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Research Article

Degree of Approximation of Conjugate of a Function Belonging to (N,p,q) Class by Matrix Summability Means of Conjugate Fourier Series

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ABSTRACT. An assumption of moderate hypothesis for series has to be generalized (N, p, q) summable by using the concept of Fourier series. Further, we develop new and well-known arbitrary results from the main result. Validation of the theorems done by the previous finding of theorems of summability. In this way, the system stability can be improved by finding the conditions for (N, p, q) summability.

Keywords: Summability, Fourier series, Nörlund Summability.

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1. INTRODUCTION

The hoary, hackneyd and hazy concept of convergence of infinite series was contingent upon robust foundation with the emergence of Cauchy's enduring work course d'Analyses algebraic in 1821 and Abel's researches(see [1]) on the binomial series in 1826. Nonetheless, it was noticed after perusal that certain non- convergent series, specially in Dynamical Astronomy provided almost correct results. In 1890 "a theory of divergent series" was propounded which was pioneering work for the first time. It was at this very time a" paper on the multiplication series ware published by no less a personality than by Cesàro(see [1]). For most of the seminal and pioneering mathematical analyst the theory of series, whose sequence of partial sums fluctuate, has been at the core of creative activity. Through relentless endeavours made by multiple eminent mathematicians that fruitful and satisfactory methods were conceived and concretised towards the fag end of the last century and in the early year of the present century. The method devised was so as to associate them with process closely associated with Cauchy's concept of convergence of which certain values may be called their sums in a reasonable manner. To elaborate and elucidate further. Summability Szàsz and Hardy(see [1]), the process of associating generalized sum, imparted a natural generalization of classical concept of convergence Hobson, Titchmarsh and is therefore responsible, within the domain applicability, an extensive and former rejected series which used to be forbidden as divergent. In this way, the idea of convergence has been deemed and dubbed as sweeping generalization, for it was needless to say, natural to peruse the possibility of putting forward the concept of convergence. Indeed just as the concept of convergence has resulted in the extension under the general title of summability (Kogbetliantz), the idea of ordinary and absolute convergence have contributed to the development of ordinary and absolute summability in the same way. Thus, the concept of uniform convergence would have definitely underscored the concept uniform summability highlighted

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prominently by analyst.

Let $\{S_m(x)\}$ denote the m^{th} partial sums of $\sum a_m$ and we define two non-negative sequences $\{p_n\}$ and $\{q_n\}$ are such that

$$P_a = \sum_{b=0}^{a} p_b$$

$$Q_a = \sum_{b}^{a} q_b$$

and

$$R_a = \sum_{b}^{a} p_{a-b} \;\; q_a$$

The sequence to sequences transformation

$$\lambda_a^{p,q}(z) = \frac{1}{R_a} \sum_{b=0}^{a} p_{a-b} q_b S_b(z) \; as \quad a \to \infty,$$

defines the sequence $\{S_b(z)\}$ of the (N,p,q) means of the sequences $\{S_b(z)\}$, generated by the sequence of coefficients $\{p_a\}$ and $\{q_a\}$.
If $\lambda_a^{p,q}(z) \to S(z)$, then the given series $\sum a_m$ is said to be (N, p, q) summable to $S(z)$ at Z in E.
If $\lambda_a^{p,q}(z) - S(z) = 0(1)$, a $\to \infty$ uniformly in a. domain E, then the infinite series $\sum a_m$ is (N, p, q) summable uniformly in a domain E to $S(z)$.

Let $\chi(y)$ be periodic function with the period 2π and Lebesgue integrable function of y in $(-\pi, \pi)$.The Fourier series of the given function $\chi(y)$ is denoted by

(1.1) $$\chi(y) \approx \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \; cos \; my + b_m sin \; my)$$

