Stability Analysis Of Three Species Ecological Commensalism

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Abstract

The present paper deals with an investigation on three species ecological commensalism. Here, all the three species are having limited resources quantized by the respective carrying capacities. The mathematical model equations constitute a set of three first order non-linear simultaneous coupled differential equations in the strengths N_1, N_2, N_3 of S_1, S_2, S_3 respectively. All possible equilibrium points of the model are identified. The system would be stable, if all the characteristic roots are negative, in case they are real and have negative real parts, in case they are complex. Criteria for the asymptotic stability of all eight equilibrium points is established and also the global stability is discussed by using suitable Liapunov's function. Further the numerical solutions for the growth rate equations are computed using Runga Kutta fourth order method.

Keywords: Asymptotically stable, Characteristic equation, Equilibrium State, Liapunov's function, Stable.

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1. Introduction

Ecology is the study of the interactions between organisms and their environment. The organisms include animals and plants, the environment includes the surroundings of animals. The study of living things (plants and animals) in connection to their environments and habits is known as ecology. This discipline of knowledge is a branch of evolutionary biology purported to explain how or to what extent the living beings are regulated in nature. Allied to the problem of population regulation is the problem of species distribution- prey-predator, competition and so on. The subject of ecology can be broadly sub-divided as auto-ecology (the study of single species populations) and synecology (the study of two or more communities). Synecological studies lead to the concept of the eco-system. This concept is a direct outcome of the intensive work of several life scientists/biologists and botanists of many generations. An eco-system may be considered as a unit that includes animals, plants and the physical environment in which these live. This area of knowledge seeks to explain how many different kinds of plants and animals can live together in the same place for many generations. Animals and plants share the same habitat. Sometimes they can only share for so long before some locally go extinct, but there are other circumstances when many different kinds persist in a habitat indefinitely. As such, ecology may also be referred as the study of distribution and abundance of species under habitat availing the same resources. The

Ecological interactions can be broadly classified as Ammensalism, Competition, Commensalism, Neutralism, Mutualism, Predation and so on. Significant researches in the area of theoretical ecology have been discussed by Gillman [3] and by Kot [4]. Several ecologists and mathematicians contributed to the growth of this area of knowledge. Mathematical ecology can be broadly divided into two main sub-divisions, Autecology and Synecology, which are described in the treatises of Anna Sher [1], Arumugam [2] and Sharma [21].

Mathematical Modeling plays a key role in providing insight into the mutual relationships (positive, negative) between the interacting species. Several authors Ma [6], Moghadas [7], Murray [8] and Sze-Bi Hsu [23] were introduced the general concepts of Modeling in Biological Science. Srinivas [22] studied the competitive ecosystem of two species and three species with limited and unlimited resources. Later, Narayan [9] studied prey-predator ecological models with partial cover for the prey and alternate food for the predator. Further, Kumar [5] studied some mathematical models of ecological commensalism. The present author Prasad [10-20] investigated continuous and discrete models on two, three and four species syn-ecosystems.

Notation

$N_i(t)$	: The population strength of S_i at time t, $i=1,2,3$
t	: Time instant
a_i	: Natural growth rate of S_i, $i=1,2,3$
a_{ii}	: Self inhibition coefficients of S_i, $i=1,2,3$
a_{12}	: Interaction coefficients of S_1 due to S_2
a_{13}	: Interaction coefficients of S_1 due to S_3
a_{23}	: Interaction coefficients of S_2 due to S_3
$k_i = \dfrac{a_i}{a_{ii}}$	: Carrying capacity of S_i, $i=1,2,3$

Further the variables N_1, N_2, N_3 are non-negative and the model parameters $a_1, a_2, a_3, a_{11}, a_{22}$, a_{12}, a_{13}, a_{23} are assumed to be non-negative constants.

2. Basic Equations :

The model equations for syn ecosystem is given by the following system of first order non-linear ordinary differential equations.

$$\frac{dN_1}{dt} = a_1\, N_1 - a_{11}\, N_1^2 + a_{12}\, N_1\, N_2 \,. \tag{1}$$

$$\frac{dN_2}{dt} = a_2\, N_2 - a_{22}\, N_2^2 \,. \tag{2}$$

$$\frac{dN_3}{dt} = a_3\, N_3 - a_{33}\, N_3^2 + a_{13}\, N_1\, N_3 + a_{23}\, N_2\, N_3 \,. \tag{3}$$

3. Equilibrium States:

The system under investigation has eight equilibrium states given by

$$\frac{dN_i}{dt} = 0 . \text{ i= 1,2,3.} \tag{4}$$

(i)Fully washed out state .

$E_1 : \bar{N}_1 = 0 , \bar{N}_2 = 0 , \bar{N}_3 = 0.$

(ii)States in which only one of the three species is survives while the other two are not .

