Mathematical Statistician and Engineering Applications
ISSN:2094-0343
2326-9865

A Note on Weakly Generalized Closed Sets in Topological Spaces

Dr. K. Karuppiah
Assistant Professor, Department of Mathematics,
Government College of Engineering, Bodinayakkanur-625 582,
Affiliated to Anna University, Chennai, Tamil Nadu, India.
E-mail: karuppiahmaths@gmail.com

Article Info
Page Number: 481 - 492
Publication Issue:
Vol 72 No. 2 (2023)

Abstract: The objective of this paper is to extend the concept of a new class of weakly generalized closed sets called weakly ρ-closed sets by applying generalized closed sets. I introduce the definition of weakly ρ-closed sets in topological spaces and study the relationships of such sets.

Article History:
Article Received: *15 October 2023*
Revised: *24 November 2023*
Accepted: *18 December 2023*

Keywords: ρ-closed, wρ-closed, gsp-closed, gs-closed, sg-closed.

2020 AMS Subject Classifications: 54A05, 54D15.

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1. INTRODUCTION

Levine [11] introduced generalized closed sets in general topology as a generalization of closed sets. Sheik John [17] introduced ω-closed sets in topological spaces. Ravi and Ganesan [16] presented and $\ddot{g}$-closed sets in general topology as one more generalization of closed sets and the new class of $\ddot{g}$-closed sets lies between the class of closed sets and the class of g-closed sets. Pious Missier et al. [15] Presented the idea of g'''-closed sets and concentrated on their most basic properties in topological spaces. Sheik John [17] introduced ω-closed sets in topological spaces. Many researchers like Veerakumar [19] introduced ĝ-closed sets in topological spaces.

The objective of this paper is to introduce for a new class of weakly generalized closed sets called weakly ρ-closed sets by applying generalized closed sets in topological spaces.

2. PRELIMINARIES

Throughout this paper (X, τ), (Y, σ) and (Z, η) (or X, Y and Z) represent topological spaces (briefly $\mathscr{TPS}$) on which no separation axioms are assumed unless otherwise mentioned. For a subset $\mathscr{K}$ of a space X, cl($\mathscr{K}$), int($\mathscr{K}$) and $\mathscr{K}^c$ or X | $\mathscr{K}$ or X – $\mathscr{K}$ denote the closure of $\mathscr{K}$, the interior of $\mathscr{K}$ and the complement of $\mathscr{K}$, respectively.

We recall the following definitions which are useful in the sequel.

Definition 2.1

A subset $\mathscr{K}$ of a space X is called:

(i) semi-open [10] if $\mathscr{K} \subseteq$ cl(int($\mathscr{K}$));

(ii) α-open [13] if $\mathcal{K} \subseteq$ int(cl(int($\mathcal{K}$)));

(iii) semi-preopen [1] if $\mathcal{K} \subseteq$ cl(int(cl($\mathcal{K}$)));

(iv) regular open [18] if $\mathcal{K}$ = int(cl($\mathcal{K}$)).

The complements of the above mentioned open sets are called their respective closed sets.

Definition 2.2

A subset $\mathcal{K}$ of a space X is called a

(i) generalized closed (briefly g-closed) [11] if cl($\mathcal{K}$) $\subseteq \mathcal{B}$ whenever $\mathcal{K} \subseteq \mathcal{B}$ and $\mathcal{B}$ is open in X.

(ii) generalized semiclosed (briefly gs-closed) [3] if scl($\mathcal{K}$) $\subseteq \mathcal{B}$ whenever $\mathcal{K} \subseteq \mathcal{B}$ and $\mathcal{B}$ is open in X.

(iii) α -generalized closed (briefly α g-closed) set [12] if α cl($\mathcal{K}$) $\subseteq \mathcal{B}$ whenever $\mathcal{K} \subseteq \mathcal{B}$ and $\mathcal{B}$ is open in X.

(iv) generalized semi-preclosed (briefly gsp-closed) set [9] if spcl($\mathcal{K}$) $\subseteq \mathcal{B}$ whenever $\mathcal{K} \subseteq \mathcal{B}$ and $\mathcal{B}$ is open in X

(v) semi-generalized closed (briefly sg-closed) [5] if scl($\mathcal{K}$) $\subseteq \mathcal{B}$ whenever $\mathcal{K} \subseteq \mathcal{B}$ and $\mathcal{B}$ is semi-open in X.