Its conjugate series is

(1.2) $$\sum_{m=1}^{\infty} (b_m cos \; my - a_m sin \; my).$$

We have

$$\xi(z) = \chi(y+z) + \chi(y-z) - 2\chi(z),$$

$$\zeta(z) = \chi(y+z) - \chi(y-z),$$

$$v(z) = \int_v^z \big|\xi(v)\big| dv,$$

$$\eta(z) = \int_o^z \big|\xi(v)\big| dv,$$

$$N_a^{p,q}(z) = \frac{1}{2\pi R_a}\sum_{b=v}^{a} p_{a-b}q_b.\frac{sin(b+\frac{1}{2})z}{sin\frac{z}{2}},$$

$$\bar{N}_a^{p,q}(z) = \frac{1}{2\pi R_a}\sum_{b=v}^{a} p_{a-b}q_b.\frac{cos(b+\frac{1}{2})z}{sin\frac{z}{2}},$$

$$\tau = \left(\frac{1}{z}\right), \text{the integral part of } \frac{1}{z},$$

$$\bar{\chi}_a(y) = \frac{\pi}{2}\int_{\frac{1}{a}}^{\pi} \xi(u)cot\frac{u}{2}du$$

and

$$\bar{\chi}_a(y) = \lim_{a\to\infty} \bar{\chi}_a(y).$$

2. Known results

There are a lot of interesting generalizations that have been achieved by many researchers (see [1-5]). Among them, Tripathi and Singh (see [6]) has established the following theorems on Nřolund summability of Fourier series and its conjugate series.

Theorem A:- Let us suppose that $A(z)$ and $B(z)$ are two positive functions of z.
If

$$v(z) = \int_0^z \Big|\xi(u)\Big|du = \left[\frac{A\left(\frac{1}{z}\right).p_\tau}{B\left(P_\tau\right)}\right] \tag{2.3}$$

and

$$A(a)P_a = O[B(P_a)] \tag{2.4}$$

as $a \to \infty$, then the given Fourier series $\chi(y)$ i.e. (1.1) at $z = y$ is summable (N, p_a) to $\chi(y)$ such that $P_a \to 0$ as $a \to \infty$, where $\{p_a\}$ is real non-negative and non-increasing sequence.

Theorem B:- Let us suppose that $A(z)$ and $B(z)$ are two positive functions of z and $\{p_n\}$ is a real non-negative and non-increasing sequences as in theorem A.
Then if,

$$\eta(z) = \int_0^z \Big|\xi(v)\Big|dv = o\left[\frac{A\left(\frac{1}{z}\right).p_\tau}{B\left(P_\tau\right)}\right] \tag{2.5}$$

as $z \to +0$, then the (1.2) of (1.1) is summable (N, p_a) to

$$\frac{1}{2\pi}\int_0^\pi \xi(z)cot\frac{z}{2}dz$$

at every point of the domain E, where this integral exists.
The above results motivates us to study the degree of approximation of a function in more generalized as popular cases.Therefore, an attempts to make an advance of this note, we study the

degree of approximation of a periodic function by (N,p,q) summability of series in the following forms:

Theorem 1:- let us suppose that $A(z)$ and $B(z)$ are two positive function depend on z and $A(z), B(z)$ and $z.\frac{A(z)}{B(z)}$ increases monotonically with z. Let $\{p_a\}$ and $\{q_a\}$ are two positive monotonically decreasing sequences of constants with their partial sums $P_a \to 0, Q_b \to 0$ as $a, b \to \infty$. If

$$\sum_{b=0}^{a} \left|\Delta\left(p_{a-b}q_b\right)\right| = O\left(R_a.a^{-1}\right) \tag{2.6}$$

$$A(a).R_a = O\left(B(R_a)^m\right) \tag{2.7}$$

as $a \to \infty$ where $m\ \epsilon[0,1]$ and

$$\upsilon(z) = \int_0^z \left|\xi(v)\right| dv = O\left[\frac{A(z)q_\tau}{\{B(R_\tau)\}^m}\right] \tag{2.8}$$

uniformly in a set E in which $\xi(y)$ is bounded, then the given series (1.1) is (N,p,q) summable in the domain E to the sum $\xi(y)$.

