DGN_1 −Topological Graph

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Abstract

In this paper, by induced alternate definition of DGN_1-open set, we established a new kind of topological structure connected with digrphs named DGN_1-topological graph. With respect to this alternated definition, we investigated into too many topology properties which were determined by a digrph.

Keywords: Digrph, Topology, Topological operator acyclic digrphs.

1. Introduction

Because of two features, graph theory is a useful mathematical tool in a range of domains and is considered as a fundamental component of different mathematics. To begin, graphs are created using mathematics. picked from a theoretical perspective. Graphs can perform topological space, collection objects, and a variety of other mathematical groups, despite being fundamental relational combinations. Another main reason is that some concepts are easier to understand when expressed by graphs. become more realistic. In terms of topollogy and graph theory, topological ideas are one of the tools used to express the graph. For example, converting a set of edges or vertices to topological space. Topology is one of the most well-known and current subjects for studying the other topological ideas in this space. Thin king on a wde region of mathematician The following are some previous literature on the subject of topological graphs. Evans and Harary [4] In 1967, they proposed the concept of topology on digrph. They observed that the set of all topologies with n vertices and the set of all graph vertices have a single relationship. "A.J.M. Khalaf and others in [1] They define the hyper-Zagreb index, first multiple Zagreb index, second multiple Zagreb index, Zagreb polynomials, and M-polynomial for micro structures of bridge graphs and present the concept of bridge graphs. And more relations between Polynomial and Topological Indices see[10,3,13,14,15]. N. De, M. Cancan, and M. Alaeiyan establish general equations for numerous degree-based topological indices of a specific network known as the mk-graph in [9] for some positive integer k. In [11, 16-22] X. Zhang and et al., they described a new method for obtaining the exact values of ECI of butterfly networks, bene network graphs, and toroidal grid graphs. Kh. Sh. Al'Dzhabri and colleagues invented a new concept in topological spaces called DG-topological spaces topology related to digrphs formed by a new open set called DG-open set, and investigated many properties of this concept using the closed link between topology and digrphs in [7]. Kh. Sh. Al'Dzhabri also introduced other sorts of open sets that are linked to graphs [5,6,8].

2.Preliminaries

Are the first step in the process. In this paper we introduces some basic definitions of graph theory[10,3] and topology[2]. A graph (resp., directed graph or digrph) $D = (V, E)$ "consists of a vertex set V and an edge set E of unordered (resp., ordered) pairs of elements of V. To avoid ambiguities, we assume that the vertex and edge sets are disjoint. We say that two vertices v and w of a graph (resp., digrph D) are adjacent if there is an edge of the form vw (rsep., $\overrightarrow{wv}$ or $\overrightarrow{vw}$) joining them, and the vertices v and w are then incident with such an edge. A subdigrph H of a digrph D is a digrph, each of whose vertices belong to V and each of whose edges belong to E. The degree of a vertex v of D is the number of edges incident with v, and written deg(v). A vertex of degree zero is an isolated vertex. In digrph, the outdegree, of a vertex v of D is the number of edges of the form $\overrightarrow{vw}$ and denoted by $d^{+}(v)$, similarly, the indegree of a vertex v of D is the number of edges of the form $\overrightarrow{wv}$, and denoted by $d^{-}(v)$,. A vertex of out-degree and in-degree are zero is an isolated vertex. A digrph is said to be symmetric digrph if $wv \in E$ implies $vw \in E$. A complete digrph is simple digrph where each pair of distinct vertices is connected by a pair of unique arcs (one in each direction). A digrph is called strongly connected digrph if there is a dipath from every vertex to every other vertices. A cyclic digrph is a non-empty directed trail in which only the first and last vertices are equal. And let X be a set. A topology on X is a set $\tau \subset \mathcal{P}(X)$ of a subsets of X, called open sets, such that: (1) $\emptyset$ and X are open. (2) The intersection of finitely many open sets is open. (3) Any union of open sets is open. A topological space is a set X together with a topology τ on X. A subset of a topological space is closed if its complement is open. In the topology on X, if $\tau = \{\emptyset, X\}$ is called the indiscrete topology, and only two subsets are open. If $\tau = \{A | A \subseteq X\}$ is called discrete topology, and all subsets are open.A topology basis is a set $\beta \subset \mathcal{P}(X)$ of a subsets of X, called basis sets, such that: (1) Any point of X lies in a basis set, i.e $X = \cup\, \beta$. (2) The intersection of any two basis sets is a union of basis sets. A topology subbasis is a set $S \subset \mathcal{P}(X)$ of a subsets of X, called subbasis sets, such that: (1) Any point of X lies in a subbasis set, "i.e $X = \cup\, S$.If β is a topology basis then the set of subset of X $\tau_{\beta} = \{\text{union of basis sets}\}$ is called the topology generated by β, and note that τ_{β} is a topology on X. If S is a topology subbasis then the set of subset of X $\tau_{S} = \{\text{union of finite intersection of subbasis sets}\}$ is called the topology generated by S, and note that τ_{S} is a topology on X. In a space X, $A \subset X$. The interior of A is the union of all open sets contained in A, $\text{Int}\, A = \cup \{U \subset A |\, U \text{ is open}\} = A^{\circ}$ .And the closure of A is the intersection of all closed sets contained A, $\text{Cl}\, A = \cap \{F \supset A |\, F \text{ is closed}\} = \overline{A}$.A subset A of a space X is called dense if $\overline{A} = X$. A point $x \in X$ is limit point of A if $U \cap (A - \{x\}) \neq \emptyset$ for all neighborhoods U of x. The limit points of A is denoted by $\acute{A}$. A two nonempty subset A and B of a space X are called separated if and only if $A \cap \overline{B} = \emptyset$ and $\overline{A} \cap B = \emptyset$, or equivalently, $(A \cap \overline{B}) \cup (\overline{A} \cap B) = \emptyset$. A topological space X is τ_0 −space or (Kolmogorov space) if for any two distinct points $x_1, x_2 \in X, x_1 \neq x_2$ there exists an open set U containing one but not both points. A topological space X is τ_1 −space if point are closed: for any two distinct points $x_1, x_2 \in X, x_1 \neq x_2$ there exists an open set U such that $x_1 \in U$ and $x_2 \notin U$. A space X is said to be Alexandroff space if and only if every point has minimal neighborhood or equivalently, has unique minimal base. This is also equivalent to the fact that the intersection of family of open sets is open. The minimal neighborhood is the intersection of all open sets containing x".

