prove that

P_b (R(b, a), b) ≤ k. max {P_b(R(a, b), a), P_b(R(b, a), b)}.

Thus

max {P_b (R(a, b), a) , P_b (R(b, a), b)} ≤ k. max {P_b(R(a, b), a), P_b(R(b, a), b)}.

It follows that R(a, b) = a = Aa and R(b, a) = b = Ab. Therefore (a, b) is common coupled fixed point of R and A for uniqueness let us suppose (w, z) be another common coupled fixed point of R and A. such that a ≠ w and b ≠ z. Now from (3.1.1), we have that

P_b(a, z) = P_b (R(a, b), R(z, w))

≤ s.P_b (R(a, b), R(z, w))

$$\leq k\max\left\{\begin{matrix} P_b\,(a,z),\ P_b\,(b,w) \\ \frac{P_b(a,a).\ \ P_b(b,b)}{1+P_b(a,b)}, \frac{P_b(z,z).\ P_b(w,w)}{1+P_b(z,w)} \end{matrix}\right\}$$

$$\leq k\max\{P_b\,(a,z),\ P_b\,(b,w)\}$$

Similarly, we can prove P_b(b, w) ≤ $k\max\{P_b\,(a,z),\ P_b\,(b,w)\}$.

Therefore,

max{ P_b(a, z), P_b(b, w) }≤ $k\max\{P_b\,(a,z),\ P_b\,(b,w)\}$

$$<\max\{P_b\,(a,\ z),\ P_b\,(b,w)\}.$$

It is a contradiction. Hence (a, b) is unique common coupled fixed point of R and A.

Now we prove that $a = b$ From (3.1.1), we have

$P_b(a, b) = P_b\,(R(a, b), R(b, a))$

≤ $s.P_b\,(R(a, b), R(b, a))$

$$\leq k\max\left\{\begin{matrix} P_b\,(a,b),\ P_b\,(b,a) \\ \frac{P_b(a,a).\ \ P_b(b,b)}{1+P_b(a,b)}, \frac{P_b(a\ ,b).\ P_b(b\ ,b)}{1+P_b(a,\ \ b)} \end{matrix}\right\}$$

$\leq k\ P_b(a,b) < P_b(a,b)$.

It follows $a = b$. Hence (a,a) is unique common coupled fixed point of R and A.

Corollary 3.2. : Let (X, P_b) be a complete Partial b- metric space, and the mapping R : $X \times X \rightarrow X$ satisfying $s.P_b\,(R(x, y), R(u, v)) \leq k.\max\{P_b(x, u), P_b(y, v)\}$.Then R has a unique coupled fixed point in X.

Example 3.3. Let $X = [0, 1)$ and a function $F : X \times X \rightarrow X$ be defined as

$F(x, y)=\frac{x^2+y^2}{4(x+y+1)}$ and $f : X \rightarrow X$ by $f(x)=\frac{x}{2}$ and $P_b: X \times X \rightarrow [0,\infty)$

$P_b(x; y) = [\max\{x, y\}]^2 + |x-y|^2$, for all $x, y \in X$ is a complete partial b- metric space on X

Now

$$P_b(F(x,y),F(u,v))=\left[max\left\{\frac{x^2+y^2}{4(x+y+1)},\ \frac{u^2+v^2}{4(u+v+1)}\right\}\right]^2+\left|\frac{x^2+y^2}{4(x+y+1)}-\frac{u^2+v^2}{4(u+v+1)}\right|^2$$

$$=\frac{1}{16}\left\{\begin{matrix}\left[max\left\{\frac{x^2}{x+y+1},\ \frac{u^2}{u+v+1}\right\}+max\left\{\frac{y^2}{x+y+1},\ \frac{v^2}{u+v+1}\right\}\right]^2\\ \left|\frac{x^2}{x+y+1}-\frac{u^2}{u+v+1}\right|^2+\left|\frac{y^2}{x+y+1}-\frac{v^2}{u+v+1}\right|^2\end{matrix}\right\}$$

