Exponential Generalized Half Logistic Distribution

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Abstract
In this work, we introduce a new distribution, the Odd Exponential Half Logistic Distribution (OEHLD), by expanding the half logistic distribution(HLD) to boost its adaptability.The survival function, the hazard rate function, the order statistics, the quantiles, the median, and the asymptotic confidence bounds are all derived.After obtaining a new distribution. The new obtained parameters are estimated by using the maximum likelihood estimation (MLE) method to fit the new distribution.To choose the most appropriate distribution for the data, a comparison was performed using goodness-of-fit metrics.

Keywords: Half Logistic Distribution, Maximum Likelihood Estimation, Simulation, Entropy, Order Statistics.

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1. Introduction

The HLD is a continuous probability distribution. The probability density function (pdf) of HLD is given by

$$f(t) = \frac{2\alpha e^{-\alpha t}}{(1+e^{-\alpha t})^2}\ t > 0, \alpha > 0 \qquad (1.1)$$

The corresponding cumulative distribution function (cdf) is given by

$$F(t) = 2\,(1+e^{-\alpha t})^{-1} - 1\ t > 0, \alpha > 0 \qquad (1.2)$$

The survival function of the HLD is given by

$$\overline{F} = 2 - 2(1+e^{-\alpha t})^{-1} \qquad (1.3)$$

The HLD ordering statistics was analyzed by Balakrishnan (1985).Olapade (2003) gave various theorems to characterize the distribution, whereas Balakrishnan and Puthenpura (1986) achieved the most accurate estimates of the distribution's location and scale parameters. Both the mean and standard deviation of the HLD were estimated using maximum likelihood methods by Balakrishnan and Wong (1991).Using both complete and censored data, Torabi and Bagheri

(2010) developed a generalized HLD.
And investigated several strategies for predicting its parameters. Researching a lifetime

requires additional distributions that are adaptable enough to work with a wide variety of data types.

In light of these issues, a number of authors have recently paid considerable attention to the HLD, proposing a number of extensions and new forms of the HLD, including the generalized half logistic distribution (GHLD), in (2016) Krishnarani propose the power half logistic distribution (PWHLD), in (2014) Olapade propose the generalized half logistic distribution (OGHLD), and in (2014) Cordeiro propose the exponentiated half logistic distribution family of distributions (EHLD-G)

The formula[6] below can be used to calculate the odd ratio, which stands in for the random variable of the producing odd distribution.

$$T = \frac{F(t)}{\overline{F}(t)} \qquad (1.4)$$

Then if T is distributed from exponential with parameter, then its pdf is given by

$$f(t) = \vartheta e^{-\vartheta t}, t > 0, \vartheta > 0 \qquad (1.5)$$

While its cdf is given by

$$F(t) = 1 - e^{-\vartheta t} \qquad (1.6)$$

One of the essential properties that we need to recall is related to the gamma pdf, that is

$$1 = \int_0^{\infty} \frac{1}{\Gamma(\eta)\beta^{\eta}} t^{\eta-1} e^{-\frac{t}{\beta}} dt \qquad (1.7)$$

2. Odd Exponential half-logistic Distribution

We present OEHLD and obtain the cdf, and pdf, respectively by using the formula of odd ratio in (1.4) to obtain the following new random variable

$$T = \frac{2\left(1+e^{-\alpha t}\right)^{-1} - 1}{2 - 2(1+e^{-\alpha t})^{-1}} = \frac{1-e^{-\alpha t}}{2e^{-\alpha t}} \qquad (2.1)$$

Then

$$F_{OEHL}(t) = 1 - e^{-\vartheta \frac{1-e^{-\alpha t}}{2e^{-\alpha t}}} \qquad (2.2)$$

is the cdf and the corresponding pdf has the following formula

$$f_{OEHL}(t) = \frac{\alpha\vartheta}{2} e^{\alpha t} e^{\frac{-\vartheta}{2} \cdot \frac{1-e^{-\alpha t}}{e^{-\alpha t}}}, t > 0, \vartheta, \alpha > 0, \text{ respectively} \qquad (2.3)$$

The survival function of the OEHLD is given as follows

$$S_{OEHL}(t) = 1 - F(t) = e^{\frac{-\vartheta}{2} \cdot \frac{1-e^{-\alpha t}}{e^{-\alpha t}}}, \ (2.4)$$

the OEHLD is applicable to survival analysis, hydrology, and economics.