3. DGN_1 −Topological Operators Associated with Digrph

K.S. Al'Dzhabri, M.F. Hani in[5] are introuced the connection between topology and graph by defined the concepts called DG-open set and the graph is required to be transitive and they demonstrated many results In this section, the graph is required not nesserary to be transitive and we give new definition named DGN_1 −topological space by define new concept called DGN_1 −open set it is differernt from the concept of DG-open set in terms of building new base of DGN_1 −topological space. A topology may be determined on V defining certain subset of V to be open with respect to a digrph $D = (V, E)$, and we introduced new concepts DGN_1 −closure, DGN_1 −kernel, DGN_1 −core, DGN_1 −limit point and DGN_1 −interior operators to investigate the connectedness of the digrph with these concepts. We demonstrated many results that differ in the method of proof from the results in the [5].

Definition 3.1:Let $D = (V, E)$ be a digrph. A subset A of V is called DGN_1 −open set if for $v_j \in A$ and an arc $v_j u_i \in E$, then $u_i \in A$.

Remark 3.2:The complement of DGN_1 −open set is called DGN_1 −closed set.

Proposition 3.3: Let $D = (V, E)$ be a digrph. A subset A of V is a DGN_1 −open set if and only if $u_i \in A$ and $v_j \in A^c$, implies $u_i v_j \notin E$. In other word a subset A of V is a DGN_1 −open set if there does not exists an arc from A to A^c.

Proof: (⇒) Suppose that A be DGN_1 −open set. And to prove that if $u_i \in A$ and $v_j \in A^c$, implies $u_i v_j \notin E$. Since A be a DGN_1 −open set then from definition (3.1) $v_j \in A$ and $v_j u_i \in E$ and we have that $u_i \in A$ and this means that if $v_j \notin A$ then $u_i \in A$ and $u_i v_j \notin E$. Hence if $u_i \in A$ and $v_j \in A^c$, implies $u_i v_j \notin E$. (⇐) Suppose that $u_i \in A$ and $v_j \in A^c$, implies $u_i v_j \notin E$. And to prove that A be a DGN_1 −open set. Since $u_i \in A$ and $v_j \in A^c$ and $u_i v_j \notin E$ and that means if $v_j u_i \in E$ and $u_i \in A$, then we have $v_j \in A$. Hence A is a DGN_1 −open set. ■

Remark 3.4: From the definition (3.1) the topology associated with the digrph $D = (V, E)$ is denoted by τ_{DGN_1} and $\tau_{DGN_1} = \{ A: A \text{ is } DGN_1 - \text{open set} \}$ and (V, τ_{DGN_1}) is called DGN_1 −topological graph.

Example 3.5: Consider the following digrph $D = (V, E)$ in figure (3.1) where $V = \{ v_1, v_2, v_3, v_4\}$ and $E = \{v_2v_1, v_2v_4, v_3v_1, v_3v_2, v_3v_4, v_4v_1\}$.

Then the topology corresponding to the above digrph $\tau_{DGN_1} = \{ \emptyset, V, \{ v_1\}, \{ v_1, v_4\}, \{ v_1, v_2, v_4\}\}$.

Example 3.6: The topology associated with the complete digrph, strongly connected digrph and cyclic digrph are Indiscreet topology.

Theorem 3.7: Let $D = (V, E)$ be a digrph. Then (V, τ_{DGN_1}) is a topology on the set V associated with the digrph $D = (V, E)$.

Proof: i) Clearly $\emptyset$ and $V \in \tau_{DGN_1}$.

ii) Let $U_i \in \tau_{DGN_1}$ $\forall\, i = 1,2,3, \dots, n$. Now let $v_j \in \cap_{i=1}^{n} U_i$ and $v_j u_i \in E$, then $v_j \in U_i$ for $i = 1,2, \dots, n$. Then $u_i \in U_i$ for all $i = 1,2, \dots, n$. Hence $u_i \in \cap_{i=1}^{n} U_i$,and therefore the family $\cap_{i=1}^{n} U_i \in \tau_{DGN_1}$.

iii) Let $\{ U_\alpha\}$ be a collection of subsets of V in τ_{DGN_1}, and let $v_j \in \cup_\alpha U_\alpha$ for all α and $v_j u_i \in E$. Then $\exists\, U_{\alpha_o} \in \{ U_\alpha\}$ such that $v_j \in U_{\alpha_o}$ with $v_j u_i \in E$.This implies that $u_i \in U_{\alpha_o}$.Hence $u_i \in \cup_\alpha U_\alpha$ for all α, so $\cup_\alpha U_\alpha \in \tau_{DGN_1}$. Hence τ_{DGN_1} is a topology on V. ■

Remark 3.8: Every DGN_1 −topological graph is Alexandorff space.

