Using Series to Develop Methods for Solving Ordinary Differential Equation

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Abstract. This paper is the study of solving ordinary differential equations using series and the study of Partial differential equations, their classification, and methods of solving them. Explaining the importance of using the Mathematica and Midea program Its effectiveness in overcoming the difficulties in performing arithmetic operations and solving ordinary and partial differential equations Reducing the error rate and saving time and effort, as the differential equations are solved in the shortest period of time With accuracy in the results obtained © and I hope to achieve the desired goal. The importance of the research stems from the following:

1. Studying some methods of solving differential equations using series and studying differential equations Injury and ways to solve it.

2. As a result of the difficulties that researchers face in performing long and complex calculations and solving Differential equations (the use of series) by the manual method in addition to falling into some.

We conclude from this paper that technology is used to solve differential equations that take a lot of time and effort to solve with the same accuracy.
Key words and phrases. differential equations. Mathematica program. Midea program. series

INTRODUCTION

Differential equations express a system Kinetic such as planetary motion, eject, vector transmission, heat spread and population growth and others. Where the differential equation governs the behavior of kinetic systems, and we may detect the behavior of this system and anticipate its behavior in the past or future by solving the differential equation. That is why we will investigate. We will explore partial differential equations and their solutions as well as solve differential equations utilizing series. One of the ways will be used, and it will be applied to one of the latest programs. (Mathematica program).

search problem:

1. In many cases, quadratic differential equations with variable coefficients cannot be solved

Closed image. The sequential solution method offers us a powerful alternative. Easy solution idea .The series is not limited to equations with variable coefficients, but of course includes equations with variable coefficients

fixed transactions.

2. Explain what is meant by partial differential equations, how to solve them, and give a brief description of their classification. more than Physical phenomena both in the field of mechanical fluid flow. optics or heat flow Electricity can be generally described by partial differential equations.

3. Not using modern technologies and linking them to practical life made the teaching method boring and unimportant.

4. The traditional methods require time and effort to solve some differential equations, especially when solving using series.

The object of this paper is

1. Study methods for solving differential equations using series.

2. Recognize partial differential equations and methods of solving them.

3. Using a program (and its meaning) and clarifying the possibility of using a computer to solve.

Previous studies :

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SOLVING NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS USING THE

NDM

In his research, he used the NDM application in solving linear differential equations.

The search problem is:

1- Mathematics is relied upon by many electronic program designers, but it is little used by students in it

2- Solving equations by traditional methods require a great deal of time and effort, especially when solving differential equations using series.

1. Basic concepts and methods for solving differential equations using series

We will learn about some important and necessary definitions and concepts that we need in this chapter until The researcher can respond and comprehend everything that will be presented later in addition to illustrative examples as well Methods for solving differential equations using series **.**

Definition 2.1.[1] let f(x) function be analytical when x = a If it can be spread over a simple series of convergent forces

$$f(x) = f(a) + (x - a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots + \frac{(x-a)^n}{n!} f^{(n)}(a) + R_{n+1}$$

Where:

$$R_{n+1} = \frac{(x-a)^{n+1}}{(n+1)} f^{(n+1)}(\delta) \qquad ,a < \delta < x$$

To study the convergence of the series, the ratio test for absolute convergence (Delembert test) can be used.

$$\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = L \quad ,L < 1$$

Example 2.2.[1] Show that a Function publishable at $f(x) = e^x$ publishable at $x = 2$

Sol :

When x=2

$$f(2) = f'(2) = f''(2) = f^{(3)}(2) = \cdots = e^2$$

Therefore, the spread of the series for function $f(x) = e^x$ around $x = 2$ is as follows:

$$f(x) = e^x + (x - 2)e^x + \frac{(x-2)^2}{2!} e^x + \cdots + \frac{(x-2)^n}{n!} e^x + R_{n+1}$$