The hazard function is given by

$$h_{OEHL} = \frac{\alpha\vartheta}{2} e^{\alpha t} \qquad (2.5)$$

and the corresponding cumulative hazard function is given by

$$H_{OEHL}(t) = \int_0^t h_{OEHL}(t)\, dt = \frac{\vartheta}{2}(e^{\alpha t} - 1) \; (2.6)$$

3. Properties of OEHLD

3.1. Moments: This section discusses the current state of OEHLD. Moments are essential to any statistical analysis, particularly applications.It can be used to examine the most significant properties and aspects of a distribution (e.g. tendency, dispersion, skewness and kurtosis).

Theorem 3.1. If T has OEHLD (α, ϑ), then the moments of random variable T, is given by the formula in (3.1)

$$E(T) = \frac{1}{\alpha}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j \binom{i}{j}\left(\frac{-\vartheta}{2}\right)^i (j-i)^{-1} \qquad (3.1)$$

$$E(T^2) = \frac{2}{\alpha^2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j \binom{i}{j}\left(\frac{-\vartheta}{2}\right)^i (j-i)^{-2} \qquad (3.2)$$

Proof. We begin with the well-known distribution of the moment of the random variable T, where the probability density function of (T) is given.

$$E(T) = \int_0^{\infty} t\, f(t; \alpha, \vartheta) dt \qquad (3.3)$$

Substituting Eq. (2.3) in Eq. (3.3) we get

$$E(T) = \int_0^{\infty} t \frac{\alpha\vartheta}{2} e^{\alpha t}\, e^{\frac{-\vartheta}{2}.\frac{1-e^{-\alpha t}}{e^{-\alpha t}}} dt$$

By using integral with by part method, we have

$u = t \rightarrow du = dt, \int dv = \int \frac{\vartheta\alpha}{2} e^{\alpha t} e^{-\frac{-\vartheta(1-e^{-\alpha t})}{2e^{-\alpha t}}} dt \rightarrow v = -e^{-\frac{-\vartheta(1-e^{-\alpha t})}{2e^{-\alpha t}}}$

Then

$$E(T) = \int_0^{\infty} e^{-\frac{-\vartheta(1-e^{-\alpha t})}{2e^{-\alpha t}}} dt$$

By using the expansion of exponential function, we obtain the following solution

$$e^{-\frac{-\vartheta(1-e^{-\alpha t})}{2e^{-\alpha t}}} = \sum_{i=1}^{\infty}\left(-\frac{\vartheta}{2}\right)^i (e^{\alpha t} - 1)^i$$

and we can also see that

$$(e^{\alpha t}-1)^i=\sum_{j=1}^{\infty}(-1)^j\binom{i}{j}e^{-\alpha(j-i)t}$$

Thus

$$E(T)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(-1)^j\binom{i}{j}\left(-\frac{\vartheta}{2}\right)^i\int_0^{\infty}e^{-\alpha(j-i)t}\,dt$$

Therefore, and by using the property of gamma $distribution\ in$ (1.7)

$$E(T)=\frac{1}{\alpha}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(-1)^j\binom{i}{j}\left(-\frac{\vartheta}{2}\right)^i(j-i)^{-1}$$

similarly, we can prove that the second moment and obtain formula (3.2)

3.2. Order Statistics: The order statistics can be derived and obtained. OEHLD as follows. Let $T_{1:n}\leq T_{2:n}\leq\cdots\leq T_{n:n}$be From a population, we can derive and retrieve the ordered sample. sales figures for the comulative distribution function of $T_{n:n}$, the n^{th} order statistics, is given by

$$F_{n:n}(t)=[F(t)]^n=\left[1-e^{-\vartheta\frac{1-e^{-\alpha t}}{2e^{-\alpha t}}}\right]^n \qquad (3.4)$$

Where the cdf of the OEHLD is F(T).

The probability distribution function the n^{th}order statistics for the OEHL random variable $t_{n:n}$ can be obtained by using Eqs.(2.1) and (2.2) we have

$$f_{n:n}(t)=n[F(t)]^{n-1}f(t)$$

$$=n\left[1-e^{-\vartheta\frac{1-e^{-\alpha t}}{2e^{-\alpha t}}}\right]^{n-1}\frac{\alpha\vartheta}{2}e^{\alpha t}e^{\frac{-\vartheta}{2}\cdot\frac{1-e^{-\alpha t}}{e^{-\alpha t}}} \qquad (3.5)$$

Where f(T) is the pdf of OEHLD.