$E_2 : \bar{N}_1 = k_1 , \bar{N}_2 = 0 , \bar{N}_3 = 0.$

$E_3 : \bar{N}_1 = 0 , \bar{N}_2 = k_2 , \bar{N}_3 = 0.$

$E_4 : \bar{N}_1 = 0 , \bar{N}_2 = 0 , \bar{N}_3 = k_3.$

(iii)States in which only two of the three species are survives while the other one is not .

$E_5 : \bar{N}_1 = k_1 + \frac{k_2\, a_{12}}{a_{11}} , \bar{N}_2 = k_2 , \bar{N}_3 = 0.$

$E_6 : \bar{N}_1 = k_1 , \bar{N}_2 = 0 , \bar{N}_3 = k_3 + \frac{k_1\, a_{13}}{a_{33}} .$

$E_7 : \bar{N}_1 = 0 , \bar{N}_2 = k_2 , \bar{N}_3 = k_3 + \frac{k_2\, a_{23}}{a_{33}} .$

(iv)The co-existent state (or) normal steady state.

$E_8 : \bar{N}_1 = k_1 + \frac{k_2\, a_{12}}{a_{11}}, \bar{N}_2 = k_2, \bar{N}_3 = \frac{k_1\, a_{13}}{a_{33}} + \left(\frac{a_{13}\, a_{12}}{a_{11}} + a_{23}\right)\frac{k_2}{a_{33}} + k_3 .$

4. Stability Analysis of the Equilibrium States:

Let $N = (N_1 , N_2 , N_3) = \bar{N} + U .$

Where $U = (u_1 , u_2 , u_3)^T$ is a small perturbation over the equilibrium state

$\bar{N} = (\bar{N}_1 , \bar{N}_2 , \bar{N}_3) .$

The basic equations are quasi-linearized to obtain the equations for the perturbed state as,

$$\text{i.e} \quad \frac{dU}{dt} = SU \tag{5}$$

where

$$S = \begin{bmatrix} a_1 - 2a_{11}\bar{N}_1 + a_{12}\bar{N}_2 & a_{12}\bar{N}_1 & 0 \\ 0 & a_2 - 2a_{22}\bar{N}_2 & 0 \\ a_{13}\bar{N}_3 & a_{23}\bar{N}_3 & a_3 - 2a_{33}\bar{N}_3 + a_{13}\bar{N}_1 + a_{23}\bar{N}_2 \end{bmatrix} \tag{6}$$

The characteristic equation for the system is $det\,[S - \lambda I] = 0.$ (7)
The equilibrium state is stable, if all the roots of the equation (6) are negative in case they are real or have negative real parts, in case they are complex.

4.1. Stability of fully washed out state:

In this case, we have

$$S = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \tag{8}$$

The characteristic equation is $(\lambda - a_1)(\lambda - a_2)(\lambda - a_3) = 0$.

The characteristic roots of above equation are a_1, a_2, a_3 .Since all the three roots are positive. Hence, the fully washed out state is unstable and the solution of the above equation are

$$u_1 = u_{10}\, e^{a_1 t} \ ; \ u_2 = u_{20}\, e^{a_2 t} \ ; \ u_3 = u_{30}\, e^{a_3 t} \ . \tag{9}$$

Where u_{10}, u_{20}, u_{30} are the initial values of u_1, u_2, u_3 respectively.

Trajectories of perturbations:

The trajectories in $u_1 - u_2$ and $u_2 - u_3$ planes are $\left[\frac{u_1}{u_{10}}\right]^{\frac{1}{a_1}} = \left[\frac{u_2}{u_{20}}\right]^{\frac{1}{a_2}} = \left[\frac{u_3}{u_{30}}\right]^{\frac{1}{a_3}}$. (10)

4.2. Equilibrium state E_2 : $\bar{N}_1 = k_1$, $\bar{N}_2 = 0$, $\bar{N}_3 = 0$.

In this state, we have

$$S = \begin{bmatrix} -a_1 & a_{12}\, k_1 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 + a_{13}\, k_1 \end{bmatrix} \tag{11}$$

Here the characteristic roots are $-a_1$, a_2 and $a_3 + a_{13}\, k_1$. Since two of these three are positive, hence the state is unstable and the solutions are ,

$$u_1 = (u_{10} - \phi_1)e^{-a_1 t} + \phi_1\, e^{a_2 t}$$

$$u_2 = u_{20}\, e^{a_2 t} \text{ and } u_3 = u_{30} e^{(a_3 + a_{13} k_1)t} \ . \tag{12}$$