The complements of the above mentioned closed sets are called their respective open sets.

3. WEAKLY ρ-CLOSED SETS

I introduce the definition of weakly ρ-closed sets in $\mathcal{TPS}$ and study the relationships of such sets.

Definition 3.1

A subset $\mathcal{K}$ of a $\mathcal{TPS}$ is called

(i) a ρ-closed (briefly, ρ-cld) if cl($\mathcal{K}$) $\subseteq \mathcal{B}$ whenever $\mathcal{K} \subseteq \mathcal{B}$ and $\mathcal{B}$ is sg-open in X.

(ii) a weakly ρ-closed (briefly, wρ-cld) if cl(int($\mathcal{K}$)) $\subseteq \mathcal{B}$ whenever $\mathcal{K} \subseteq \mathcal{B}$ and $\mathcal{B}$ is sg-open in X.

The complements of the above mentioned closed sets are called their respective open sets.

Theorem 3.2

Any closed is wρ-closed but converse is not true.

Proof

Let $\mathcal{K}$ be a closed set. Then cl($\mathcal{K}$) = $\mathcal{K}$. Let $\mathcal{K} \subseteq \mathcal{B}$ and $\mathcal{B}$ be sg-open. Since int($\mathcal{K}$) $\subseteq \mathcal{K}$, cl(int($\mathcal{K}$)) $\subseteq$ cl($\mathcal{K}$) = $\mathcal{K}$. We have cl(int($\mathcal{K}$)) $\subseteq \mathcal{K} \subseteq \mathcal{B}$ whenever $\mathcal{K} \subseteq \mathcal{B}$ and $\mathcal{B}$ is sg-open. Hence $\mathcal{K}$ is wρ-closed.

Example 3.3

Let X = {k_1, k_2, k_3} and τ = {ϕ, {k_1}, {k_2}, {k_1, k_2}, X}. Then the set {k_1, k_2} is wρ-cld set but not closed in X.

Theorem 3.4

Any ρ-closed is wρ-closed but converse is not true.

Proof

It is obviously.

Example 3.5

Let X = $\{k_1, k_2, k_3\}$ and $\tau = \{\phi, \{k_1\}, \{k_2\}, \{k_1, k_2\}, X\}$. The set $\{k_1, k_2\}$ is wρ-closed set but not ρ-closed in X.

Theorem 3.6

Any regular closed is wρ-closed but converse is not true.

Proof

Let $\mathcal{K}$ be any regular closed set and let $\mathcal{B}$ be gs-open set containing $\mathcal{K}$. Since $\mathcal{K}$ is regular closed, we have $\mathcal{K} = cl(int(\mathcal{K})) \subseteq \mathcal{B}$. Thus, $\mathcal{K}$ is wρ-closed.

Example 3.7

Let X = $\{k_1, k_2, k_3\}$ and $\tau = \{\phi, \{k_1\}, \{k_2\}, \{k_1, k_2\}, X\}$. The set $\{k_1\}$ is wρ-closed but not regular closed in X.

Theorem 3.8

Any wρ-closed is gsp-closed but converse is not true.

Proof

Let $\mathcal{K}$ be any wρ-closed and $\mathcal{B}$ be open set containing $\mathcal{K}$. Then $\mathcal{B}$ is a sg-open containing $\mathcal{K}$ and $cl(int(\mathcal{K})) \subseteq \mathcal{B}$. Since $\mathcal{B}$ is open, we get $int(cl(int(\mathcal{K}))) \subseteq \mathcal{B}$ which implies $spcl(\mathcal{K}) = \mathcal{K} \cup int(cl(int(\mathcal{K}))) \subseteq \mathcal{B}$. Thus, $\mathcal{K}$ is gsp-closed.

Example 3.9

Let X = $\{k_1, k_2, k_3\}$ and $\tau = \{\phi, \{k_1\}, \{k_2\}, \{k_1, k_2\}, X\}$. Then the set $\{k_1\}$ is gsp-closed but not wρ-closed.