Remark 3.9: Let $D = (V, E)$ be a digrph. We construct a DGN_1 −subbase S_{DGN_1} for a DGN_1 −topological graph τ_{DGN_1} on V by $S_{DGN_1} = \{u_i\} \cup \{v_j: u_i v_j \in E \text{ where } u_i, v_j \in V\}$. We define a DGN_1 −base β_{DGN_1} using the finite intersection of members of S_{DGN_1}. And from the arbitrary union of members of β_{DGN_1} we have the DGN_1 −topological graph τ_{DGN_1} on V.

Example 3.10: Consider the following digrph $D = (V, E)$ in figure (3.2) where $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{v_1v_4, v_2v_1, v_2v_3, v_2v_4, v_3v_4, v_5v_1, v_5v_4\}$.

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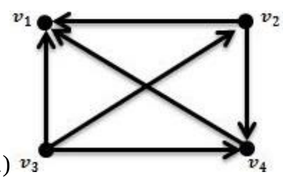
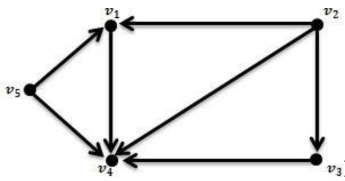


Figure(3.1)



Figure(3.2).

Then the DGN_1 −subbase to above digrph is $S_{\mathrm{DGN}_1} = \{\{v_4\},\{v_1,v_4\},\{v_3,v_4\},\{v_1,v_4,v_5\},\{v_1,v_2,v_3,v_4\}\}$ And $\beta_{\mathrm{DGN}_1} = \{V,\{v_4\},\{v_1,v_4\},\{v_3,v_4\},\{v_1,v_4,v_5\},\{v_1,v_2,v_3,v_4\}\}$. And $\tau_{\mathrm{DGN}_1} = \{\emptyset,V,\{v_4\},\{v_1,v_4\},\{v_3,v_4\},\{v_1,v_3,v_4\},\{v_1,v_4,v_5\},\{v_1,v_2,v_3,v_4\},\{v_1,v_3,v_4,v_5\}\}$"

Proposition 3.11: Let $D = (V,E)$, and let u_i and u_j be a fixed vertices of a set V. Then u_i is in degree to u_j if and only if for each subset $A \subseteq V$ such that $u_j \in A$ and $u_i \notin A$, there exists an arc from A^c to A.

Proof:(⇒) To prove there exists an arc from A^c to A, if $u_i = u_j$, the proposition is obviously. Now suppose that u_i is in degree to u_j for $u_i \neq u_j$. Thus there exists a dipath of finite length from u_i to u_j. Let A be a subset of V such that $u_j \in A$ and $u_i \notin A$. From definition of a dipath, a dipath is an ordered tuple of finite length, let u_k, the k the vertex in this tuple, be the first vertex of this dipath which is belongs to A and other vertices are not in A, and also $u_j \neq u_k$. Thus $u_{k-1} \notin A$ and we have the required are namely $u_{k-1}u_k$. Hence there exists an arc from A^c to A.

(⇐)Now suppose that u_i and u_j be a distinct fixed vertices of a set V, and $A \subseteq V$ with $u_j \in A$ and $u_i \notin A$, and suppose that exists an arc from A^c to A.Now from the set $A_i = \{u_d : \Gamma(i,d)\}$. Suppose that $u_i \notin A_i$, then by hypothesis, there exists an arc from ${A_i}^c$ to A_i, say $u_r u_s$ such that $u_s \in A_i$and $u_r \in {A_i}^c$. But u_i is in degree to u_s and this means $u_i = u_s$ or there exists a finite dipath from u_i to u_s and thus there exists a finite dipath from u_i to u_r which containing the vertex u_s. Thus u_i is in degree to u_r, and therefore $u_r \in A_i$ which is a contradiction.Hence $u_i \in A_i$ i.e u_i is in degree to u_j.■

Proposition 3.12: Let $D = (V,E)$, and let u_i and u_j be a fixed vertices of a set V. Then u_i is in degree to u_j if and only if each DGN_1 −closed set W, such that $u_j \in W$, implies $u_i \in W$; or equivalently each DGN_1 −open set M, such that $u_i \in M$, implies $u_j \in M$.

Proof: (⇒)Let u_i and u_j be a fixed vertices of the V and suppose that u_i is in degree to u_j.Let W be an DGN_1 −closed set such that $u_j \in W$. If $u_i \notin W$ that is means $u_i \in W^c$, then by proposition (3.11) there exists an arc from W^c to W, which implies that W is not DGN_1 −closed set. Hence $u_i \in W$.