Where $\phi_1 = \frac{a_{12} k_1 u_{20}}{a_1 + a_2} > 0$

Trajectories of perturbation

The trajectories in $u_1 - u_2$ and $u_2 - u_3$ planes are

$$u_1 = (u_{10} - \phi_1)\left[\frac{u_2}{u_{20}}\right]^{\frac{-a_1}{a_2}} + \phi_1 \left[\frac{u_2}{u_{20}}\right] \text{ and } \left[\frac{u_2}{u_{20}}\right]^{\frac{1}{a_2}} = \left[\frac{u_3}{u_{30}}\right]^{\frac{1}{a_3 + a_{13} k_1}} . \tag{13}$$

4.3. Equilibrium state E_3 : $\bar{N}_1 = 0$, $\bar{N}_2 = k_2$, $\bar{N}_3 = 0$.

After linearization, we get

$$S = \begin{bmatrix} a_1 + a_{12} k_2 & 0 & 0 \\ 0 & -a_2 & 0 \\ 0 & 0 & a_3 + a_{23} k_2 \end{bmatrix} \tag{14}$$

Here the characteristic roots are $a_1 + a_{12} k_2$, $-a_2$ and $a_3 + a_{23} k_2$. Since two of these three are positive, hence the state is unstable and the solutions are,

$$u_1 = u_{10}\, e^{(a_1 + a_{12} k_2)t} \ , \quad u_2 = u_{20} e^{-a_2 t} \text{ and } u_3 = u_{30} e^{(a_3 + a_{23} k_2)t} \ . \tag{15}$$

Trajectories of perturbations:

The trajectories in $u_1 - u_2$ and $u_2 - u_3$ planes are

$$\left[\frac{u_1}{u_{10}}\right]^{\frac{1}{a_1 + a_{12} k_2}} = \left[\frac{u_2}{u_{20}}\right]^{\frac{-1}{a_2}} \text{ and } \left[\frac{u_2}{u_{20}}\right]^{\frac{-1}{a_2}} = \left[\frac{u_3}{u_{30}}\right]^{\frac{1}{a_3 + a_{23} k_2}} \tag{16}$$

4.4. Equilibrium state E_4 : $\bar{N}_1 = 0$, $\bar{N}_2 = 0$, $\bar{N}_3 = k_3$.

In this state, after linearization

$$S = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ a_{13}k_3 & a_{23}k_3 & -a_3 \end{bmatrix} \tag{17}$$

Here the characteristic roots are a_1 , a_2 and $-a_3$. Since two of these three are positive, hence the state is unstable and the solutions are,

$u_1 = u_{10}\, e^{a_1 t}$, $u_2 = u_{20}\, e^{a_2 t}$ and

$$u_3 = (u_{30} - \phi_2 - \phi_3)e^{-a_3 t} + \phi_2 e^{a_1 t} + \phi_3 e^{a_2 t} \tag{18}$$

Where $\phi_2 = \frac{a_{13}k_3u_{10}}{a_1+a_3} > 0$ and $\phi_3 = \frac{a_{23}k_3u_{20}}{a_2+a_3} > 0$.

Trajectories of perturbations:

The trajectories in $u_1 - u_2$ and $u_2 - u_3$ planes are

$\left[\frac{u_1}{u_{10}}\right]^{\frac{1}{a_1}} = \left[\frac{u_2}{u_{20}}\right]^{\frac{1}{a_2}}$ and

$$u_3 = (u_{30} - \phi_2 - \phi_3)\left[\frac{u_2}{u_{20}}\right]^{\frac{-a_3}{a_2}} + \phi_2 \left[\frac{u_2}{u_{20}}\right]^{\frac{a_1}{a_2}} + \phi_3 \left[\frac{u_2}{u_{20}}\right] . \tag{19}$$

4.5. Equilibrium state E_5 : $\bar{N}_1 = k_1 + \frac{k_2\, a_{12}}{a_{11}}$, $\bar{N}_2 = k_2$, $\bar{N}_3 = 0$.

After linearization, we have

$$S = \begin{bmatrix} -\tau & a_{12}\left(k_1 + \frac{k_2 a_{12}}{a_{11}}\right) & 0 \\ 0 & -a_2 & 0 \\ 0 & 0 & a_3 + a_{23}k_2 + \frac{a_{13}\tau}{a_{11}} \end{bmatrix} \tag{20}$$

Where $\tau = a_1 + a_{12}k_2 > 0$

Here the characteristic roots are $-\tau$, $-a_2$ and $a_3 + a_{23}k_2 + \frac{a_{13}\tau}{a_{11}}$. Since two of these three are negative, hence the state is unstable and the solutions are ,

$u_1 = (u_{10} - \phi_4)e^{-\tau t} + \phi_4 e^{-a_2 t}$.

$$u_2 = u_{20}e^{-a_2 t} \quad \text{and } u_3 = u_{30}e^{\phi_5 t} \tag{21}$$

where $\phi_4 = \frac{a_{12}u_{20}\tau}{a_{11}(\tau - a_2)}$, $\tau \neq a_2$ and