Theorem 3.10

If a subset $\mathcal{K}$ of a $\mathcal{TPS}$ X is both closed and α g-closed, then it is wρ-closed in X.

Proof

Let $\mathcal{K}$ be an α g-closed set in X and $\mathcal{B}$ be an open set containing $\mathcal{K}$. Then $\mathcal{B} \supseteq \alpha\, cl(\mathcal{K}) = \mathcal{K} \cup cl(int(cl(\mathcal{K})))$. Since $\mathcal{K}$ is closed, $\mathcal{B} \supseteq cl(int(\mathcal{K}))$ and hence $\mathcal{K}$ is wρ-closed in X.

Theorem 3.11

If a subset $\mathcal{K}$ of a $\mathcal{TPS}$ X is both open and wρ-closed, then it is closed.

Proof

Since $\mathcal{K}$ is both open and wρ-closed, $\mathcal{K} \supseteq cl(int(\mathcal{K})) = cl(\mathcal{K})$ and hence $\mathcal{K}$ is closed in X.

Corollary 3.12

If a subset $\mathcal{K}$ of a $\mathcal{TPS}$ X is both open and wρ-closed, then it is both regular open and regular closed in X.

Theorem 3.13

Suppose that $\mathcal{B} \subseteq \mathcal{K} \subseteq X$, $\mathcal{B}$ is a gs-closed relative to $\mathcal{K}$ and that $\mathcal{K}$ is both open and sg-closed subset of X. Then $\mathcal{B}$ is gs-closed relative to X.

Proof

Let $\mathcal{B} \subseteq O$ and suppose that O is open in X. Then $\mathcal{B} \subseteq \mathcal{K} \cap O$ and $scl_A(\mathcal{B}) \subseteq \mathcal{K} \cap O$. It follows then that $\mathcal{K} \cap scl(\mathcal{B}) \subseteq \mathcal{K} \cap O$ and $\mathcal{K} \subseteq O \cup (scl(\mathcal{B}))^c$. Since $\mathcal{K}$ is sg-closed in X, we have $scl(\mathcal{K}) \subseteq O \cup (scl(\mathcal{B}))^c$ since the union of open set and semi-open set is semi-open. Therefore $scl(\mathcal{B}) \subseteq scl(\mathcal{K}) \subseteq O \cup (scl(\mathcal{B}))^c$ and consequently, $scl(\mathcal{B}) \subseteq O$. Then $\mathcal{B}$ is gs-closed relative to X.

Corollary 3.14

Let $\mathcal{K}$ be both open and sg-closed and suppose that F is closed. Then $\mathcal{K} \cap F$ is gs-closed.

Proof

$\mathcal{K} \cap F$ is closed in $\mathcal{K}$ and hence gs-closed in $\mathcal{K}$ (Apply Theorem 3.13).

Theorem 3.15

A set $\mathcal{K}$ is wρ-closed if and only if $cl(int(\mathcal{K})) - \mathcal{K}$ contains no non-empty gs-closed.

Proof

Necessity: Let F be a gs-closed such that $F \subseteq cl(int(\mathcal{K})) - \mathcal{K}$. Since F^c is gs-open and $\mathcal{K} \subseteq F^c$, from the definition of wρ-closed it follows that $cl(int(\mathcal{K})) \subseteq F^c$. ie. $F \subseteq (cl(int(\mathcal{K})))^c$. This implies that $F \subseteq (cl(int(\mathcal{K}))) \cap (cl(int(\mathcal{K})))^c = \phi$.

Sufficiency: Let $\mathcal{K} \subseteq G$, where G is both closed and sg-open set in X. If $cl(int(\mathcal{K}))$ is not contained in G, then $cl(int(\mathcal{K})) \cap G^c$ is a non-empty gs-closed subset of $cl(int(\mathcal{K})) - \mathcal{K}$, we obtain a contradiction. This proves the sufficiency and hence the theorem.

Theorem 3.16

Let X be a $\mathcal{TPS}$ and $\mathcal{K} \subseteq Y \subseteq X$. If $\mathcal{K}$ is open and wρ-closed in X, then $\mathcal{K}$ is wρ-closed relative to Y.