$$\leq\frac{1}{16}\left\{\begin{matrix}\left[max\left\{\frac{x^2}{x+1},\ \frac{u^2}{u+1}\right\}+max\left\{\frac{y^2}{y+1},\ \frac{v^2}{v+1}\right\}\right]^2\\ \left|\frac{x^2}{x+1}-\frac{u^2}{u+1}\right|^2+\left|\frac{y^2}{y+1}-\frac{v^2}{v+1}\right|^2\end{matrix}\right\}$$

$$\leq\frac{1}{16}\left\{\begin{matrix}\left[max\left\{\frac{x}{x+1},\ \frac{u}{u+1}\right\}+max\left\{\frac{y}{y+1},\ \frac{v}{v+1}\right\}\right]^2\\ \left|\frac{x}{x+1}-\frac{u}{u+1}\right|^2+\left|\frac{y}{y+1}-\frac{v}{v+1}\right|^2\end{matrix}\right\}$$

$$\leq\frac{1}{4}\left\{\begin{matrix}\left[max\left\{\frac{x}{x+1},\ \frac{u}{u+1}\right\}\right]^2+\left|\frac{x}{x+1}-\frac{u}{u+1}\right|^2\\ \left[max\left\{\frac{y}{y+1},\ \frac{v}{v+1}\right\}\right]^2+\left|\frac{y}{y+1}-\frac{v}{v+1}\right|^2\end{matrix}\right\}$$

$$=\frac{1}{4}(P_b\,(fx, fu)+P_b(fy, fv))\leq\frac{1}{4}\max\{P_b\,(fx, fu), P_b(fy, fv)\}$$

$$\leq\frac{k}{s}\max\left\{\begin{matrix}P_b(fx,fu),P_b(fy,\ fv),\\ \frac{P_b(fx,\ F(x,y))P_b(fu,\ \ F(u,v)).}{1+P_b(fx,\ fu)}\quad\frac{P_b(fy,\ F(y,x)),P_b(fv,F(v,u))}{1+P_b(fy,\ fv)}\end{matrix}\right\}$$

Hence all conditions of Theorem 3.1 are holds and (0, 0) is unique common coupled fixed point of F and f.

3.1. Application to Integral Equations:

In this section, we study the existence of a unique solution to an initial value problem, as an application to Theorem 3.1 Consider the initial value problem

$x'(t) = f(t, x(t), x(t)),\ t \in I = [0, 1],\ x(0) = x_0$ (3.13)

where $f : I \times [\frac{x_0}{4}, \infty) \times [\frac{x_0}{4}, \infty) \to [\frac{x_0}{4}, \infty)$ and $x_0 \in R$.

Theorem 3.4. Consider the initial value problem 3.13 with $f \in C(I \times [\frac{x_0}{4}, \infty) \times [\frac{x_0}{4}, \infty))$ and

$$\int_0^t f(s, x(s), y(s))ds \leq max \begin{Bmatrix} \frac{1}{4}\int_0^t f(s.x(s), x(s))ds - \frac{9x_0}{16} \\ \frac{1}{4}\int_0^t f(s.y(s), y(s))ds - \frac{9x_0}{16} \end{Bmatrix}$$

Then there exists unique solution in $C(I, [\frac{x_0}{4}, \infty))$ for the initial value problem 3.13

Proof. The integral equation corresponding to initial value problem 3.13 is

$$x(t)=x_0+\int_0^t f(s, x(s), x(s))ds \quad (3.14)$$

Let $X= C(I, [\frac{x_0}{4}, \infty))$ and $P_b(x; y) = [\max\{x, y\}]^2 + |x - y|^2$, for all $x, y \in X$. Define $A : X \to X$ by $A(x)(t) = \frac{x(t)}{2}$ and $F : X \times X \to X$ by $F(x, y)(t)= x_0+\int_0^t f(s, x(s), y(s))ds$.

