Mathematical Statistician and Engineering Applications
ISSN:2094-0343
2326-9865

# Nonsingular Modules Over Serial Rings

**Manish Kumar Jha**

Past Research Scholar

P.G. Dept. of Mathematics

L.N.M.U., Darbhanga

***Article Info***
***Page Number:273 -277***
***Publication Issue:***
***Vol 68 No. 1 (2019)***

***Article History***
***Article Received: 09 September 2019***
***Revised: 16 October 2019***
***Accepted: 21 November 2019***
***Publication: 28 December 2019***

**Abstract :**

Non-Singular Modules over serial rings are equivalent to Modules with a unique in decomposable decomposition. They form closed subset of the module lattice. The ring of endomorphism of non-singular modules is serial ring.

**Key word**-Serial ring, non-singular modules, module lattice, Jacobson radical.

**Introduction :**

Serial rings have been studied developed criteria which ensure that a uniform module over a serial ring with Krull dimension is uniserial. We generalize these results, and in particular free them from the Krull dimensional hypothesis. At the same time we provide simplified proofs. This is accomplished by formulating the statements and arguments in terms of variants of non-singularity.

We emphasize that all these criteria are sufficient but not necessary. Effective necessary and sufficient tests, for when a single uniform module over a serial ring is uniserial, appear not to be known.

Along the way, we show that any semi prime right serial is a direct sum of prime rings, and we obtain facts about the spectrum of a serial ring. The later ones will also be useful in subsequent paper, Which gives a fairly comprehensive structure theory for arbitrary serial rings.

**Preliminaries**

All rings considered here have an identity element, and all modules are unitary right modules, unless specified differently.

For an R-module M, Z(M) denote the singular sub module and the injective hull. $N \subseteq M$, $N \subset M$, and $N \subseteq' M$ indicate that N is a submodule, proper sub module, and essential submodule of M, respectively.

A nonzero cyclic module with a unique maximal submodule is called a local module. A nonzero x of an arbitrary module is called local if xR is a local module.

We call a module uniserial if its submodules form a chain, and serial if it is the direct sum of uniserial modules. A ring is called right (left) serial if it is serial as ring (left) module over itself; it is called serial if is left and right serial. Every left or right serial ring is semi perfect. Over a right serial ring, any element of any module is the sum of local elements. In a uniserial module, any element is local.

J will always denote the Jacobson radical of the ring R. The letter e, with and without subscripts, is reserved for indecomposable idempotents of R.

The transfinite powers of the Jacobson radical are defined inductively as $J(\alpha) = \bigcap_{\beta<\alpha, n\in N} J(\beta)^n$, for ordinals, $\alpha$, $\beta$. (that is simplifies to $J(\alpha) = \bigcap_{n\in N} J(\alpha-1)^n$ if $\alpha$ is non-limit ordinal, and to $J(\alpha) = \bigcap_{\beta<\alpha} J(\beta)$ if $\alpha$ is a limit ordinal.)

Theorem 1.1 Let R be a serial ring, P a finitely generated projective R-moudle, and M a finitely generated submodule of P. Then there is decomposition $P= P_1 \oplus \dots \oplus P_n$ into indecomposable projective such that $M=M \cap P_1 \oplus \dots \oplus M \cap P_n$.

Proposition 1.2 For any R-module M over a serial R, M/Z(M) is a non-singular R/Z(R)-module. In particular, R/Z (R) itself is a (right) nonsingular ring.

Proposition 1.3 Every nonsingular uniform module over a serial ring is uniserial.

Theory 1.4 A serial ring is left and right nonsingular if it is left and/or right semihereditary. If so, then every finitely generated nonsingular module is projective.

Recall that a ring is called right Goldie if it has finite Goldie dimension, and ascending chain condition on right annihilators.

A right nonsingular ring with finite Goldie dimension is right Goldie. The converse holds true provided the ring is semi prime.