Theorem 2:- Let $\{p_a\}$ and $\{q_b\}$ are the same as in the theorem 1, satisfying the condition (2.6) and $A(z)$ and $B(z)$ are the same as above satisfying the condition (2.7), if

$$\eta(z) = \int_0^z \left|\xi(v)\right| dv = O\left[\frac{A(z)q_\tau}{\{B(R_\tau)\}^m},\right. \tag{2.9}$$

as $z \to \infty$, m $\epsilon[0,1]$ uniformly in the domain E, then the (1.2) of (1.1) is (N,p,q) summable uniformly in E to the sum

$$\bar{\chi}(y) = \frac{1}{2\pi}\int_0^\pi \chi(u)\ cot\frac{u}{2}du \tag{2.10}$$

whenever the integral exists uniformly in the domain E.
To prove the theorems, we follow a series of lemmas.
Lemma1 (see [7]): For $0 \le a < b \le \infty,\ \ 0 \le z \le 2\pi$,

$$\left|\int_{j=a}^{b} p_{m-j}\ \ q_j e^{ijz}\right| < CR_\tau.$$

Lemma 2:- If $R_a \to \infty$, $a \to \infty$ and the conditions (2.6) is satisfied, then

$$aq_a = O(R_a), a \to \infty$$

Proof:- we have

$$\sum_{b=1}^{a} \left|\Delta(p_{a-b}\ q_b)\right| = O(R_a a^{-1})$$

$$\Rightarrow \sum_{b=1}^{n} b\left|\Delta(p_{a-b}\ q_b)\right| = O(R_a)$$

Now,

$$\sum_{b\geq 1}^{a-2} \Delta b. \sum_{m=1}^{b} \Delta(p_{a-m}\ q_m) + (a+b)\sum_{b=1}^{a-1} \Delta(p_{a-b}\ q_b)$$

$$= \sum_{b=1}^{a-b} \Delta a(p_{a-1}\ q_1 - p_0\ q_a) + (a+1)(p_{a-1}\ q_1 - p_0\ q_a)$$

$$= \sum_{b=1}^{a} p_{a-b}\ q_a - p_a q_o - a q_a p_o$$

and therefore

$$a\ q_a\ p_o = R_a - \sum_{k=1}^{a-b} b\Delta(p_{a-b}\ q_b) - p_a\ q_o$$

$$\Rightarrow a\ q_a = O(R_a), a \to \infty.$$

Lemma 3 (see [6]):- If $0 \leq z \leq \frac{1}{a}$, then $N_a^{p,q}(z) = O(a)$
Proof:- we have

$$\left| N_a^{p,q}(z) \right| \frac{1}{2\pi R_n} \left| \sum_{b=0}^{a} p_{a-b}\ q_b. \frac{sin(b+\frac{1}{2})z}{sin\frac{z}{2}} \right|$$

$$= O\left\{ \frac{1}{R_a} \left| \sum_{b=0}^{a} (2b+1) p_{a-b}\ q_b \right| \right\}$$

$$= O\left[\frac{2a+1}{R_a} \sum_{b=1}^{a} \sum^{a-b} q_b \right]$$

$$= O(a), a \to \infty$$

Lemma 4:- For $\frac{1}{a} \leq z \leq \delta < \pi$,

$$N_a^{p,q}(a) = O\left(\frac{R_\tau}{z R_a} \right)$$

Proof:- We have

$$N_a^{p,q}(z) = \frac{1}{2\pi R_a} \sum_{b=0}^{a} p_{a-b}\ q_b \frac{sin(b+\frac{1}{2})Z}{sin\frac{z}{2}}$$

$$\Rightarrow \left| N_a^{p,q}(z) = \frac{1}{2\pi R_a} \right| \sum_{b=0}^{a} p_{a-b}\ q_b. \frac{sin(b+\frac{1}{2})z}{sin\frac{z}{2}} \right|$$

$$= \frac{1}{2\pi R_a} \times \frac{1}{|sin\frac{z}{2}|} \left| I_m \sum_{b=0}^{a} p_{a-b}\ q_b.exp(i(a-b)+\frac{1}{2}z) \right|$$

$$= \frac{1}{2\pi R_a} \times \frac{1}{|Sin\frac{z}{2}|} \times \left| I_m \left(e^{\frac{\iota z}{2}} \sum_{b=0}^{a} p_{a-b}\ q_b e^{\iota(a-b)z} \right) \right|$$

$$\leq \frac{1}{2\pi R_a} \times \frac{1}{z} \left| \sum_{b=0}^{a} p_{a-b}\ q_b e^{i(a-b)z} \right|$$

$$= O\left(\frac{R_\tau}{R_a z}\right) \text{(by lemma 1)}$$

Lemnma 5:- For $0 \leq z \leq \frac{1}{a}$,

$$\bar{N}_a^{p,q}(z) = O(a).$$

The proof is similar to that of lemma 3.