Now

$$P_b(F(x, y)(t), F(u, v)(t))= \left[\max\{F(x,y) - \frac{x_0}{4}, F(u,v) - \frac{x_0}{4}\}\right]^2 + |F(x,y) - F(u,v)|^2$$

$$= \left[\max\left\{\frac{3x_0}{4} + \int_0^t f(s, x(s), y(s))ds, \frac{3x_0}{4} + \int_0^t f(s, u(s), v(s))ds\right\}\right]^2$$

$$+\left|x_0 + \int_0^t f(s, x(s), y(s))ds - x_0 + \int_0^t f(s, u(s), v(s))ds\right|^2$$

$$\leq \left[\max\begin{Bmatrix} \frac{3x_0}{4} + max\begin{Bmatrix} \frac{1}{4}\int_0^t f(s.x(s), x(s))ds - \frac{9x_0}{16} \\ \frac{1}{4}\int_0^t f(s.y(s), y(s))ds - \frac{9x_0}{16}\end{Bmatrix} \\ \frac{3x_0}{4} + max\begin{Bmatrix} \frac{1}{4}\int_0^t f(s.u(s), u(s))ds - \frac{9x_0}{16} \\ \frac{1}{4}\int_0^t f(s.v(s), v(s))ds - \frac{9x_0}{16}\end{Bmatrix}\end{Bmatrix}\right]^2$$

$$+\left| max\begin{Bmatrix}\frac{1}{4}\int_0^t f\big(s.x(s),x(s)\big)ds-\frac{9x_0}{16}\\ \frac{1}{4}\int_0^t f\big(s.y(s),y(s)\big)ds-\frac{9x_0}{16}\end{Bmatrix}- max\begin{Bmatrix}\frac{1}{4}\int_0^t f\big(s.u(s),u(s)\big)ds-\frac{9x_0}{16}\\ \frac{1}{4}\int_0^t f\big(s.v(s),v(s)\big)ds-\frac{9x_0}{16}\end{Bmatrix}\right|^2$$

$$=\left[\max\begin{Bmatrix}\max\left\{\frac{x(t)}{4}-\frac{x_0}{16},\frac{y(t)}{4}-\frac{x_0}{16}\right\}\\ \max\left\{\frac{u(t)}{4}-\frac{x_0}{16},\frac{v(t)}{4}-\frac{x_0}{16}\right\}\end{Bmatrix}\right]^2$$

$$+\left|max\left\{\frac{x(t)}{4}-\frac{13x_0}{16},\frac{y(t)}{4}-\frac{13x_0}{16}\right\}-max\left\{\frac{u(t)}{4}-\frac{13x_0}{16},\frac{v(t)}{4}-\frac{13x_0}{16}\right\}\right|^2$$

$$\leq\frac{1}{4}\max\begin{Bmatrix}\left[\max\{\frac{x(t)}{2}-\frac{x_0}{4},\frac{u(t)}{2}-\frac{x_0}{4}\}\right]^2+\left|\frac{x(t)}{2}-\frac{u(t)}{4}\right|^2\\ \left[\max\{\frac{y(t)}{2}-\frac{x_0}{4},\frac{v(t)}{2}-\frac{x_0}{4}\}\right]^2+\left|\frac{y(t)}{2}-\frac{v(t)}{4}\right|^2\end{Bmatrix}$$

$$=\frac{1}{4}\max\{P_b(Ax(t), Au(t)), P_b(Ay(t), Av(t))\}$$

$$\leq\frac{k}{s}\max\begin{Bmatrix}P_b(Ax(t),Au(t)),P_b(Ay(t),\ Av(t)),\\ \frac{P_b(Ax(t),\ R(x,y)(t))\ P_b(Au(t),\ R(u,v)(t)).}{1+P_b(Ax(t),\ Au(t))}\ \frac{P_b(Ay(t),\ R(y,x)(t))\ P_b(Av(t),\ R(v,u)(t))}{1+P_b(Ay(t),Av(t))}\end{Bmatrix}$$

Thus F satisfies the condition of Theorem (3.1), we conclude that F has a unique coupled fixed point (x, y) with x = y. In particular x(t) is the unique solution of the integral equation (3.14).

3.2. Applications to Homotopy:

In this section, we study the existence of a unique solution to Homotopy theory.

Theorem 3.5. Let (X, P_b) be a complete partial -b-metric space, $\overline{U}$ be closed subset of X such that $U \subseteq \overline{U}$.

Suppose $H : \overline{U}\times \overline{U} \times [0, 1] \rightarrow X$ be an operator such that the following conditions are satisfying,

(i) $x \neq H(x, y, \lambda)$ & $y \neq H(y, x, \lambda)$ for each $x, y \in \partial U$ and $\lambda \in [0, 1]$, (here ∂U denotes the boundary of U in X),

(ii) $sP_b(H(x, y, \lambda), H(u, v, \lambda)) \leq k. \max \{ P_b(x, u), P_b(y, v)\}$

(iii) there exists $M \geq 0$ such that $P_b(H(x, y, \lambda), H(x, y, \mu)) \leq M|\lambda - \mu|$ for every $x \in U$ and $\lambda, \mu \in [0, 1]$.