Consequently, a semiprime serial ring is left and/or right Goldie if it is left and/or right nonsingular. Iff it is left and/or right semihereditary, iff it has ascending chain condition on left and/or right annihilators.

Proposition 1.5 If xR is a local module over a semi perfect ring R, then there is an indecomposable idempotent e such that x=xe.

Proof. Let $f : P \to xR$ be a projective cover, and define $g : R \to xR$ via $g(r) = xr$. There exists an epimorphism $h : R \to P$ such that $fh = g$. Then h splits : $R = eR \oplus \ker h$. Consequently $x = g(1) = fh(1) = fh(e) = fh(1)e = xe$.

**2. Non- Singular Modules Over Serial Rings**

This section contains a few technical observations, concerning the modules of the title.

Lemma 2.1 Let I be an ideal, and A a right ideal, of an arbitrary ring, such that $I \subseteq A \subseteq R$.

(i) If $A/I \subseteq' R/I$, then $A \subseteq' R$.

(ii) If M is a nonsingular R-module and MI = 0, then M is a non-singular R/I-module.

Lemma 2.2 A nonzero projective module over a serial ring is not singular.

Proof. As a serial ring is semi perfect, a projective module is the direct sum of sub modules of the form eR. The annihilator (1-e) R of e is not essential, and therefore $e \notin Z(eR)$.

Lemma 2.3 Over a serial ring, a module is nonsingular if every local/finitely generated sub modules is projective.

Proof. Let N is a finitely generated sub modules of the nonsingular R-module M, and let Z = Z (R). Clearly NZ=0, and therefore N is non-singular R/Z-module, by (2.1). Now R/Z is a nonsingular ring. Therefore N is a projective R/Z-module. Thus N is a direct sum of sub modules of the form eR/eZ. Since these eR/eZ are even nonsingular and R-modules, that is implies that's eZ=0. Consequently N is a projective R-module.

Lemma 2.4 For an ideal I of a serial ring R, the following are equivalent :

(i) For every indecomposable idempotent e, eI=0 or every finitely generated (=cyclic) submodule of eR/eI is a project R/I-module.

(ii) For every indecomposable idempotent e, eI=0 or eR/eI is a non-singular R/I-module.

(iii) For every indecomposable idempotent e, and every x $\in$ eR - eI, xI = eI holds.

(iv) For every uniserial module M and x $\in$ M - MI, xI = MI holds.

Proof. (i) and (iv) for given x $\in$ M - MI, we have to show xI $\supseteq$ yI for all y $\in$ M. This is trivially true if xR $\supseteq$ yR or if yI =0. We are left with the case xR $\subset$ yR and yI $\neq$ 0.

The (1.5) we have e with y $\approx$ ye. Clearly eI $\neq$ 0. The obvious empimorphism eR $\rightarrow$ yR induces an isomorphism eR/eI $\cong$ yR/yI. The inclusion xR $\subset$ yR induces a homomorphism xR/xI $\rightarrow$ yR/yI $\cong$ eR/eI. Its image, a cyclic submodule of eR/eI, is a projective R/I-module, by hypothesis. Thus the homomorphism splits, and we conclude xI = yI, xR is uniserial.

(iv) implies (iii) : Trivial.

(iii) implies (i) : Let eI $\neq$ 0 and consider arbitrary x $\in$ eR - eI. By hypothesis, xI = eI $\neq$ 0. With x = xe′ from (1.5), we obtain again e′ R/e′I $\cong$ xR/xI. Thus the arbitrary cyclic sub module xR/xI = xR/el of eR/eI is a projective R/I-module.

DEFINITION. We call an ideal I of a serial ring R (right) almost non-singular if it has the equivalent.

Examples. The class of almost nonsingular ideals is closed under arbitrary sums and down directed intersections.

That every nonsingular ideal is almost nonsingular. In particular, every Goldie semiprime ideal, and specifically J(R) = J(0), is almost nonsingular.