Lemma 6:- For $\frac{1}{a} \leq z \leq \delta < \pi$,

$$\bar{N}_a^{p,q}(z) = o\left(\frac{R_\tau}{zR_a}\right), \quad a \to \infty.$$

Proof:

We have

$$\bar{N}_a^{(p.q)}(z) = \frac{1}{2\pi R_a} \sum_{b=0}^{a} p_{a-b}\ q_b x \quad \frac{cos(b+\frac{1}{2})z}{sin\frac{z}{2}}$$

$$\Rightarrow \left|\bar{N}_a^{p,q}(z)\right| = \frac{1}{2\pi R_a} \left| \sum_{b=0}^{a} p_{a-b}\ q_b . \frac{cos\left(a-b+\frac{1}{2}\right)z}{sin\frac{z}{2}} \right|$$

$$\leq \frac{1}{2\pi R_a} \times \frac{1}{|sin\frac{z}{2}|} \times \left| R_e \sum_{b=0}^{a} p_{a-b}\ q_b e^{i(a-b+\frac{1}{2})z} \right|$$

$$= \frac{1}{2\pi R_a} \times \frac{1}{|sin\frac{z}{2}|} \times \left| R_e \left(e^{\frac{iz}{2}} \sum_{b=0}^{a} p_{a-b}\ q_b\ e^{i(a-b)z} \right) \right|$$

$$\leq \frac{1}{2\pi R_a} \left| \sum_{b=0}^{a} p_{a-b}\ q_b\ e^{i(a-b)z} \right|$$

$$= o\left(\frac{R_\tau}{zR_a}\right), \qquad \text{(by lemma 4)}$$

Proof of theorem 1: Let $G_a(y)$ denote the a^{th} partial sum of (1.1) at a point $z = y$ in a domain E, then

$$G_a(y) - \chi(y) = \frac{1}{2\pi} \int_0^\pi \left\{ \frac{\xi(z).sin\left(a+\frac{1}{2}\right)z}{sin\frac{z}{2}} \right\} dz.$$

Thus by following (**), we may write

$$\lambda_a^{p,q}(y) - \chi(y) = \frac{1}{R_a} \sum_{b=0}^{a} p_{a-b}\ q_b \left\{ G_b(z) - \chi(y) \right\} \frac{1}{2\pi R_a} \sum_{b=0}^{a} p_{a-b}\ q_b$$

$$\times \int_0^\pi \xi . \frac{sin\left(b+\frac{1}{2}\right)z}{sin\frac{z}{2}}dz$$

$$= \int_0^\pi (z)\left\{\frac{1}{2\pi R_a}\sum_{b=0}^{a} p_{a-b}\ q_b\ \times \frac{sin\left(b+\frac{1}{2}\right)z}{sin\frac{z}{2}}\right\}dz$$

$$= \int_0^\pi \chi(z)N_a^{p,q}(z)dz$$

(2.11) $$= \left(\int_0^{\frac{1}{a}} + \int_{\frac{1}{a}}^{\delta} + \int_\delta^\pi\right)\chi(z)\ N_a^{p,q}(z)dz$$

(2.12) $$= B_1 + B_2 + B_3 \quad \text{(say)}.$$

We have to show that

(2.13) $$\int_0^\pi (z)N_a^{p,q}(z)dz = (1), \text{as } \ a \to \infty$$

uniformly in E.

$$Now, \quad 0 \le z \le \frac{1}{a},$$

$$B_1 = \int_0^{\frac{1}{a}} \xi(z)N_a^{p,q}(z)dz$$

$$\Rightarrow |B_1| = \left|\int_0^{\frac{1}{a}} \xi(z)N_a^{p,q}(z)dz\right|$$

$$= O(a)\int_0^{\frac{1}{a}} \chi(z)dz, \quad \text{(using lemma (3))}$$

$$= (a).O\left[\frac{A(z).\ q_a}{B(R-a)^m}\right], \quad \text{(using 8)},$$

$$= O\left[\frac{A(z)\ R_a}{B(R-a)^m}\right], \quad \text{(using lemma (2))},$$

(2.14) $$= O(1), a \to \infty$$

unifrmly in the domain E.