(⇐)Now to prove u_i is in degree to u_j, suppose that W is a DGN_1 −closed set such that $u_j \in W$ then $u_i \in W$ this means there does not exist a DGN_1 −closed set W such that $u_j \in W$ but $u_i \notin W$. And a set W is a DGN_1 −closed if and only if W^c is a DGN_1 −open set. Thus there does not exist a DGN_1 −open set W^c such that $u_i \in W^c$ but $u_j \notin W^c$. Now assume that each DGN_1 −open set M, such that $u_i \in M$, implies $u_j \in M$ and hence each DGN_1 −open set M, such that $u_i \in M$ but $u_j \notin M$ is not DGN_1 −open set. Hence M^c is not DGN_1 −closed set. Thus for each set M^c, there exists an arc from M to M^c, then by proposition (3.11) u_i is in degree to u_j. ■

Definition 3.13: "Let $D = (V,E)$ be a digrph and $(V,\tau_{\mathrm{DGN}_1})$ be a DGN_1 −topological graph. The DGN_1 −closure of a subset A of V is the intersection of all DGN_1 − closed subset of V containing A, and denoted by $\mathrm{Cl}_{\mathrm{DGN}_1}(A)$. i.e. $\mathrm{Cl}_{\mathrm{DGN}_1}(A) = \cap\{F : F \text{ is } \mathrm{DGN}_1 - \text{closed}, A \subseteq F \subseteq V\}$.

Example 3.14: Consider the digrph $D = (V,E)$ in figure (3.3) where $V = \{v_1,v_2,v_3,v_4,v_5\}$ and $E = \{v_1v_3, v_2v_1, v_2v_3, v_2v_4, v_2v_5, v_4v_3, v_4v_5, v_5v_3\}$

Then the topology associated to above digrph is $\tau_{DGN_1} = \{ \emptyset, V, \{v_3\}, \{v_1, v_3\}, \{v_3, v_5\}, \{v_1, v_3, v_5\}, \{ v_3, v_4, v_5\}, \{v_1, v_3, v_4, v_5\} \}$. Then $Cl_{DGN_1}(\{v_2, v_5\}) = \{ v_2, v_4, v_5\}$.

Remark 3.15: $Cl_{DGN_1}(A)$ is the set of all vertices which are lead to A.

Example 3.16: Consider the above digrph $D = (V, E)$ in figure (3.3). Let $A = \{v_1, v_4\}$. Then $Cl_{DGN_1}(A) = Cl_{DGN_1}(\{v_1, v_4\})$ the set of all vertices which are lead to $\{v_1, v_4\}$ is $\{v_1, v_2, v_4\}$.

Remark 3.17: Let (V, τ_{DGN_1}) be a DGN_1 −topological graph associated with the digrph $D = (V, E)$ and let $A, B \subseteq V$. Then:

i)The DGN_1 −closure set is always DGN_1 − closed set.

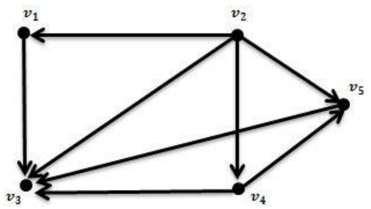
ii)$Cl_{DGN_1}(A)$ is the smallest DGN_1 − closed set contained A.

iii)$Cl_{DGN_1}(A) = A$ if and only if A is DGN_1 − closed set.

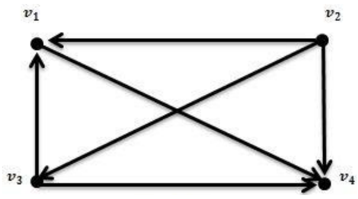
iv)Since τ_{DGN_1} satisfy the completely additive closure then $Cl_{DGN_1}(A \cup B) = Cl_{DGN_1}(A) \cup Cl_{DGN_1}(B)$ Now in the following definition we define a new operators:

Definition 3.18: Let $D = (V, E)$ be a digrph and (V, τ_{DGN_1}) be a DGN_1 −topological graph. The DGN_1 −kernal of a subset A of V is the intersection of all DGN_1 − open subset of V containing A, and denoted by $Kl_{DGN_1}(A)$. i.e $Kl_{DGN_1}(A) = \cap \{U: U \text{ is } DGN_1 - \text{open}, A \subseteq U \subseteq V \}$.

Example 3.19: Consider the digrph $D = (V, E)$ in figure (3.4) where $V = \{ v_1, v_2, v_3, v_4\}$ and $E = \{v_1v_4, v_2v_1, v_2v_3, v_2v_4, v_3v_1, v_3v_4\}$.



Figure(3.3)



Figure(3.4)

The topology associated to above digrph is $\tau_{DGN_1} = \{ \emptyset, V, \{v_4\}, \{v_1, v_4\}, \{v_1, v_3, v_4\} \}$. Then $Kl_{DGN_1}(\{v_1, v_3\}) = \{ v_1, v_3, v_4\}$.

Remark 3.20: Let (V, τ_{DGN_1}) be a DGN_1 −topological graph associated with the digrph $D = (V, E)$ and let $A, B \subseteq V$. Then:

i)From Remark (3.8) the topology τ_{DGN_1} on a digrph $D = (V)$ has completely additive closure i.e (The intersection of an arbitrary DGN_1 − open set is DGN_1 − open set) and we see that the DGN_1 −kernal of a set is DGN_1 − open set and that a set A is DGN_1 − open set if and only if $A = Kl_{DGN_1}(A)$.

ii)Since τ_{DGN_1} satisfies the completely additive closure then $Kl_{DGN_1}(A \cup B) = Kl_{DGN_1}(A) \cup Kl_{DGN_1}(B)$.