Then H(., 0) has a coupled fixed point if and only if H(., 1) has a coupled fixed point.

Proof. Consider the set $A = \{\lambda \in [0, 1] : x = H(x, y, \lambda)$ for some $x, y \in U\}$.

Since H(., 0) has a coupled fixed point in U, we have that $0 \in A$. So that A is non empty set. We will show that A is both open and closed in [0, 1] and so by the connectedness we have that A = [0, 1]. As a result, H(., 1) has a coupled fixed point in U. First we show that A is closed in [0, 1].

To see this let $\{\lambda_n\} \subseteq A$ with $\lambda_n \to \lambda \in [0, 1]$ as $n \to \infty$.

We must show that $\lambda \in A$. Since $\lambda_n \in A$ for n = 1, 2, 3, · · · ,

there exist $x_n, y_n \in U$ with $x_n = H(x_n, y_n, \lambda_n)$, $y_n = H(y_n, x_n, \lambda_n)$.

Consider

$$P_b(x_n, x_{n+1}) = P_b(H(x_n, y_n, \lambda_n), H(x_{n+1}, y_{n+1}, \lambda_{n+1}))$$

$$\leq s\left\{\begin{array}{c} P_b(H(x_n, y_n, \lambda_n), H(x_{n+1}, y_{n+1}, \lambda_n)) \\ +P_b(H(x_{n+1}, y_{n+1}, \lambda_n), H(x_{n+1}, y_{n+1}, \lambda_{n+1})) \\ -P_b(H(x_{n+1}, y_{n+1}, \lambda_n), H(x_{n+1}, y_{n+1}, \lambda_n)) \end{array}\right\}$$

$$\leq sP_b(H(x_n, y_n, \lambda_n), H(x_{n+1}, y_{n+1}, \lambda_{n+1}))+sM|\lambda_n - \lambda_{n+1}|$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} P_b(x_n, x_{n+1}) \leq \lim_{n\to\infty} sP_b(H(x_n, y_n, \lambda_n), H(x_{n+1}, y_{n+1}, \lambda_{n+1}))+0.$$

From (ii) we obtain

$$\lim_{n\to\infty} P_b(x_n, x_{n+1}) \leq \lim_{n\to\infty} sP_b(H(x_n, y_n, \lambda_n), H(x_{n+1}, y_{n+1}, \lambda_{n+1}))$$

$$\leq k \lim_{n\to\infty} \max\{P_b(x_n, x_{n+1}), P_b(y_n, y_{n+1})\}$$

Similarly, we can prove

$$\lim_{n\to\infty} P_b(y_n, y_{n+1}) \leq k \lim_{n\to\infty} \max\{P_b(x_n, x_{n+1}), P_b(y_n, y_{n+1})\}.$$

It follows that $\lim_{n\to\infty} P_b(x_n, x_{n+1}) = 0 = \lim_{n\to\infty} P_b(y_n, y_{n+1})$ (3.15)

From (p_{b2}) , $\lim_{n\to\infty} P(x_n, x_n) = 0 = \lim_{n\to\infty} P(y_n, y_n)$ (3.16)

By definition of d_{p_b} , we obtain

$$\lim_{n\to\infty} d_{p_b}(x_n, x_{n+1}) = 0 = \lim_{n\to\infty} d_{p_b}(y_n, y_{n+1}) \quad (3.17)$$

Now we prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in (X, dp). On contrary suppose that $\{x_n\}$ or $\{y_n\}$ is not Cauchy. There exists an $\epsilon > 0$ and monotone increasing sequence of natural numbers $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k$,

$$\max\{d_{p_b}(x_{m_k}, x_{n_k}), d_{p_b}(y_{m_k}, y_{n_k})\} \geq \epsilon. \quad (3.18)$$

and $\max\{d_{p_b}(x_{m_k}, x_{n_{k-1}}), d_{p_b}(y_{m_k}, y_{n_{k-1}})\} < \epsilon.$ (3.19)