For $\frac{1}{a} \le z \le \delta$,

$$B_2 = \int_{\frac{1}{a}}^{\delta} \xi(z)N_a^{p,q}(z)dz$$

$$\Rightarrow |B_2| = O\left[\int_{\frac{1}{a}}^{\delta} \left|\xi(z)\right|\left|N_a^{p,q}(z)\right|dz\right]$$

$$= O\left(\frac{1}{R_a}\right)\int_{\frac{1}{a}}^{\delta}\left|\xi(z)\right|\frac{R_\tau}{z}dz$$

$$= O\left[\frac{1}{R_a}\left\{\upsilon(z)\frac{R_\tau}{z}\right\}_{\frac{1}{a}}^{\delta}\right] + O\left[\frac{1}{R_a}\int_{\frac{1}{a}}^{\delta}\left\{\upsilon(z)\frac{R_\tau}{z^2}dz\right\}\right] + O\left(\frac{1}{R_a}\int_{\frac{1}{a}}^{\delta}\upsilon(z)\frac{1}{z}d\left(R_\tau\right)\right)$$

(2.15) $$= B_{2.1} + B_{2.2} + B_{2.3} \quad \text{(say)}$$

Now,

$$B_{2.1} = \left[\frac{1}{R_a}\left\{\upsilon(z)\frac{R_\tau}{z}\right\}_{\frac{1}{a}}^{\delta}\right]$$

$$= O\left(\frac{1}{a}\right) + O\left[\frac{aA(a)\ q_a}{\left\{B\left(R_a\right)\right\}^m}\right]$$

$$= O\left(R_a^{-1}\right) + O\left[\frac{A(a)R_a}{\left\{\mu\left(R_a\right)\right\}^m}\right]$$

(2.16) $$= O(1), a \to \infty$$

uniformly in E.

$$B_{2.2} = O\left(R_a^{-1}\right)\int_{\frac{1}{a}}^{\delta}\left[\upsilon(z)\frac{R_\tau}{z^2}dz\right]$$

$$= O(1) + O\left[\frac{1}{R_a}\sum_{c-1}^{a-1}\int_{c}^{c+1}\upsilon\left(\frac{1}{v}\right)R_{[v]}dv\right]$$

But

$$\int_{c}^{c+1}\upsilon\left(\frac{1}{v}\right)R_{[v]}dv \le \upsilon\left(\frac{1}{c}\right)R_c = \left[\frac{A(c)\ q_c\ R_c}{\{B(R_c)\}^m}\right]$$

$$= O(q_c), c \to \infty \ \text{ unioformly in E.}$$

Thus,

$$B_{2.2} = O(1) + O(R_a^{-1}).O\left(\sum_{c=1}^{q-1} q_c\right)$$

$$= O(1) + O(1), as \to \infty \ \text{ uniformly in E,}$$

For

$$B_{2.3} = O\left[R_a^{-1}\int_{\frac{1}{a}}^{\delta}\upsilon(z)\frac{1}{z}dR_\tau\right]$$

$$= O\left[R_a^{-1}\int_{\frac{1}{a}}^{\delta} \upsilon\left(\frac{1}{u}\right) z\, dR_{[v]}\right]$$

$$= O(1) + O\left(R_a^{-1}\left\{\sum_{c=1}^{a-1} R_c\, \upsilon\left(\frac{1}{c}\right)\right\}\right)$$

$$= O(1) + O\left[\frac{1}{R_a}\sum_{c=1}^{a-1}\frac{A(c)R_c\, q_c}{\{B(R_c)\}^m}\right]$$

$$= O(1) + O(1)$$

(2.17) $$= O(1), \quad as \to \infty$$

uniformly i E.
Then (14) becomes

(2.18) $$B_2 = O(1), as\ a \to \infty$$

uniformly in the domain E.
Again,

$$B_3 = \int_{\delta}^{\pi} \chi(z) N_a^{p,q}(z)dz$$

$$\Rightarrow |B_3| = O\left[\int_{\delta}^{\pi}\left|\chi(z)\right|\left|N_a^{p,q}(z)\right|dz\right]$$

(2.19) $$= O(1), as\ a \to \infty$$

uniformly in the domain E by virtue of Riemann- Lebesgue theorem and regularity of the method of summation.
Now, using the help of (13), (18) and (19), we get

$$\int_0^{\pi} \xi(z) N_a^{p,q}(z)dz = O(1),$$

$a \to \infty$ uniformly in the domain E.