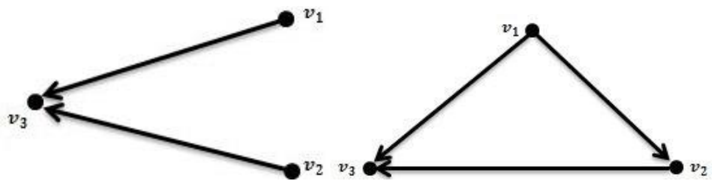
Definition 3.21: Let $D = (V, E)$ be a digrph and (V, τ_{DGN_1}) be a DGN_1 −topological graph. The DGN_1 −core of a subset A of V is the intersection of all subsets of V containing A which are DGN_1 − closed or DGN_1 −open,and denoted by $C_{DGN_1}(A)$. i.e $C_{DGN_1}(A) = \cap \{U: U \text{ is } DGN_1 - \text{closed or } DGN_1 - \text{open}, A \subseteq U \subseteq V \}$.

Remark 3.22: Note that $C_{DGN_1}(A) = Cl_{DGN_1}(A) \cap Kl_{DGN_1}(A)$."

Example 3.23: Consider the digrph $D = (V, E)$, in figure (3.5) where $V = \{ v_1, v_2, v_3\}$ and $E = \{v_1v_3, v_2v_3\}$.

The topology associated to above digrph is $\tau_{DGN_1} = \{ \emptyset, V, \{v_3\}, \{v_1, v_3\}, \{v_2, v_3\}\}$ Then $C_{DGN_1}(\{v_1\}) = \{v_1\} \cap \{v_1, v_3\} = \{v_1\}$

Theorem 3.24: Let (V, τ_{DGN_1}) be a DGN_1 −topological graph associated with the digrph $D = (V, E)$. Then any vertex u_i of $D(V)$,
i)$Cl_{DGN_1}(u_i) = \{u_j: \Gamma(j, i)\}$, in another words, $Cl_{DGN_1}(u_i)$ is the set of vertices of the digrph $D(V)$ which are indegree to u_i.
ii)$Kl_{DGN_1}(u_i) = \{u_j: \Gamma(i, j)\}$, in another words, $Kl_{DGN_1}(u_i)$ is the set of vertices of the digrph $D(V)$ which are outdegree from u_i.
iii)$C_{DGN_1}(u_i) = \{u_j: \Gamma^*(i, j)\}$. In another words, $C_{DGN_1}(u_i)$ is the set of vertices of the digrph $D(V)$ which are symmetrically indegrable to u_i.
Proof: i)From definition (3.13), $Cl_{DGN_1}(u_i) = \cap \{F: F \text{ is } DGN_1 - \text{closed and } u_i \in F\}$ and that is $Cl_{DGN_1}(u_i)$ is the set of all vertices u_j such that $DGN_1 -$ closed set F such that $u_j \in F$, then $u_i \in F$ and from proposition (3.12) we have $Cl_{DGN_1}(u_i) = \{u_j: \Gamma(j, i)\}$.
ii)Similarly by using definition (3.18) and proposition (3.12) we have that $Kl_{DGN_1}(u_i) = \{u_j: \Gamma(i, j)\}$. iii)By using Remark (3.22) and Theorem (3.24(i)(ii)) we have that $C_{DGN_1}(u_i) = Cl_{DGN_1}(u_i) \cap Kl_{DGN_1}(u_i) = \{u_j: \Gamma(j, i)\} \cap \{u_j: \Gamma(i, j)\}=\{u_j: \Gamma(j, i) \text{and} \Gamma(i, j)\}=\{u_j: \Gamma^*(i, j)\}$. ■
Theorem 3.25: Let (V, τ_{DGN_1}) be a DGN_1 −topological graph associated with the digrph $D = (V, E)$. Then for any $A \subseteq V$,
i)$Cl_{DGN_1}(A) = \{u_j: \Gamma(j, i)$, for some $u_i \in A\}$,
ii)$Kl_{DGN_1}(A) = \{u_j: \Gamma(i, j)$, for some $u_i \in A\}$ and
iii)$C_{DGN_1}(A) = \{u_j: \Gamma(j, i)$ for some $u_i \in A$ and $\Gamma(k, j)$ for some $u_k \in A\}$.
Proof: i) Since using Remark (3.17(iv)) and Theorem (3.24 (i)) we have $Cl_{DGN_1}(A) = Cl_{DGN_1}(\cup \{u_i\}: u_i \in A) = \cup \{Cl_{DGN_1}(u_i): u_i \in A\} = \cup \{u_j: \Gamma(j, i), u_i \in A\} = \{u_j: \Gamma(j, i)$, for some $u_i \in A\}$.
ii)Similarly by using Remark (3.20(ii)) and theorem (3.24 (ii)) we have
$Kl_{DGN_1}(A) = \{u_j: \Gamma(i, j)$, for some $u_i \in A\}$.
iii)By using Remark (3.22) and Theorem(3.25 (i),(ii)) we have $C_{DGN_1}(A) = \{u_j: \Gamma(j, i)$, for some $u_i \in A$ and $\Gamma(k, j)$, for some $u_k \in A\}$. ■
Definition 3.26: Let $D = (V, E)$ be a digrph and (V, τ_{DGN_1}) be a DGN_1 −topological graph a subset A of V is called DGN_1 −dense if $Cl_{DGN_1}(A) = V$.
Example 3.27: Consider the digrph $D = (V, E)$, in figure (3.6) where $V = \{v_1, v_2, v_3\}$ and $E = \{v_1v_2, v_1v_3, v_2v_3\}$.