Where

$$R_{n+1} = \frac{(x - 2)^{n+1}}{(n+1)!} e^{\delta} \qquad 2 < \delta < x$$

Remark 2.3.[6] All terms and functions are analytic for any $x \in R$

$$e^x = \sum_{n=0} \frac{x^n}{n!}$$

$$\text{Sin } x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\text{Cos } x = \sum_{n=0}^{\infty} \frac{(-1)^n \; x^{2n}}{(2n)!}$$

The standard and logarithmic functions are analytical within the limits of their scope, meaning that:

1- Function $\frac{1}{(x-a)^n}$ be analytical when $x \in R - \{a\}$

2- Function $\ln x$ be analytical when $x > 0$

3- Function $\tan x$ be analytical when $x \in R - \left\{(2n+1)\frac{\pi}{2} \;, n = 0, \pm 1, \ldots\right\}$

Example 2.4.[2] Find the solution to equation $f(x) = \text{Sin } x$ using the Maclaurin expansion

Sol:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n\,(x_0)}{n!}$$

$$f(x) = \frac{f^0(0)}{0!} x^0 + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \cdots \quad (2.1)$$

$$f^0(x) = \text{Sin}(x) \Rightarrow f^0(0) = \text{Sin } 0 = 0$$

$$f'(x) = \text{Cos }(x) \Rightarrow f'(0) = \text{COS } 0 = 1$$

$$f''(x) = \text{Sin}(x) \Rightarrow f''(0) = -\text{Sin } 0 = 0$$

$$f^{(3)}(x) = \text{Cos }(x) \Rightarrow f^{(3)}(0) = -\text{COS } 0 = -1$$

$$f^{(4)}(x) = \text{Sin}(x) \Rightarrow f^{(4)}(0) = \text{Sin } 0 = 0$$

Theorem 2.5.[3] If $f(x)$ Knowing in the form of a series

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n \; \ldots\ldots (2.2)$$

Such that $|x - x_0| < R$

$$a_0 = f(x_0), \qquad a_1 = f'(x_1), \qquad a_2 = f''\left(\frac{f''(x)}{2!}\right),$$

Proof :

By differentiating equation (2.1)

$$f'(x) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \cdots + n a_n (x - x_0)^{(n-1)}$$

$$f''(x) = 2a_2(x - x_0) + 3!\,a_3(x - x_0)^2 + \cdots + n(n-1)a_n(x - x_0)^{(n-2)}$$

$$f'''(x) = 3!\,a_3 + \cdots + n(n-1)(n-2)a_n(x - x_0)^{(n-2)}$$

$$f^n(x) = n!\,a_n + \cdots + n(n+1)!\,a_{n+1}(x - x_0) + \cdots$$

When $x = x_0$ We get:

$$f(x) = a_0 \qquad , f'(x_0) = a_1 \qquad , f''(x_0) = 2!\,a_2$$

$$f'''(x_0) = 3!\,a_2 , \dots\dots\dots \; , f^n(x_0) = n!\,a_n$$

And from that we get:

$$a_0 = f(x_0), \qquad a_1 = f'(x_1), \qquad a_2 = f''\left(\frac{f''(x)}{2!}\right),$$

$$a_3 = \frac{f'''(x_0)}{3!} \quad , \dots\dots\dots \; , \quad a_n = \frac{f^{(n)}(x_0)}{n!}$$

From this, equation (2.1) can be written as follows:

$$f(x) = f(x_0) + f(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3$$

$$+ \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

where $|x - x_0| < R$

Theorem 2.6.[7] If $F(x)$ and $G(x)$ are a series defined as :

$$F(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots \qquad |x - x_0| < R$$

$$G(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)^2 + \cdots \qquad |x - x_0| < R$$

$\therefore F(x) = G(x)$ if $a_0 = b_0$, $a_1 = b_1$, ...