Proof of the theorem 2:-

Let $\bar{G}_a(y)$ denote the a^{th} partial sum of (1.2) at the point $z = y$ in E, then

$$\bar{G}_a(y) = \frac{1}{2\pi}\int_0^{\pi} \xi(z)\frac{cos\frac{t}{2} - cos(a+\frac{1}{2})z}{sin\frac{z}{2}}dz.$$

Thus, by the following (**), we may write

$$\lambda_a^{-p,q}(y) - \frac{1}{2\pi}\int_0^{\pi}\zeta(z)\cot\frac{z}{2}dz = \frac{1}{R_a}\sum_{b=0}^{a} p_{a-b}\, q_b\, \bar{G}_a(z) - \frac{1}{2\pi}\int_0^{\pi}\zeta(z)\cot\frac{z}{2}dz$$

$$= \int_0^{\pi}\frac{1}{2\pi R_a}\sum_{b=0}^{a} p_{a-b}\, q_b \times \left\{\frac{\cos(b+\frac{1}{2})z}{\sin\frac{z}{2}}\right\}\zeta(z)dz$$

$$= \frac{-1}{2\pi}\int_0^{\pi}\zeta(z)\cot\frac{z}{2}dz - \int_0^{\pi}\zeta(z).\frac{1}{2\pi R_a}\sum_{b=0}^{a}p_{a-b}\ q_b.\frac{\cos(b+\frac{1}{2})z}{\sin\frac{z}{2}}dz$$

$$= -\int_0^{\pi}\zeta(z)\bar{N}_a^{p,q}(z)dz$$

$$= -D \quad \text{say,}$$

where

$$D = \int_0^{\pi}\zeta(z)\bar{N}_a^{p,q}(z)dz.$$

now we have to show that

(2.20) $$\int_0^{\pi}\zeta(z)\bar{N}_a^{p,q}zdz = O(1)$$

as $a \to \infty$, uniformly in given E.
For $0 < \delta < \pi$,

$$D = \int_0^{\pi}\zeta(z)\bar{N}_a^{p,q}(z)dz$$

$$= \left\{\int_0^{\frac{1}{a}} + \int_{\frac{1}{a}}^{\pi} + \int_{\delta}^{\pi}\right\}\zeta(z)\bar{N}_a^{p,q}(z)dz$$

(2.21) $$= D_1 + D_2 + D_3 \ \ (say)$$

For D_1,

$$D_1 = \int_0^{\frac{1}{a}}\zeta(z)\bar{N}_a^{p,q}(z)dz$$

$$\Rightarrow |D_1| = O\left[\int_0^{\frac{1}{a}}\left|\zeta(z)\right|\left|\bar{N}_a^{p,q}(z)\right|dz\right]$$

$$= O\left[a\int_0^{\frac{1}{a}}\left|\zeta(z)\right|dz\right]$$

$$= O(a).O\left[\frac{A(a)q_a}{\{B(R_a)\}^m}\right]\text{(using lemma (5) and } eq^n \text{ (8))}$$

$$= O\left[\frac{A(a)R_a}{\{B(R_a)\}^m}\right], \text{ (using (8))}$$

(2.22) $$= O(1), as\ a \to \infty$$

uniformly in a domain E.
Also,

$$D_2 = \int_{\frac{1}{a}}^{\delta}\zeta(z)\bar{N}_a^{p,q}(z)dz$$

$$\Rightarrow |D_2| = O\left[\int_{\frac{1}{a}}^{\delta} \left|\zeta(z)\right|\left|\bar{N}_a^{p,q}(z)\right| dz\right]$$

$$= O\left[R_a^{-1}\int_{\frac{1}{a}}^{\delta} \left|\zeta(z)\right|\frac{R_\tau}{z} dz\right]$$

$$= O(1), a \to \infty \tag{2.23}$$

uniformly in a domain E.
Similarly, we can show that

$$|D_3| = O(1), a \to \infty \tag{2.24}$$

uniformly in a domain E by virtue of Riemann-Lebesgue theorem and regularity of the method of summation.
In this way, using the help of (20), (21), (22), (23) and (24) we get

$$\int_0^{\pi} \zeta(z)\bar{N}_a^{p,q}(z)dz = O(1),$$

as $a \to \infty$, uniformly in a domain E.

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