From (3.18) and (3.19), we obtain

$\epsilon \leq \max\{d_{p_b}(x_{m_k}, x_{n_k}), d_{p_b}(y_{m_k}, y_{n_k})\}$

$\leq$ s. $\max\{d_{p_b}(x_{m_k}, x_{n_{k-1}}), d_{p_b}(y_{m_k}, y_{n_{k-1}})\}$+s $\max\{d_{p_b}(x_{n_{k-1}}, x_{n_k}), d_{p_b}(y_{n_k}, y_{n_{k-1}})\}$

<s. ϵ+ s $\max\{d_{p_b}(x_{n_{k-1}}, x_{n_k}), d_{p_b}(y_{n_k}, y_{n_{k-1}})\}$

Taking upper limit as k → ∞ and from (3.15) , we have that

$$\epsilon \leq \lim_{k\to\infty} \sup max\{d_{p_b}(x_{m_k}, x_{n_k}), d_{p_b}(y_{m_k}, y_{n_k})\} \leq s\epsilon \quad (3.20)$$

Also

$\epsilon \leq \max\{d_{p_b}(x_{m_k}, x_{n_k}), d_{p_b}(y_{m_k}, y_{n_k})\}$

$\leq$ s. $\max\{d_{p_b}(x_{n_k}, x_{m_{k+1}}), d_{p_b}(y_{n_k}, y_{m_{k+1}})\}$+s $\max\{d_{p_b}(x_{m_{k+1}}, x_{m_k}), d_{p_b}(y_{m_{k+1}}, y_{m_k})\}$

Taking upper limit as k → ∞ and from (3.17) , we have tha

$$\frac{\epsilon}{s} \leq \lim_{k\to\infty} \sup max\{d_{p_b}(x_{m_{k+1}}, x_{n_k}), d_{p_b}(y_{m_{k+1}}, y_{n_k})\} \quad (3.21)$$

On other hand

$max\{d_{p_b}(x_{m_{k+1}}, x_{n_k}), d_{p_b}(y_{m_{k+1}}, y_{n_k})\}$

$\leq$ s. $\max\{d_{p_b}(x_{n_k}, x_{m_k}), d_{p_b}(y_{n_{k+1}}, y_{m_k})\}$+s $\max\{d_{p_b}(x_{m_{k+1}}, x_{m_k}), d_{p_b}(y_{m_{k+1}}, y_{m_k})\}$

Taking upper limit as k → ∞ and from (3.17) , we have that

$$\lim_{k\to\infty} \sup max\{d_{p_b}(x_{m_{k+1}}, x_{n_k}), d_{p_b}(y_{m_{k+1}}, y_{n_k})\} \leq \epsilon s^2 \quad (3.22)$$

also, from (3.18), we have that

$\epsilon \leq \max\{d_{p_b}(x_{m_k}, x_{n_k}), d_{p_b}(y_{m_k}, y_{n_k})\}$

$\leq$ s. $\max\{d_{p_b}(x_{n_k}, x_{n_{k+1}}), d_{p_b}(y_{n_k}, y_{n_{k+1}})\}$+s $\max\{d_{p_b}(x_{m_{k+1}}, x_{m_k}), d_{p_b}(y_{m_{k+1}}, y_{m_k})\}$

$$\leq \max\left\{\begin{matrix} s.max\{d_{p_b}(x_{n_k}, x_{n_{k+1}}), d_{p_b}(y_{n_k}, y_{n_{k+1}})\} \\ +s^2 max\{d_{p_b}(x_{n_{k+1}}, x_{m_{k+2}}), d_{p_b}(y_{n_{k+1}}, y_{m_{k+2}})\} \\ +s^2 max\{d_{p_b}(x_{m_{k+2}}, x_{m_k}), d_{p_b}(y_{m_{k+2}}, y_{m_k})\} \end{matrix}\right\}$$

$$\leq \max\left\{\begin{matrix} s.max\{d_{p_b}(x_{n_k}, x_{n_{k+1}}), d_{p_b}(y_{n_k}, y_{n_{k+1}})\} \\ +s^2 max\{d_{p_b}(x_{m_{k+1}}, x_{m_{k+2}}), d_{p_b}(y_{m_{k+1}}, y_{m_{k+2}})\} \\ +s^3 max\{d_{p_b}(x_{m_{k+2}}, x_{m_{k+1}}), d_{p_b}(y_{m_{k+2}}, y_{m_{k+1}})\} \\ +s^3 max\{d_{p_b}(x_{m_{k+1}}, x_{m_k}), d_{p_b}(y_{m_{k+1}}, y_{m_k})\} \end{matrix}\right\}$$