$\phi_5 = \frac{a_3a_{11}a_{22} + a_2(a_{13}a_{12} + a_{23}a_{11}) + a_1a_{13}a_{22}}{a_{11}a_{22}} > 0.$

Trajectories of perturbations:

The trajectories in $u_1 - u_2$ and $u_2 - u_3$ planes are

$$u_1 = (u_{10} - \phi_4)\left[\frac{u_2}{u_{20}}\right]^{\frac{\tau}{a_2}} + \phi_4\left[\frac{u_2}{u_{20}}\right] \text{ and } \left[\frac{u_2}{u_{20}}\right]^{\frac{-1}{a_2}} = \left[\frac{u_3}{u_{30}}\right]^{\frac{1}{\phi_5}}. \tag{22}$$

4.6. Equilibrium state E_6 : $\bar{N}_1 = k_1$, $\bar{N}_2 = 0$, $\bar{N}_3 = k_3 + \frac{k_1 a_{13}}{a_{33}}$.

Here,

$$S = \begin{bmatrix} -a_1 & a_{12}k_1 & 0 \\ 0 & a_2 & 0 \\ a_{13}\left(k_3 + \frac{k_1 a_{13}}{a_{33}}\right) & a_{23}\left(k_3 + \frac{k_1 a_{13}}{a_{33}}\right) & -\tau_1 \end{bmatrix} \tag{23}$$

Where $\tau_1 = a_3 + a_{13}k_1 > 0$

Here the characteristic roots are $-a_1$, a_2 and $-\tau_1$. Since two of these three are negative, hence the state is unstable. The equation yield the solutions,

$u_1 = (u_{10} - \phi_6)e^{-a_1 t} + \phi_6 e^{a_2 t}$, $u_2 = u_{20}\, e^{a_2 t}$

and $u_3 = (u_{30} - R_1 - R_2)e^{-\tau_1 t} + R_1 e^{-a_1 t} + R_2 e^{a_2 t}$ (24)

where $\phi_6 = \frac{a_{12}k_1 u_{20}}{a_1 + a_2} > 0$ 0

and $R_1 = \frac{a_{13}\tau_1(u_{10} - \phi_6)}{a_{33}(\tau_1 - a_1)} > 0$; $\tau_1 \neq a_1$, $R_2 = \frac{a_{13}\tau_1\phi_6 + a_{23}\tau_1 u_{20}}{a_{33}(\tau_1 + a_2)} > 0$.

Trajectories of perturbations:

The trajectories in $u_1 - u_2$ and $u_2 - u_3$ planes are

$$u_1 = (u_{10} - \phi_6)\left[\frac{u_2}{u_{20}}\right]^{\frac{-a_1}{a_2}} + \phi_6\left[\frac{u_2}{u_{20}}\right] \quad \text{and}$$

$$u_3 = (u_{30} - R_1 - R_2)\left[\frac{u_2}{u_{20}}\right]^{\frac{-\tau_1}{a_2}} + R_1\left[\frac{u_2}{u_{20}}\right]^{\frac{-a_1}{a_2}} + R_2\left[\frac{u_2}{u_{20}}\right]. \tag{25}$$

4.7. Equilibrium state E_7 : $\bar{N}_1 = 0$, $\bar{N}_2 = k_2$, $\bar{N}_3 = k_3 + \frac{k_2 a_{23}}{a_{33}}$.

In this case, we have

$$S = \begin{bmatrix} a_1 + a_{12}k_2 & 0 & 0 \\ 0 & -a_2 & 0 \\ a_{13}\left(k_3 + \frac{k_2 a_{23}}{a_{33}}\right) & a_{23}\left(k_3 + \frac{k_2 a_{23}}{a_{33}}\right) & -\tau_2 \end{bmatrix} \tag{26}$$

Where $\tau_2 = a_3 + a_{23}k_2 > 0$.

Here the characteristic roots are $a_1 + a_{12}k_2$, $-a_2$ and $-\tau_2$. Since two of these three are negative , hence the state is unstable and the solutions are , $u_1 = u_{10}e^{(a_1 + a_{12}k_2)t}$, $u_2 = u_{20}e^{-a_2 t}$ and

$u_3 = (u_{30} - R_3 - R_4)e^{-\tau_2 t} + R_3 e^{(a_1 + a_{12}k_2)t} + R_4 e^{-a_2 t}$ (27)

Where $R_3 = \frac{a_{13}\tau_2 u_{10}}{a_{33}(a_1 + a_{12}k_2 + \tau_2)} > 0$ and $R_4 = \frac{a_{23}\tau_2 u_{20}}{a_{33}(\tau_2 - a_2)} > 0$, $\tau_2 \neq a_2$

Trajectories of perturbations:

The trajectories in $u_1 - u_2$ and $u_2 - u_3$ planes are

$\left[\frac{u_1}{u_{10}}\right]^{\frac{1}{a_1+a_{12}k_2}} = \left[\frac{u_2}{u_{20}}\right]^{\frac{-1}{a_2}}$ and

$$u_3 = (u_{30} - R_3 - R_4)\left[\frac{u_2}{u_{20}}\right]^{\frac{\tau_2}{a_2}} + R_3\left[\frac{u_2}{u_{20}}\right]^{\frac{-(a_1+a_{12}k_2)}{a_2}} + R_4\left[\frac{u_2}{u_{20}}\right]. \tag{28}$$

4.8. Normal Steady State

$$S = \begin{bmatrix} -\tau_3 & a_{12}\overline{N}_1 & 0 \\ 0 & -a_2 & 0 \\ a_{13}\overline{N}_3 & a_{23}\overline{N}_3 & -\left[a_3 + a_{13}k_1 + \left(\frac{a_{13}a_{12}}{a_{11}} + a_{23}\right)k_2\right] \end{bmatrix} \tag{29}$$

Where $\tau_3 = a_1 + a_{12}k_2 > 0$.

Here the characteristic roots are $-\tau_3$, $-a_2$ and $-\left[a_3 + a_{13}k_1 + \left(\frac{a_{13}a_{12}}{a_{11}} + a_{23}\right)k_2\right]$. these are all negative, hence the state is **stable**. The equations yield the solutions as,

$$u_1 = (u_{10} - R_5)e^{-\tau_3 t} + R_5 e^{-a_2 t}$$

$$u_2 = u_{20}e^{-a_2 t} \text{ and } u_3 = (u_{30} - R_6 + R_7)e^{\psi t} + R_6 e^{-\tau_3 t} - R_7 e^{-a_2 t} \tag{30}$$

where $R_5 = \frac{a_{12}\tau_3 u_{20}}{a_{11}(\tau_3 - a_2)} > 0$, $\tau_3 \neq a_2$ and $\tau_3 = a_1 + a_{12}k_2$

$$\psi = \frac{a_3 a_{11} a_{22} + a_2(a_{13}a_{12} + a_{23}a_{11}) + a_1 a_{13} a_{22} - 2a_{33}a_{22}a_{11}\mu}{a_{11}a_{22}} > 0,$$

$$\mu = \frac{k_1\, a_{13}}{a_{33}} + \left(\frac{a_{13}\, a_{12}}{a_{11}} + a_{23}\right)\frac{k_2}{a_{33}} + k_3\, ,\ R_6 = \frac{a_{13}\mu(R_5 - u_{10})}{\tau_3 + \psi} > 0$$

and $R_7 = \frac{R_5 + a_{23}\mu u_{20}}{a_2 + \psi} > 0$

Trajectories of perturbations:

The trajectories in $u_1 - u_2$ and $u_2 - u_3$ planes are

$$u_1 = (u_{10} - R_5)\left[\frac{u_2}{u_{20}}\right]^{\frac{\tau_3}{a_2}} + R_5\left[\frac{u_2}{u_{20}}\right] \text{ and}$$

$$u_3 = (u_{30} - R_6 + R_7)\left[\frac{u_2}{u_{20}}\right]^{\frac{-\psi}{a_2}} + R_6\left[\frac{u_2}{u_{20}}\right]^{\frac{\tau_3}{a_2}} - R_7\left[\frac{u_2}{u_{20}}\right]. \tag{31}$$

5. Liapunov's function for global stability :

In section 4, we discussed the local stability of all eight equilibrium states. From which only the normal steady state is **stable** and rest of them are **unstable**. We now examine the global stability of dynamical system (1), (2) and (3) at this state by suitable Liapunov's function.

Theorem : The equilibrium state is E_8 : $(\overline{N}_1, \overline{N}_2, \overline{N}_3)$ globally asymptotically stable .

Proof : Let us consider the following Liapunovs function .

$$L(N_1, N_2, N_3) = N_1 - \overline{N}_1 - \overline{N}_1 \ln\left(\frac{N_1}{\overline{N}_1}\right) + l_1\left[N_2 - \overline{N}_2 - \overline{N}_2 ln\left(\frac{N_2}{\overline{N}_2}\right)\right]$$
$$+ l_2\left[N_3 - \overline{N}_3 - \overline{N}_3\, ln\left(\frac{N_3}{\overline{N}_3}\right)\right]. \tag{32}$$

where l_1 and l_2 are suitable constants to be determined as in the subsequent steps. Now the time derivative of L , along with solutions of (1),(2) and (3) can be written as ,

$$\frac{dL}{dt} = \left(\frac{N_1 - \bar{N}_1}{N_1}\right)\frac{dN_1}{dt} + l_1\left(\frac{N_2 - \bar{N}_2}{N_2}\right)\frac{dN_2}{dt} + l_2\left(\frac{N_3 - \bar{N}_3}{N_3}\right)\frac{dN_3}{dt} \qquad (33)$$