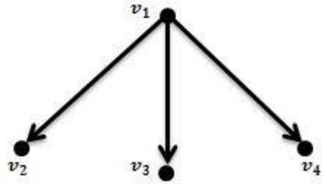
Figure(3.5) Figure(3.6)

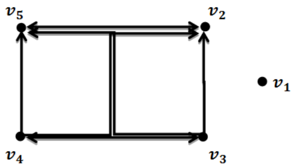
Which has the topology associated to above digrph is $\tau_{DGN_1} = \{\emptyset, V, \{v_3\}, \{v_2, v_3\}\}$.Then $A = \{v_2, v_3\}$ is dense since $Cl_{DGN_1}(\{v_2, v_3\}) = V$.
Definition 3.28: Let $D = (V, E)$ be a digrph and (V, τ_{DGN_1}) be a DGN_1 −topological graph A_i and A_j are called DGN_1 −separated in (V, τ_{DGN_1}) if either $Cl_{DGN_1}(A_i) \cap A_j = \emptyset$ and $A_i \cap Cl_{DGN_1}(A_j) = \emptyset$ or $(Cl_{DGN_1}(A_i) \cap A_j) \cup (A_i \cap Cl_{DGN_1}(A_j)) = \emptyset$.
Example 3.29: Consider the digrph $D = (V, E)$, in figure (3.7) where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{v_1v_2, v_1v_3, v_1v_4\}$
The topology associated to above digrph is:
$\tau_{DGN_1} = \{\emptyset, V, \{v_2\}, \{v_3\}, \{v_4\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}, , \{v_2, v_3, v_4\}\}$.

Then $A_1 = \{v_2\}$ and $A_2 = \{v_4\}$ are DGN_1 −separated in (V, τ_{DGN_1}) since $Cl_{DGN_1}(\{v_2\}) \cap \{v_4\} = \emptyset$ and $\{v_2\} \cap Cl_{DGN_1}(\{v_4\}) = \emptyset$

Definition 3.30: Let $D = (V, E)$ be a digrph and (V, τ_{DGN_1}) be a DGN_1 −topological graph. The point $v_j \in V$ is called DGN_1 −limit point of a subset A of V if for every DGN_1 − open set U containing v_j, then $(U - \{v_j\}) \cap A = \emptyset$. The set of all DGN_1 −limit point of A is denoted by $Lim_{DGN_1}(A)$.

Example 3.31: Consider the following digrph $D = (V, E)$ in figure (3.8) where $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{v_2v_5, v_3v_2, v_3v_4, v_3v_5, v_4v_2, v_4v_3, v_4v_5, v_5v_2\}$.





Figure(3.7) Figure(3.8)

The topology associated to above digrph is
$\tau_{DGN_1} = \{\emptyset, V, \{v_1\}, \{v_2, v_5\}, \{v_1, v_2, v_5\}, \{v_2, v_3, v_4, v_5\}\}$. Then $Lim_{DGN_1}(\{v_2, v_3\}) = \{v_3, v_4, v_5\}$.

Theorem 3.32: Let (V, τ_{DGN_1}) be a DGN_1 −topological graph associated with the digrph $D = (V, E)$ and let $A \subseteq V$. Then $A \cup Lim_{DGN_1}(A)$ is DGN_1 − closed set. Proof: Let $v_j \in A \cup Lim_{DGN_1}(A)$ and $u_i \in (A \cup Lim_{DGN_1}(A))^c$. To prove $u_iv_j \notin E$, since $u_i \in (A \cup Lim_{DGN_1}(A))^c$ then $u_i \notin A \cup Lim_{DGN_1}(A)$, so $u_i \notin A \wedge u_i \notin Lim_{DGN_1}(A)$. Then there exist a DGN_1 − open set U containing u_i, such that $(U - \{u_i\}) \cap A = \emptyset$. Then we get a DGN_1 − open set U containing u_i but not v_j and $u_iv_j \notin E$. Hence $A \cup Lim_{DGN_1}(A)$ is DGN_1 − closed set. ■

Theorem 3.33: Let (V, τ_{DGN_1}) be a DGN_1 −topological graph associated with the digrph $D = (V, E)$ and let $A \subseteq V$. Then $Lim_{DGN_1}(A) \subseteq A$ if and only if A is DGN_1 − closed set.Proof: (⇒)Suppose that $Lim_{DGN_1}(A) \subseteq A$ and to prove that A is DGN_1 − closed set, let $v_j \in A$ and $u_i \in A^c$. since $u_i \in A^c$ then $u_i \notin A$, then $u_i \notin Lim_{DGN_1}(A)$ since $Lim_{DGN_1}(A) \subseteq A$. Then there exist a DGN_1 − open set U containing u_i, such that $(U - \{u_i\}) \cap A = \emptyset$. Then $u_iv_j \notin E$. Hence A is DGN_1 − closed set.(⇐)Now assume that A is DGN_1 − closed.To prove $Lim_{DGN_1}(A) \subseteq A$, let $u_i \notin A$. Then $u_i \in A^c$. Since A^c is DGN_1 − open set containing u_i and $A^c \cap A = \emptyset$ then $u_i \notin Lim_{DGN_1}(A)$. Hence $Lim_{DGN_1}(A) \subseteq A$. ■