Proof

Let $\mathcal{K} \subseteq Y \cap G$ where G is gs-open in X. Since $\mathcal{K}$ is w$\tilde{a}$-closed in X, $\mathcal{K} \subseteq G$ implies $cl(int(\mathcal{K})) \subseteq G$. That is $Y \cap (cl(int(\mathcal{K}))) \subseteq Y \cap G$ where $Y \cap cl(int(\mathcal{K}))$ is closure of interior of $\mathcal{K}$ in Y. Thus, $\mathcal{K}$ is wρ-closed relative to Y.

Theorem 3.17

If a subset $\mathcal{K}$ of a $\mathcal{TPS}$ X is nowhere dense, then it is wρ-closed.

Proof

Since int($\mathcal{K}$) $\subseteq$ int(cl($\mathcal{K}$)) and $\mathcal{K}$ is nowhere dense, int($\mathcal{K}$) = ϕ. Therefore cl(int($\mathcal{K}$)) = ϕ and hence $\mathcal{K}$ is wρ-closed in X.

The converse of Theorem 3.17 need not be true as seen in the following example.

Example 3.18

Let X = {k_1, k_2, k_3} and τ = {ϕ, {k_1}, {k_2, k_3}, X}. Then the set {k_1} is wρ-closed but not nowhere dense in X.

Remark 3.19

The following examples show that w$\tilde{\alpha}$-closed and semi-closedness are independent.

Example 3.20

Let X = {k_1, k_2, k_3} and τ = {ϕ, {k_1}, {k_2, k_3}, X}. The set {k_2} is wρ-closed but not semi-closed in X.

Example 3.21

Let X = {k_1, k_2, k_3} and τ = {ϕ, {k_1}, {k_2}, {k_1, k_2}, X}. Then the set {k_2} is semi-closed set but not wρ-closed in X.

Definition 3.22

A subset $\mathcal{K}$ of a $\mathcal{TPS}$ X is called wρ-open set if $\mathcal{K}^c$ is wρ-closed in X.

Theorem 3.23

Any open set is wρ-open.

Proof

Let $\mathcal{K}$ be an open set in a $\mathcal{TPS}$ X. Then $\mathcal{K}^c$ is closed in X. By Theorem 3.2 it follows that $\mathcal{K}^c$ is w$\tilde{\alpha}$-closed in X. Hence $\mathcal{K}$ is wρ-open in X.

The converse of Theorem 3.23 need not be true as seen in the following example.

Example 3.24

Let X = {k_1, k_2, k_3} and τ = {ϕ, {k_1}, {k_2}, {k_1, k_2}, X}. The set {k_3} is wρ-open set but it is not open in X.

Proposition 3.25

(i) Any wρ-open is wρ-open but converse is not true.

(ii) Any regular open is wρ-open but converse is not true.

(iii) Any g-open is wρ-open but converse is not true.

(iv) Any wρ-open is gsp-open but converse is not true.

It can be shown that the converse of (i), (ii), (iii) and (iv) need not be true.

Theorem 3.26

A subset $\mathcal{K}$ of a $\mathcal{TPS}$ X is wρ-open if G $\subseteq$ int(cl($\mathcal{K}$)) whenever G $\subseteq$ $\mathcal{K}$ and G is gs-closed.

Proof

Let $\mathcal{K}$ be any wρ-open. Then $\mathcal{K}^c$ is wρ-closed. Let G be a sg-closed contained in $\mathcal{K}$. Then G^c is a sg-open set containing $\mathcal{K}^c$. Since $\mathcal{K}^c$ is wρ-closed, we have $cl(int(\mathcal{K}^c)) \subseteq G^c$. Therefore $G \subseteq int(cl(\mathcal{K}))$.

Conversely, we suppose that $G \subseteq int(cl(\mathcal{K}))$ whenever $G \subseteq \mathcal{K}$ and G is sg-closed. Then G^c is a sg-open set containing $\mathcal{K}^c$ and $G^c \supseteq (int(cl(\mathcal{K})))^c$. It follows that $G^c \supseteq cl(int(\mathcal{K}^c))$. Hence $\mathcal{K}^c$ is w$\tilde{a}$-closed and so $\mathcal{K}$ is wρ-open.

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