2.6. Solving Differential Equations by Using Series [1]

There are certain differential equations that are impossible to solve using conventional methods., so we resort to solving using series and we will study the solution when x_0 is an ordinary point in two ways:

2.6.1. differential progression method [8]

In this method, the differential equations are given with the initial conditions, from which we get $y(x_0)$, $y'(x_0)$, $y''(x_0)$, ... by the progression of differential, and it be will show in the following example:

Example 2.7.[3] Resolve the following differential equation using differential progression

$$y'' - (x + 1)y' + x^2y = x$$

Where

$$y(0) = 1 \text{ ,and } y'(0) = 1$$

Sol:

When searching for this equation find:

$$F_0(x) = x^2 \quad , \qquad F_1(x) = (-x-1) \; , \qquad Q(x) = x$$

Where

$$F_0(x) \,,\, F_1 \,(x) \;\, , F_2(x) \qquad \text{It is publishable}$$

From the initial condition $x_0 = 0$ we will apply the Maclauron series.

$$y(x) = y(0) + y'(0) + \frac{y''(0)}{2\,!} x^2 + \frac{y^{(3)}(0)}{3\,!} x^3 + \cdots ..$$

From the initial conditions

$$y = 1 \qquad ,y' = 1 \qquad ,x_0 = 0$$

Substituting in the differential equation, we get:

$$y'' - 1 = 0 \;\rightarrow\; y'' = 1$$

$$y''(0) = 1$$

Differentiate again:

$$y^{(3)} - (x + 1)\, y'' + x^2 y' + 2xy = 1$$

When

$$y = 1 ,\; y' = 1 \;\; ,x = 0 \;\; ,y'' = 1$$

We get :

$$y^{(3)} = 3$$

Differentiate again:

$$y^{(4)} - (x + 1)\, y^{(3)} - y'' + x^2 y'' + 4xy' - 2y = 0$$

Now when :

$$y = \; y' = \; y'' = 1 \;\; ,x_0 = 0 \; ,y^{(3)} = 3$$

We get :

$$y^{(4)} = 3$$

Substituting into the Maclaurin series

$$y(x) = 1 + \frac{x^2}{2!} + \frac{3x^2}{2*3} + \frac{3x^4}{2*3*2} + \cdots$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^2}{2} + \frac{x^4}{8} + \cdots$$

Example 2.8.[9] Resolve the following differential equation

$$y^{(3)} = \frac{1}{x} y' + \frac{1}{x^2} y = 0$$

Where

$$y(1) = 1 \ \text{, and } \ y' = 0 \text{, and } \ y''(1) = 0 \text{, and } \ x \neq 0$$

Sol :

$$F_0(x) = \frac{1}{x^2} \ , F_1(x) = \frac{1}{x} \ \ , Q(x) = 0$$

Where $F_0(x)$, $F_1(x)$, It is publishable From the initial condition $x_0 = 1$ we will apply the Maclauron series. find that

$$y^{(3)} - 1 = 0 \Rightarrow y^{(3)} = 1$$
$$\therefore y^{(3)}(1) = 1$$

$$x^2 y^{(3)} + xy' - y = 0$$

$$\Rightarrow x^2 y^{(4)} + 2xy^{(3)} - xy''$$

$$x = 1, y = y'', y^{(3)} = 1,\ y' = 0$$

find that

$$y^{(4)} = -3$$

$$x^2 y^{(5)} + 2xy^{(4)} + 2y^{(3)} + xy^{(3)} + y'' = 0$$

$$\therefore y^{(5)} = 8$$

2.6.2. Transaction comparison method [4]

Through this method, we impose the following solution:

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_3 x^3 + \cdots$$

to $y^{(n)}$ and substitute in the original equation and then we compare the coefficients to get:

$$a_1, a_2, a_3, \ldots..$$

Example 2.9.[4] Using the coefficient comparison method, solve the following equation:

$$y'' - (x+1)y' + x^2y = x$$

When

$$y''(0) = 1 \quad , y' = 1$$

Sol:

The solution will be as follows:

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$$

$$y'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots$$

$$y''(x) = a_2x + 6a_3x + 12a_4x^2 + \cdots$$

This is replaced by the given equation ,

$$(2a_2x + 6a_3x + 12a_4x^2 + \cdots)$$

$$-(x+1)(a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots)$$

$$+x^2(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots) = x$$

We arrange the equation as follows :

$$(2a_2 - a_1) + (6a_3 - 2a_2 - a_1 - 1)x + (12a_4 - 3a_3 - 2a_2 + a_0)x^2 + \cdots = 0$$

$$2a_2 - a_1 = 0 \to a_2 = \frac{a_1}{2}$$

$$6a_3 - 2a_2 - a_1 - 1 = 0 \to a_3 = \frac{2a_2 - a_1 - 1}{6}$$

$$12a_4 - 3a_3 - 2a_2 + a_0 = 0 \to a_4 = \frac{3a_3 - 2a_2 + a_0}{12}$$

From the condition .

$$a_0 = y(0) = 0$$

$$a_1 = y'(0) = 0$$

We get :

$$a_2 = \frac{1}{2} \qquad , a_3 = \frac{1}{2} \qquad , a_4 = \frac{1}{2}$$

We substitute this with our hypothesis :

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{8} + \cdots$$

2.6.3. Legender Differential Equation [5] :

Know the Legender equation as follows:

$$(1-x)^2 y'' - 2xy' + k(k+1)y = 0 \qquad (2.2)$$

So that (K) a real number

So

$$y'' = \frac{2x}{1+x^2}y' + \frac{K(K+1)}{1+x^2}\ y = 0$$

$$\frac{K(K+1)}{1+x^2} = K(K+1)(1+x^2+x^4+\cdots)$$

$$\frac{-2x}{1+x^2} = -2x + (1+x^2+x^4+\cdots)$$

When

$|x| < 1$

$$y(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + a_{n+1} x^{n+1} + a_{n+2} x^{n+2} + \cdots$$

$$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots + na_n x^{n-1} + (n+1)a_{n+1}x^n + (n+2)a_{n+2}x^{n+1} + \ldots$$

$$y''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + \cdots + n(n-1)a_n x^{n-2} + n(n+1)a_{n+1}x^{n-1}$$

$$+(n+1)(n+2)a_{n+2}x^n + \cdots$$

Substitute into equation (2.1)

$$1 + x^2)[2a_2 x + 6a_3 x + 12a_4 x^2 + \cdots + n(n-1)a_n x^{n-2} + n(n+1)a_{n+1}x^{n-1} + a_{n+1}x^n + (n+1)(n+2)a_{n+2}x^n + \cdots]$$

$$-2x[a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots + na_n x^{n-1} + (n+1)a_{n+1}x^n + (n+2)a_{n+2}x^{n+1} + \cdots] + k(k+1)[a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \cdots] = 0$$

$$[2a_2 + k(k+1)a_0] + [6a_3 2a_1 + k(k+1)a_1]x$$

$$[12a_4 - 2a_2 - 42a_2 + k(k+1)2a_2] + \cdots$$

$$[(n+1)(n+2)a_{n+2} - n(n-1)a_n - 2a_n$$

$$+k(k+1)a_n]x^n = 0$$

And from there will get:

$$a_2 = -\frac{k(k+1)}{2}a_0$$

$$a_{3=}\frac{2-k(k+1)}{6}a_1 \rightarrow a_{3=}-\frac{[k(k+1)-2]}{6}a_1 = \frac{-k^2+k+2}{6}a_1$$

$$\rightarrow a_{3=}\frac{-(k+1)(k+2)}{3!}a_1$$

$$a_{4=}\frac{6-k(k+1)}{12}a_2 = \frac{-[k^2+k+6]}{12}a_2 = \frac{-(k+2)(k+3)}{12}.\frac{-k(k+1)}{2}a_0$$

$$\rightarrow a_{4=}\frac{+k(k+1)(k-2)(k+3)}{4!}a_0$$

$$a_{4=}\frac{12-k(k+1)}{20}a_3 = \frac{-(k-3)(k+4)}{20}a_3 \rightarrow$$

$$a_5 = \frac{(k+1)(k-2)(k+3)(k+4)}{5!}a_1$$

$$a_6 = \frac{k(k+1)(k-2)(k+3)(k-4)(k-5)}{6!}a_1$$

$$\therefore\ a_{2n} = \frac{(-1)^n k(k+1)(k-2)(k+3)(k-4)\ldots\ldots(k+2n-1)}{2n!}a_0$$

$$\therefore\ a_{2n+1} = \frac{(-1)^n k(k+1)(k-2)(k+3)(k-4)\ldots\ldots(k+2n)}{(2n+1)!}a_1$$