$$\frac{dL}{dt} = -a_{11}[(N_1 - \bar{N}_1)^2] + a_{12}[(N_1 - \bar{N}_1)(N_2 - \bar{N}_2)] + l_1[-a_{22}(N_2 - \bar{N}_2)^2] + l_2\left\{\begin{array}{c}[-a_{33}(N_3 - \bar{N}_3)^2] + a_{13}[(N_1 - \bar{N}_1)(N_3 - \bar{N}_3)] \\ +a_{23}[(N_2 - \bar{N}_2)(N_3 - \bar{N}_3)]\end{array}\right\}$$

$$\begin{aligned}\frac{dL}{dt} = &-\left[\sqrt{a_{11}}(N_1 - \bar{N}_1) + \sqrt{l_1 a_{22}}(N_2 - \bar{N}_2) + \sqrt{l_2 a_{33}}(N_3 - \bar{N}_3)\right]^2 \\ &+\left(2\sqrt{l_1 a_{11} a_{22}} + a_{12}\right)(N_1 - \bar{N}_1)(N_2 - \bar{N}_2) \\ &+\left(2\sqrt{l_2 a_{11} a_{33}} + l_2 a_{13}\right)(N_1 - \bar{N}_1)(N_3 - \bar{N}_3) \\ &+\left(2\sqrt{l_1 l_2 a_{22} a_{33}} + l_2 a_{23}\right)(N_2 - \bar{N}_2)(N_3 - \bar{N}_3) \qquad (34)\end{aligned}$$

The positive constants l_1 and l_2 as so chosen that, the coefficients of $(N_1 - \bar{N}_1)(N_2 - \bar{N}_2)$, $(N_2 - \bar{N}_2)(N_3 - \bar{N}_3)$ and $(N_1 - \bar{N}_1)(N_3 - \bar{N}_3)$ are vanish in (34)

Then we get $l_1 = \frac{a_{12}^2}{4a_{11}a_{12}} > 0$ and $l_2 = \frac{4a_{11}a_{33}}{a_{12}^2} > 0$ with $a_{12}^3 a_{22} = 4a_{11}^2 a_{23}^2$.

$$\frac{dL}{dt} = -\left[\sqrt{a_{11}}(N_1 - \bar{N}_1) + \frac{a_{12}\sqrt{a_{22}}}{2\sqrt{a_{11}a_{12}}}(N_2 - \bar{N}_2) + \frac{2a_{33}\sqrt{a_{11}}}{a_{12}}(N_3 - \bar{N}_3)\right]^2 \qquad (35)$$

Which is negative definite. Hence, the normal steady state is **globally asymptotically stable.**

6. Numerical approach

The numerical solutions of the growth rate equations computed employing the fourth order Runge-Kutta method for specific values of the various parameters that characterize the model and the initial conditions. The results are illustrated in Figures 1 to 4.

Table

Fig. no.	a_1	a_2	a_3	a_{11}	a_{22}	a_{33}	a_{12}	a_{13}	a_{23}	N_1	N_2	N_3	t^*
1	0.84	3.81	22.79	2.64	1.48	32.05	5.6	10	17.08	20	20	20	--
2	4.92	5.6	10.28	1.48	10.28	4.68	3.48	0.92	6.68	2.12	0.44	2.96	0.19
3	0.31	3.01	0.23	5.71	4.01	2.7	5.47	1.08	3.24	1.85	1.47	5.26	0.25
4	1.39	2.57	0.11	3.08	0.51	0.13	4.4	0.44	0.04	3.54	2.31	0.92	0.93 &0.71