The cdf of the first order statistics $t_{1:n}$,, is obtained as

$$f_{1:n}(t)=[1-F(t)]^n=\left[e^{-\vartheta\frac{1-e^{-\alpha t}}{2e^{-\alpha t}}}\right]^n \qquad (3.6)$$

and the pdf of $t_{1:n}$, is given by

$$f_{1:n}(t)=n[1-F(t)]^{n-1}-f(t)$$

$$=n\left[e^{-\vartheta\frac{1-e^{-\alpha t}}{2e^{-\alpha t}}}\right]^{n-1}\frac{-\alpha\vartheta}{2}e^{\alpha t}e^{\frac{-\vartheta}{2}\cdot\frac{1-e^{-\alpha t}}{e^{-\alpha t}}} \qquad (3.7)$$

Then, the pdf of the k^{th} order statistics $t_{k:n}$,, is obtained as

$$f_{t_{k:n}}(t)=\frac{n!}{(k-1)!\,(n-k)!}f(t)[F(t)]^{k-1}[1-F(t)]^{n-k} \qquad (3.8)$$

We are able to get the pdf of the k^{th} order statistics as follows:

$$f_{k:n}(t)=\frac{n!}{(k-1)!\,(n-k)!}\left[1-e^{-\vartheta\frac{1-e^{-\alpha t}}{2e^{-\alpha t}}}\right]^{k-1}\left[e^{\frac{-\vartheta}{2}\cdot\frac{1-e^{-\alpha t}}{e^{-\alpha t}}}\right]^{n-k} \qquad (3.9)$$

3.3 Quantile and Median: In this section, specific formulas for the quantile and median are provided. OEHLD are determined.The percentile t of the OEHLD is given by

$$F(t)=q, 0<q<1 \ (3.10)$$

From Eq. (2.1), t can be obtained as follows.

$$t = \frac{1}{\alpha} ln\left[\frac{\vartheta - 2ln(1-q)}{\vartheta}\right] \quad (3.11)$$

Setting q = 0.5 in Eq. (3.11), we get the median of OEHLD as follows.

$$t = \frac{1}{\alpha} ln\left[1 + \frac{2\ln(2)}{\vartheta}\right] \quad (3.12)$$

3.4. Maximum Likelihood Estimates: While there is much debate regarding which estimating technique is best, MLE is now the most used because it yields the most data about the estimated parameters' characteristics.

On top of that we can write the probability function for a sample as:

$$\prod_{i=1}^{n} f(t_i) = \frac{\alpha^n \vartheta^n}{2^n} e^{\alpha \sum_{i=1}^{n} t_i} . e^{\frac{-\vartheta}{2}\sum_{i=1}^{n}\frac{1-e^{-\alpha t_i}}{e^{-\alpha t_i}}} \quad (3.13)$$

Then, the loglikelihood function is obtained as

$$l = nlog\alpha + nlog\vartheta - nlog2 + \alpha \sum_{i=1}^{n} t_i - \frac{\vartheta}{2}\sum_{i=1}^{n}\frac{1-e^{-\alpha t_i}}{e^{-\alpha t_i}} \quad (3.14)$$

Therefore, we can fined the MLE's parameter values by taken the first derivative of equation (3.14) with respect to parameters that amount to zero.

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} t_i - \frac{\vartheta}{2}\sum_{i=1}^{n}\frac{t_i}{e^{-\alpha t_i}} = 0 (3.15)$$

$$\frac{\partial l}{\partial \vartheta} = \frac{n}{\vartheta} - \frac{1}{2}\sum_{i=1}^{n}\frac{1-e^{-\alpha t_i}}{e^{-\alpha t_i}} = 0 (3.16)$$

Assuming that α is known, we can obtain the MLE of ϑ as

$$\hat{\vartheta} = \frac{2n}{\sum_{i=1}^{n}\frac{1-e^{-\alpha t_i}}{e^{-\alpha t_i}}}$$