So the general solution is:

$y(x) = a_0 S_k(x) + a_1 T_k(x)$

$$y(x) = a_0\left[1-\frac{k(k+1)x^2}{2!}+k(k+1)(k-2)(k+3)x^4+\cdots + (-1)^n\frac{k(k+1)(k-2)(k+3)(k-4)\ldots\ldots(k+2n-1)}{2n!}x^{2n}\right] + a_1\left[x-\frac{(k-1)(k+2)}{3!}x^3+\frac{k(k+1)(k-2)(k+3)(k-4)}{5n!}x^5 + \cdots + (-1)^n\frac{k(k-1)(k+2)(k-3)(k+4)\ldots\ldots(k+2n)}{(2n+1)!}x^{2n+1}\right]$$

3. Using Mathematica to solve differential equations

We will apply some of the examples shown during the research using the Mathematica program and clarify the possibility of using the computer for the solution in the shortest period of time, less effort and more accuracy.

3.1. Application of Mathematica program to solve differential equations using series.

Application 1:

When applying the example 2.4 that was solved manually using the differential progression method, using the Mathematica program, the differential equation is written as shown below:

```
In[48]:= Series[Sin[x], {x, 0, 15}]
```

$$\text{Out[48]=}\; x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5040}+\frac{x^9}{362\,880}-\frac{x^{11}}{39\,916\,800}+\frac{x^{13}}{6\,227\,020\,800}-\frac{x^{15}}{1\,307\,674\,368\,000}+O[x]^{16}$$

Application 2:

Applying the example 2.2 that was solved manually with Taylor series, using Mathematica software, the differential equation is written as shown below:

```
In[50]:= Series[Exp[x], {x, 2, 10}]
```

$$\text{Out[50]=}\; e^2+e^2\,(x-2)+\frac{1}{2}e^2\,(x-2)^2+\frac{1}{6}e^2\,(x-2)^3+\frac{1}{24}e^2\,(x-2)^4+\frac{1}{120}e^2\,(x-2)^5+\frac{1}{720}e^2\,(x-2)^6+\frac{e^2\,(x-2)^7}{5040}+\frac{e^2\,(x-2)^8}{40\,320}+\frac{e^2\,(x-2)^9}{362\,880}+\frac{e^2\,(x-2)^{10}}{3\,628\,800}+O[x-2]^{11}$$

Application3:

When applying the example 2.7 which was solved manually using the differential progression method, using Mathematica software, the differential equation is written as shown below:

```
lde = {y''[x] - (x + 1) * y'[x] + x^2 y[x] == x, y'[0] == 1, y[0] == 1};
Series[DifferentialRoot[Function @@ {{y, x}, lde}][x], {x, 0, 10}]
```

$$1+x+\frac{x^2}{2}+\frac{x^3}{2}+\frac{x^4}{8}+\frac{x^5}{20}+\frac{x^6}{120}-\frac{x^7}{210}-\frac{13\,x^8}{6720}-\frac{83\,x^9}{60\,480}-\frac{9\,x^{10}}{22\,400}+O[x]^{11}$$

Application4:

When applying the example 2.8 which was solved manually using the differential progression method, using Mathematica software, the differential equation is written as shown below:

```
lde = {y'''[x] + (1/x) y'[x] - (1/x^2) y[x] == 0, y''[1] == 1, y'[1] == 0, y[1] == 1};
Series[DifferentialRoot[Function @@ {{y, x}, lde}][x], {x, 1, 5}]
```

$$1+\frac{1}{2}(x-1)^2+\frac{1}{6}(x-1)^3-\frac{1}{8}(x-1)^4+\frac{1}{15}(x-1)^5+O[x-1]^6$$

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