Definition 3.34: Let $D = (V, E)$ be a digrph and (V, τ_{DGN_1}) be a DGN_1 −topological graph. The DGN_1 −interior of a subset A of V is the union of all DGN_1 − open subset of V contained in A, and denoted by $Int_{DGN_1}(A)$. i.e $Int_{DGN_1}(A) = \cup\{U: U \text{ is } DGN_1 - \text{open}, U \subseteq A \subseteq V\}$.Example 3.35: Consider the digrph $D = (V, E)$ in figure (3.9) where $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{v_1v_3, v_2v_1, v_2v_3, v_2v_4, v_2v_5, v_4v_3, v_4v_5, v_5v_3\}$.
The topology associated to above digrph is:
$\tau_{DGN_1} = \{\emptyset, V, \{v_3\}, \{v_1, v_3\}, \{v_3, v_5\}, \{v_1, v_3, v_5\}, \{v_3, v_4, v_5\}, \{v_1, v_3, v_4, v_5\}\}$.
Then$Int_{DGN_1}(\{v_1, v_3, v_4\}) = \{v_1, v_3\}$.

Remark 3.36: $Int_{DGN_1}(A)$ is the set of all vertices which are reachable to $V - A$.

Example 3.37: Consider the above digrph $D = (V, E)$ in figure (3.9).Let $A = \{v_2, v_3, v_4\}$. Then $Int_{DGN_1}(A) = Int_{DGN_1}(\{v_2, v_3, v_4\})$ is the set of all vertices which are reachable to $\{v_1, v_5\}$ is $\{v_3\}$.

Remark 3.38: Let (V, τ_{DGN_1}) be a DGN_1 −topological graph associated with the digrph $D = (V, E)$ and let $A, B \subseteq V$. Then:

i) The DGN_1 −interior set is always DGN_1 − open set.

ii) $Int_{DGN_1}(A) = V - Cl_{DGN_1}(A^c)$.

iii) $\mathrm{Int}_{\mathrm{DGN}_1}(\mathrm{A})$ is the largest DGN_1- open set contained in A.
iv) $\mathrm{Int}_{\mathrm{DGN}_1}(\mathrm{A}) = \mathrm{A}$ if and only if A is DGN_1- open set.
v) $\mathrm{Int}_{\mathrm{DGN}_1}\left(\mathrm{Int}_{\mathrm{DGN}_1}(\mathrm{A})\right) = \mathrm{Int}_{\mathrm{DGN}_1}(\mathrm{A})$.
vi) $\mathrm{Int}_{\mathrm{DGN}_1}(\mathrm{A} \cap \mathrm{B}) = \mathrm{Int}_{\mathrm{DGN}_1}(\mathrm{A}) \cap \mathrm{Int}_{\mathrm{DGN}_1}(\mathrm{B})$".

Theorem 3.39: Let $\left(\mathrm{V}, \tau_{\mathrm{DGN}_1}\right)$ be a DGN_1-topological graph associated with the digrph $\mathrm{D} = (\mathrm{V}, \mathrm{E})$. Then $(\mathrm{Int}_{\mathrm{DGN}_1}(\mathrm{A}))^{\mathrm{c}} = \mathrm{Cl}_{\mathrm{DGN}_1}(\mathrm{A}^{\mathrm{c}})$.

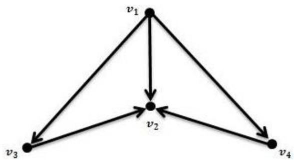
Proof: It is clear from Remark (3.38(ii)). ■

4 - On DGN_1-Separation Axioms

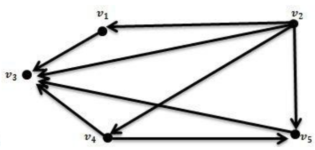
In this section, we introduce the following two DGN_1-Separation Axioms $\mathrm{DGN}_1-\tau_0$ space and $\mathrm{DGN}_1-\tau_1$ space and we prove some theorems of a digrph min terms of DGN_1-Separation Axioms satisfied by the DGN_1-topological graph which is determined by that digrph.

Definition 4.1: Let $\mathrm{D} = (\mathrm{V}, \mathrm{E})$ be a digrph and $\left(\mathrm{V}, \tau_{\mathrm{DGN}_1}\right)$ an DGN_1-topological graph. then $\left(\mathrm{V}, \tau_{\mathrm{DGN}_1}\right)$ is a $\mathrm{DGN}_1-\tau_0$ space if for every $\mathrm{u_i}, \mathrm{u_j} \in \mathrm{V}, \mathrm{u_i} \neq \mathrm{u_j}$ there exists a DGN_1-open set U such that either $\mathrm{u_i} \in \mathrm{U}$ or $\mathrm{u_j} \in \mathrm{U}$.