By taking above values we have drawn some figures

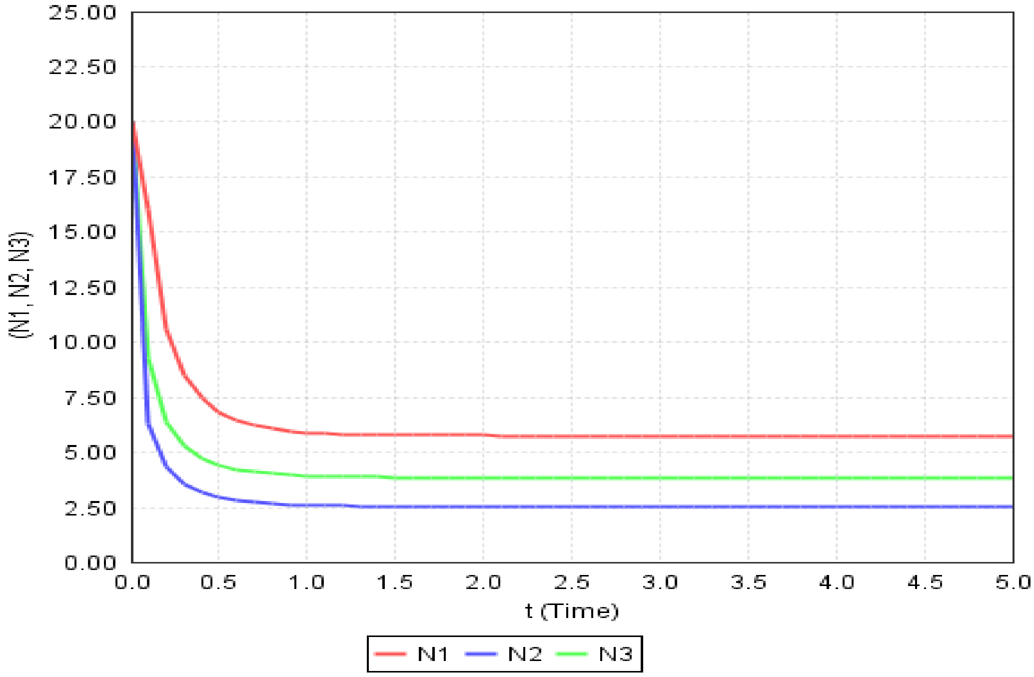

Figure 1

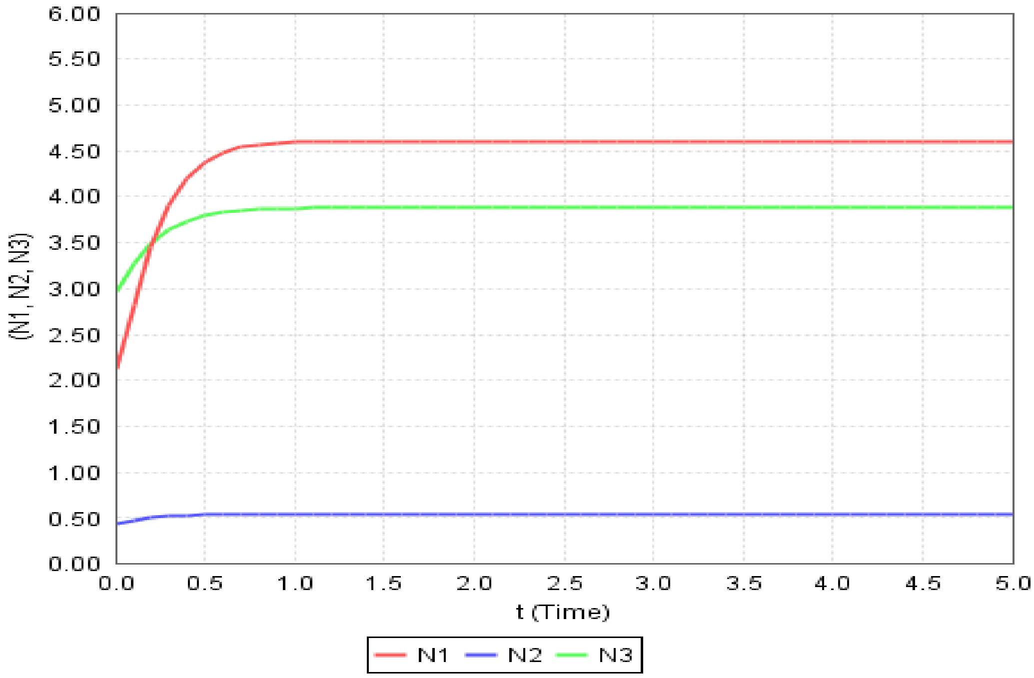

Figure 2

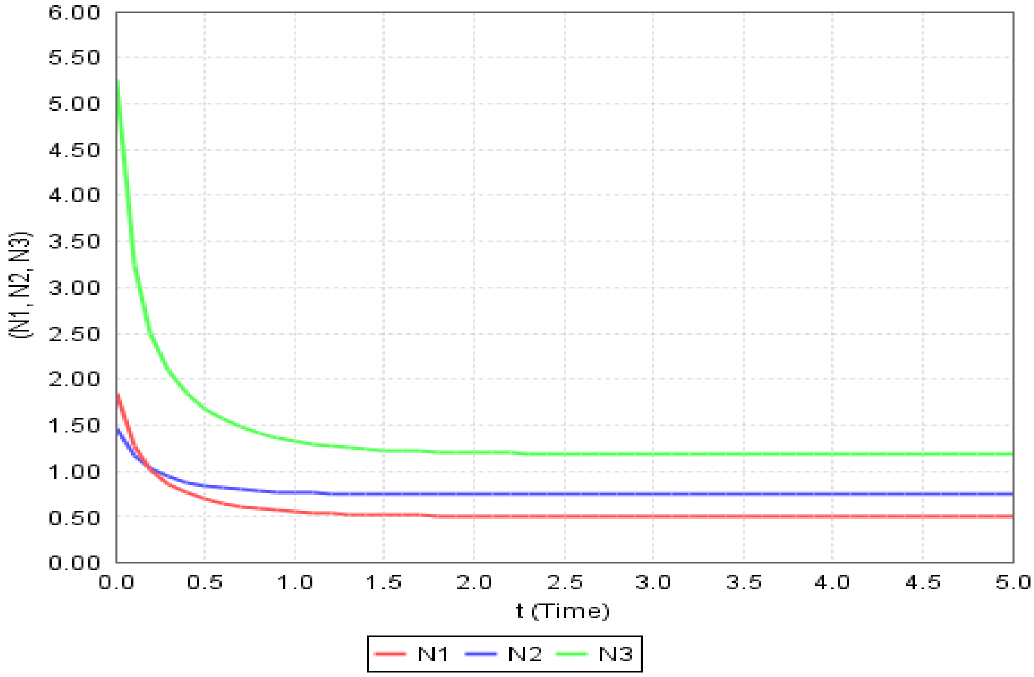

Figure 3

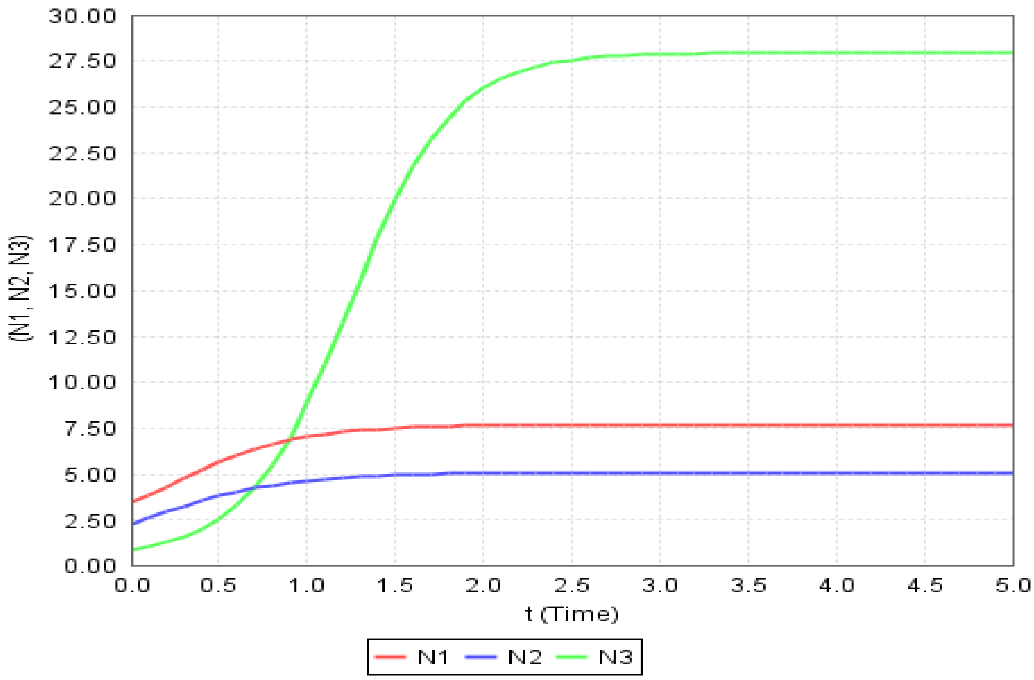

Figure 4

7. Observations

Case 1: In this case the initial conditions of the three species S_1, S_2, S_3 are identical. It is evident that all the three species asymptotically converge to the equilibrium point. Further we notice that the third species has the greatest natural growth rate. This is illustrated in Figure 1.

Case 2: In this case the initial conditions of S_2, S_1, S_3 are in increasing order. The natural growth rate of S_3 and the self inhibition coefficients of S_2 are identical. Further the third species dominates over the first species up to the time instant $t^* = 0.19$ after which the dominance is reversed as shown in Figure 2.

Case 3: In this case the third species has the least natural birth rate and all the three species decrease initially. The first species dominates over the second species initially till the time instant $t^* = 0.25$ and thereafter the dominance is reversed. Further it is evident that all the three species asymptotically converge to the equilibrium point. This is shown in Figure 3.

Case 4: In this case the third species has the least natural growth. This is a situation at the initial conditions of S_1, S_2, S_3 are in decreasing order. Initially the first and second species dominates over the third up to the time $t^* = 0.93$ and $t^* = 0.71$ after which the dominance is reversed. (Figure 4).

8. Conclusion

In this paper, we discussed the stability analysis of three species ecological commensalism. The model equations constitute a set of three first order non-linear coupled differential equations. All possible equilibrium states of the model are identified and the local stability is discussed. It is observed that, in all eight equilibrium states, only the normal state is locally stable. Further, the global stability of the system is established with the aid of suitably constructed Liapunov's function and the growth rates of the species are numerically estimated using Runge-Kutta fourth order method.

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