Taking upper limit as k → ∞ and from (3.17) , we have that

$$\frac{\epsilon}{s^3} \leq \lim_{k\to\infty} \sup max\{d_{p_b}(x_{n_{k+1}}, x_{m_{k+2}}), d_{p_b}(y_{n_{k+1}}, y_{m_{k+2}})\}.$$

On the other hand

$$max\{d_{p_b}(x_{n_{k+1}},x_{m_{k+2}}),d_{p_b}(y_{n_{k+1}},y_{m_{k+2}})\}$$

$$\leq s.\max\{d_{p_b}(x_{n_k},x_{n_{k+1}}),d_{p_b}(y_{n_k},y_{n_{k+1}})\}+s\max\{d_{p_b}(x_{m_{k+2}},x_{n_k}),d_{p_b}(y_{m_{k+2}},y_{n_k})\}$$

$$\leq\max\left\{\begin{array}{c} s.max\{d_{p_b}(x_{n_{k+1}},x_{m_{k+2}}),d_{p_b}(y_{n_{k+1}},y_{m_{k+2}})\} \\ +s^2max\{d_{p_b}(x_{n_k},x_{m_k}),d_{p_b}(y_{n_k},y_{m_k})\} \\ +s^3max\{d_{p_b}(x_{m_k},x_{m_{k+1}}),d_{p_b}(y_{m_{k+1}},y_{m_k})\} \\ +s^3max\{d_{p_b}(x_{m_{k+1}},x_{m_{k+2}}),d_{p_b}(y_{m_{k+1}},y_{m_{k+2}})\} \end{array}\right\}$$

Taking upper limit as $k\to\infty$ and from (3.17), (3.20) we have that

$$\lim_{k\to\infty}\sup max\{d_{p_b}(x_{n_{k+1}},x_{m_{k+2}}),d_{p_b}(y_{n_{k+1}},y_{m_{k+2}})\} \leq \epsilon\, s^3 \tag{3.23}$$

Now

$$P_b(x_{n_{k+1}},x_{m_{k+2}})=P_b(H(x_{n_{k+1}},y_{n_{k+1}},\lambda_{n_{k+1}}),H(x_{m_{k+2}},y_{m_{k+2}},\lambda_{m_{k+2}}))$$

$$\leq s.\left\{\begin{array}{c} P_b(H(x_{n_{k+1}},y_{n_{k+1}},\lambda_{n_{k+1}}),H(x_{n_{k+1}},y_{n_{k+1}},\lambda_{m_{k+2}})) \\ +P_b(H(x_{n_{k+1}},y_{n_{k+1}},\lambda_{m_{k+2}}),H(x_{m_{k+2}},y_{m_{k+2}},\lambda_{m_{k+2}})) \\ -P_b(H(x_{n_{k+1}},y_{n_{k+1}},\lambda_{m_{k+2}}),H(x_{n_{k+1}},y_{n_{k+1}},\lambda_{m_{k+2}})) \end{array}\right\}$$

$$\leq s.M|\lambda_{n_{k+1}}-\lambda_{m_{k+2}}|+sP_b(H(x_{n_{k+1}},y_{n_{k+1}},\lambda_{m_{k+2}}),H(x_{m_{k+2}},y_{m_{k+2}},\lambda_{m_{k+2}}))$$

$$\leq s.M|\lambda_{n_{k+1}}-\lambda_{m_{k+2}}|+k.\max\{d_{p_b}(x_{n_{k+1}},x_{m_{k+2}}),d_{p_b}(y_{n_{k+1}},y_{m_{k+2}})\}.$$

Similarly, we can prove

$$P_b(y_{n_{k+1}},y_{m_{k+2}})\leq s.M|\lambda_{n_{k+1}}-\lambda_{m_{k+2}}|+k.\max\{d_{p_b}(x_{n_{k+1}},x_{m_{k+2}}),d_{p_b}(y_{n_{k+1}},y_{m_{k+2}})\}$$