Example 4.2: Consider the digrph $\mathrm{D} = (\mathrm{V}, \mathrm{E})$ in figure (4.1) where $\mathrm{V} = \{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, \mathrm{v}_4\}$ and $\mathrm{E} = \{\mathrm{v}_1\mathrm{v}_2, \mathrm{v}_1\mathrm{v}_3, \mathrm{v}_1\mathrm{v}_4, \mathrm{v}_3\mathrm{v}_2, \mathrm{v}_4\mathrm{v}_2\}$.



Figure(3.9)



Figure(3.10)

Then the topology associated to above digrph is $\tau_{\mathrm{DGN}_1} = \{\emptyset, \mathrm{V}, \{\mathrm{v}_2\}, \{\mathrm{v}_2, \mathrm{v}_3\}, \{\mathrm{v}_2, \mathrm{v}_4\}, \{\mathrm{v}_2, \mathrm{v}_3, \mathrm{v}_4\}\}$. Since for each two different vertices, there exists a DGN_1-open set containing one of them and does not contain the other. Then $\left(\mathrm{V}, \tau_{\mathrm{DGN}_1}\right)$ is a $\mathrm{DGN}_1-\tau_0$ space.

Definition 4.3: Let $\mathrm{D} = (\mathrm{V}, \mathrm{E})$ be a digrph and $\left(\mathrm{V}, \tau_{\mathrm{DGN}_1}\right)$ an DGN_1-topological graph. then $\left(\mathrm{V}, \tau_{\mathrm{DGN}_1}\right)$ is a $\mathrm{DGN}_1-\tau_1$ space if for any $\mathrm{u_i}, \mathrm{u_j} \in \mathrm{V}, \mathrm{u_i} \neq \mathrm{u_j}$ there exists two DGN_1-open sets U and H such that $(\mathrm{u_i} \in \mathrm{U}, \mathrm{u_j} \notin \mathrm{U})$ and $(\mathrm{u_i} \notin \mathrm{H}, \mathrm{u_j} \in \mathrm{H})$.

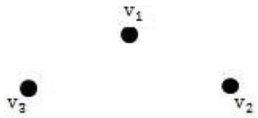
Remark 4.4: A DGN_1-topological graph $\left(\mathrm{V}, \tau_{\mathrm{DGN}_1}\right)$ is a $\mathrm{DGN}_1-\tau_1$ space if each set which consists of a single vertex is DGN_1-closed set.

Example 4.5: Consider the null digrph $\mathrm{D} = (\mathrm{V}, \mathrm{E})$ in figure (4.2) where $\mathrm{V} = \{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3\}$

Then the topology associated to above digrph is $\tau_{\mathrm{DGN}_1} = \left\{\emptyset, \mathrm{V}, \{\mathrm{v}_1\}, \{\mathrm{v}_2\}, \{\mathrm{v}_3\}, \{\mathrm{v}_1, \mathrm{v}_2\}, \{\mathrm{v}_1, \mathrm{v}_3\}, \{\mathrm{v}_2, \mathrm{v}_3\}\right\}$. Since each set which consists of a single vertex is DGN_1-closed set. Then $\left(\mathrm{V}, \tau_{\mathrm{DGN}_1}\right)$ is a $\mathrm{DGN}_1-\tau_1$ space.

Proposition 4.6: Let $\left(\mathrm{V}, \tau_{\mathrm{DGN}_1}\right)$ be a DGN_1-topological graph associated with the digrph $\mathrm{D} = (\mathrm{V}, \mathrm{E})$.Then every $\mathrm{DGN}_1-\tau_1$ space is $\mathrm{DGN}_1-\tau_0$ space. Proof: Let $\left(\mathrm{V}, \tau_{\mathrm{DGN}_1}\right)$ be a $\mathrm{DGN}_1-\tau_1$ space and we will prove that $\left(\mathrm{V}, \tau_{\mathrm{DGN}_1}\right)$ is a $\mathrm{DGN}_1-\tau_0$ space. Let $\mathrm{v}_1, \mathrm{v}_2 \in \mathrm{V}$ such that $\mathrm{v}_1 \neq \mathrm{v}_2$, since $\left(\mathrm{V}, \tau_{\mathrm{DGN}_1}\right)$ is a $\mathrm{DGN}_1-\tau_1$ space, then there exists two DGN_1-open sets U and H such that $(\mathrm{v}_1 \in \mathrm{U}, \mathrm{v}_2 \notin \mathrm{U})$ and $(\mathrm{v}_1 \notin \mathrm{H}, \mathrm{v}_2 \in \mathrm{H})$. That means there exists a DGN_1-open set containing one of two vertices and does not contains the other vertex. Hence $\left(\mathrm{V}, \tau_{\mathrm{DGN}_1}\right)$ is a $\mathrm{DGN}_1-\tau_0$ space. ■

Remark 4.7: The converse of the above proposition is not true in general from the following example. Example 4.8: Consider the digrph $\mathrm{D} = (\mathrm{V}, \mathrm{E})$ in figure (4.3) where $\mathrm{V} = \{\mathrm{v}_1, \mathrm{v}_2\}$ and $\mathrm{E} = \{\mathrm{v}_1\mathrm{v}_2\}$



Figure(4.2) Figure(4.3)

Then the topology associated to above digrph is $\tau_{DGN_1} = \{\emptyset, V, \{v_2\}\}$. We note that (V, τ_{DGN_1}) is a $DGN_1 - \tau_0$ space but not $DGN_1 - \tau_1$ space.

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