3.5. Asymptotic confidence bounds: Now we asymptotic confidence intervals for these parameters can be calculated. when $\alpha > 0, \vartheta > 0$ *as* the MLEs of $\alpha > 0$. and $\vartheta > 0$ cannot be calculated exactly using the covariance and variance matricesI^{-1}, see [7], where I^{-1} is defined as the complement of the observed information matrix:

$$I^{-1} = \begin{pmatrix} -\frac{\partial^2 l}{\partial \alpha^2} & -\frac{\partial^2 l}{\partial \alpha \partial \vartheta} \\ -\frac{\partial^2 l}{\partial \vartheta \partial \alpha} & -\frac{\partial^2 l}{\partial \vartheta^2} \end{pmatrix}^{-1}$$

the unknwown parameters

$$= \begin{pmatrix} -E\left(\frac{\partial^2 l}{\partial \alpha^2}\right) & -E\left(\frac{\partial^2 l}{\partial \alpha \partial \vartheta}\right) \\ -E\left(\frac{\partial^2 l}{\partial \vartheta \partial \alpha}\right) & -E\left(\frac{\partial^2 l}{\partial \vartheta^2}\right) \end{pmatrix} = \begin{pmatrix} var(\hat{\alpha}) & cov(\hat{\alpha}, \hat{\vartheta}) \\ cov(\hat{\vartheta}, \hat{\alpha}) & var(\hat{\vartheta}) \end{pmatrix}. (3.17)$$

The second partial derivatives included in I are given as follows:

$$\frac{\partial^2 l}{\partial \alpha^2} = -\frac{n}{\alpha^2} - \frac{\vartheta}{2}\sum_{i=1}^{n}\frac{t_i^{\,2}}{e^{-\alpha t_i}} \quad (3.18)$$

$$\frac{\partial^2 l}{\partial\alpha\partial\vartheta} = -\frac{1}{2}\sum_{i=1}^{n}\frac{t_i}{e^{-\alpha t_i}} \tag{3.19}$$

$$\frac{\partial^2 l}{\partial\vartheta^2} = -\frac{n}{\vartheta^2} \tag{3.20}$$

$$\frac{\partial^2 l}{\partial\vartheta\partial\alpha} = -\frac{1}{2}\sum_{i=1}^{n}\frac{t_i}{e^{-\alpha t_i}} \tag{3.21}$$

We can derive the $(1-\delta)100\%$ estimates of the parameters' reliability $\alpha$and ϑ by using variance matrix as in the following forms

$$\left(\hat{\alpha} \pm Z_{\frac{\delta}{2}}\sqrt{var(\hat{\alpha})}, \hat{\vartheta} \pm Z_{\frac{\delta}{2}}\sqrt{var(\hat{\vartheta})}\right)$$

By using gamma function in (1.7) we have

$$var(\hat{\alpha}) = -E\left(\frac{\partial^2 l}{\partial\alpha^2}\right) = -E\left(-\frac{n}{\alpha^2} - \frac{\vartheta}{2}\sum_{i=1}^{n}\frac{{t_i}^2}{e^{-\alpha t_i}}\right) = \frac{n}{\alpha^2} + \frac{\vartheta}{2}\sum_{i=1}^{n}E\left(\frac{{t_i}^2}{e^{-\alpha t_i}}\right)$$

$$= \frac{n}{\alpha^2} + \frac{\vartheta}{2}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{z=0}^{\infty}(-1)^{i+j}\binom{i}{j}\left(\frac{-\vartheta}{2}\right)^{i}\frac{\alpha^z}{z!}\Gamma(z+3)[\alpha(j-i)]^{-(z+3)}$$

where $E\left(\frac{t^2}{e^{-\alpha t}}\right) = \sum_{z=0}^{\infty}\frac{\alpha^z}{z!}\int_0^{\infty} t^{z+2+1-1}\ \frac{\alpha\vartheta}{2}e^{\alpha t}\, e^{\frac{-\vartheta(1-e^{-\alpha t})}{2e^{-\alpha t}}}dt$, and $Z_{\frac{\delta}{2}}$ is the upper $\left(\frac{\delta}{2}\right) - th$ percentile of the standard normal distribution.

4. Application: In this part, We consider the data in [7], and we have made a comparison between the results of the measure of goodness for these data and Table 4.1 shows the results of that comparison. The values in that table show that OEHLD is the best of HLD

Table 4.1. Goodness of fit criteria AIC, BIC, CAIC, HQIC.

Model	$\hat{l}$	**AIC**	**BIC**	**CAIC**	**HQIC**
HLD	-92.50209	189.00418	189.97399	190.33751	188.64512
OEHLD	-77.21517	158.43034	159.40015	159.76367	158.07128

4. Conclusion

By extending the HLD, we derive the OEHLD in this study.Some statistical features of this distribution, including survival, hazard rate, and order statistics, are explored.The OEHLD parameters are determined using the greatest likelihood technique.The results of the goodness-of-fit test indicate that the suggested distribution is adaptable and a viable alternative to existing ones.

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