Thus

$$\max\left\{\begin{array}{c} P_b(x_{n_{k+1}},x_{m_{k+2}}), \\ P_b(y_{n_{k+1}},y_{m_{k+2}}) \end{array}\right\}\leq s.M|\lambda_{n_{k+1}}-\lambda_{m_{k+2}}|+k.\max\{d_{p_b}(x_{n_{k+1}},x_{m_{k+2}}),d_{p_b}(y_{n_{k+1}},y_{m_{k+2}})\}.$$

Taking upper limit as $k\to\infty$ and from (3.23) we have that $\epsilon.s^3\leq k\epsilon s^3$.

It follows that $k\geq 1$, is contradiction to $k\in(0,1)$. Hence $\{x_n\}$ and $\{y_n\}$ are Cauchy sequence in (X, d_p) and

$$\lim_{n,m\to\infty}d_p(x_n,x_m)=0=\lim_{n,m\to\infty}d_p(y_n,y_m).$$

By definition of d_{p_b} and (3.11), we get $\lim_{n,m\to\infty}p(x_n,x_m)=0=\lim_{n,m\to\infty}p(y_n,y_m)$.

From Lemma 2.10 $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in (X, P_b). Since (X, P_b) is complete, there exists u, v ∈ U with

$$p_b(u, u)= \lim_{n\to\infty} p_b(x_n,\ u)=\lim_{n\to\infty} p_b(x_{n+1}, u)=\lim_{n,m\to\infty} p(x_n, x_m)=0. \quad (3.24)$$

$$p_b(v, v)= \lim_{n\to\infty} p_b(y_n,\ v)=\lim_{n\to\infty} p_b(y_{n+1}, v)=\lim_{n,m\to\infty} p(y_n, y_m)=0. \quad (3.25)$$

From Lemma 2.10, we get $\lim_{n\to\infty} p_b(x_n,\ H(u,v,\lambda))=p_b(u,\ H(u,v,\lambda))$.

Now

$$p_b(x_n, H(u,v,\lambda))=p_b(H(x_n\ y_n, \lambda_n),\ H(u,v,\lambda))$$

$$\leq s.\begin{Bmatrix} p_b(H(x_n\ y_n,\lambda_n),\ H(x_n\ y_n,\lambda)) + p_b(H(x_n\ y_n,\lambda),\ H(u,v,\lambda)) \\ -p_b(H(x_n\ y_n,\lambda),\ H(x_n\ y_n,\lambda)) \end{Bmatrix}$$

$$\leq s.M|\lambda_n - \lambda| + sp_b(H(x_n\ y_n,\lambda),\ H(u,v,\lambda))$$

Letting n → ∞, we obtain

$$p_b(u, H(u,v,\lambda)) \leq \lim_{n\to\infty} s.p_b\ (H(x_n, y_n, \lambda),\ H(u,v,\lambda))$$

$$\leq \lim_{n\to\infty} kmax\{p_b(x_n,u), p_b(y_n,v)\}=0.$$

It follows that $p_b(u, H(u, v, \lambda)) = 0$. So that u = H(u, v, λ). Similarly v = H(v, u, λ). Thus λ ∈ A.

Hence A is closed in [0, 1]. Let λ0 ∈ A. Then there exists x_0, y_0 ∈ U with $x_0 = H(x_0, y_0, \lambda_0)$.

Since U is open, then there exists r > 0 such that $B_{p_b}(x_0, r) \subseteq U$.

Choose $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$ such that $|\lambda - \lambda_0| \leq \frac{1}{M^n} < \epsilon$.

Then for x∈ $\overline{B_{p_b}(x_0, r)}$=$\{x \in x\ /p_b(x,x_0) \leq r + p_b(x_0,x_0)\}$.

$$P_b(H(x, y, \lambda), x_0) = P_b(H(x, y, \lambda), H(x_0, x_0, \lambda_0))$$

$$\leq s.\begin{Bmatrix} p_b(x\ , y, \lambda),\ H(x,\ y,\ \lambda_0)) + p_b(H(x,\ y,\ \lambda_0),\ H(x_0\ y_0,\lambda_0)) \\ -p_b(H(x,\ y,\ \lambda_0),\ H(x,\ y,\ \lambda_0)) \end{Bmatrix}$$

$$\leq s.M|\lambda_0 - \lambda| + sp_b(H(x_0\ y_0,\lambda_0),\ H(x,y,\lambda))$$

$$\leq s.\frac{1}{M^{n-1}} + sp_b(H(x_0\ y_0,\lambda_0),\ H(x,y,\lambda)).$$

Letting n → ∞, we obtain

$$P_b(H(x, y, \lambda), x_0) \leq sp_b(H(x_0\ y_0,\lambda_0),\ H(x,y,\lambda))$$

$$\leq k\ \max\{p_b(x,x_0), p_b(y,y_0)\}.$$

Similarly, we can prove

$P_b(H(y, x, \lambda), y_0) \leq k \max\{p_b(x, x_0), p_b(y, y_0)\}$.

Thus

$$\max\{ P_b(H(x, y, \lambda), x_0), p_b(H(x,y,\lambda), y_0) \} \leq k \max\{p_b(x, x_0), p_b(y, y_0)\}$$
$$\leq k \max\{ r + p_b(x_0, x_0), r + p_b(y_0, y_0)\}.$$

Thus for each fixed $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$, $H(., \lambda) : \overline{B_{p_b}(x_0, r)} \to \overline{B_{p_b}(x_0, r)}$. Since also (ii) holds, then all conditions of Corollary 3.2 are satisfied. Thus we deduce that $H(., \lambda)$ has a coupled fixed point in $\overline{U}$. But this coupled fixed point must be in U since (i) holds. Thus $\lambda \in A$ for any $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$. Hence $(\lambda_0 - \epsilon, \lambda_0 + \epsilon) \subseteq A$ and therefore A is open in [0, 1]. For the reverse implication, we use the same strategy.

Conclusion: We ensured the existence and uniqueness of a common Coupled fixed point for two mappings in the class of partial b-metric spaces. Two illustrated applications have been provided.

REFERENCES

1. S. Czerwik, Contraction mappings in b-metric spaces, Acta Math. Inform. Univ. Osrav., 1 (1993), 5-11.
2. S. Czerwik, Nonlinear set-valued contraction mappings in b-metric spaces, Atti Sem. Mat. Fis.Univ. Modena. 46 (1998), 263 - 276.
3. S. G. Matthews, Partal metric topology, Proc. 8th Summer Conference on General Topology and Applications, Ann. N.Y. Acad. Sci., 728 (1994), 183-197.
4. Z. Mustafa, J. R. Roshan, V. Parvaneh, Z. Kadelburg, Some common fixed point result in ordered partial b-metric spaces, Journal of Inequalities and Applications, (2013), 2013:562.
5. S. Shukla, Partial b-metric spaces and fixed point theorems, Mediterranean Journal of Mathematics, doi:101007/s00009-013-0327-4, (2013).
6. T.G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal. 65 (2006) 13791393.
7. KPR Rao, GNV Kishore, Kenan Tas, S Satyanaraya and D Ram Prasad Applications and common coupled fixed point results in ordered partial metric spaces, Fixed Point Theory and Applications (2017) 2017:17.DOI 10.1186/s13663-017-0610-3.

8. Abbas, M, Alikhan, M, Radenovic, S: Common coupled fixed point theorems in cone metric spaces for ω-compatible mappings, Appl. Math. Comput. 217(1), 195-202 (2010).

9. Aydi, H: Some coupled fixed point results on partial metric spaces, Int. J. Math. Math. Sci. 2011, Article ID 647091 (2011). doi:10.1155/2011/647091.

10. Abdeljawad, T. Coupled fixed point theorems for partially contractive type mappings, Fixed Point Theory Appl. 2012, 148 (2012).

11. Radenovc, S. Some coupled coincidence points results of monotone mappings in partially ordered metric spaces, Int. J. Anal. Appl. 5(2), 174-184 (2014).

12. Rao. KPR, Kishore.GNV, Raju.VCC, A coupled fixed point theorem for two pairs of w-compatible maps using altering distance function in partial metric space, J. Adv. Res. Pure Math. 4(4), 96-114 (2012).