In particular this shows that all J($\alpha$), $\alpha > 0$ are almost nonsingular. It also shows that $\bigcap_{n \in N} I^n$ is almost nonsingular for any ideal I. Specially every idempotent ideal is almost nonsingular.

Corolllary 2.5 If I is a right almost nonsingular ideal is a serial ring R, and M a uniserial R-module with MI $\neq$ 0, then M/MI is a nonsingular R/I-module.

Proof. Consider any $0 \neq \bar{x} \in$ M/MI, and select e with xe = x by (1.5). By (2.4) we have 0 $\neq$ MI = xI = xeI, in particular eI $\neq$ 0. that's yields eR/eI $\cong$ xR/xI = xR/MI = $\bar{x}R$ , as before. Thus cyclic submodule of M/MI is a projective R/I-module, and therefore M/MI itself is a non-singular R/I-module.

**3. Prime Ideal In Serial Rings**

We develop some elementary facts about the subject of the title. More details will be given in a subsequent paper.

Lemma 3.1 Any two incomparable prime ideals of a serial ring are co-maximal.

Proof. For any indecomposable idempotent e, we have eP $\subseteq$ eQ or eQ $\subseteq$ eP. If P, Q are incomparable prime ideals, we deduce e $\in$ Q or e $\in$ P, respectively. Therefore, in any case, e(P+Q) = eR. Consequently P + Q = R.

Proposition 3.2 Let P, Q be prime ideals of a right serial ring R. Assume there is a uniserial (or hollow) module M such that MP $\subset$ M and MQ $\subset$ M. Then P, Q are comparable.

We fix, for the following discussion, a specific decomposition 1 = $\sum_i^n = 1^e i$ of the identity of a right serial ring R into indecomposable orthogonal idempotent.

Lemma 3.3 Let P, Q be prime ideals of the right serial ring Rs.

(i) If $e_i \in P$ implies $e_i \in Q$ , then P, Q are comparable.

(ii) If , in addition, there is some $e_i \in Q - P$, then $P \subset Q$.

Proof. (i) Suppose $P \not\subseteq Q$. Then there is $e_i$ such the $e_i P \supset e_i Q$. Thus $e_i \notin Q$ (as otherwise $e_i P \supset e_i Q = e_i R$). By hypothesis we conclude $e_i \notin P$. Thus $P \supseteq e_i P \supset e_i Q$ implies $P \supseteq Q$.

(ii) Now obvious.

We associate, with any subset T of { $e_1$, ...., $e_n$}, the collection P (T) of prime ideals P with P $\cap$ { $e_1$, ......., $e_n$} = T. Note that P (T) may be empty. By (3.3), each P (T) is a chain; we shall call the nonempty P (T) the towers of spec R. Again by (3.3), for two towers P ($T_i$), $T_1 \subset T_2$ holds if $P_1 \subset P_2$ holds for some $P_i \in P(T_i)$.

We record a number of simple observations concerning the inclusion graph T = { T : P (T) $\neq \emptyset$}. Everything follows easily from (3.1) - (3.3).

(i) T is a finite disjoint union of rooted trees.

(ii) The roots correspond to the minimal prime ideals.

(iii) The leaves correspond to the maximal ideals.

(iv) The trees branch properly (i.e., at least three incident edges) at each vertex which is neither a root nor a leaf.

(v) The spectrum is obtained by replacing the vertices T by the towers P (T).

Proposition 3.4 Every right serial semiprime ring is a direct sum of prime rings.

Proof As the spectrum consists of finitely many chains, there are only finitely many minimal primes, say $Q_1$, ...., $Q_m$. Their intersection $\cap_{i=1}^{m} \ Q_i$ equals zero.

Definition. Let P, Q be prime ideals of a right serial ring R. We call Q a (right) successor of P [and P a (right) predecessor of Q] if $ePQ \subset eP \subset eR$ holds for some indecomposable idempotent e of Rs.

Lemma 3.5 Let Q, P be two incomparable prime ideals of a right serial ring R. If Q is a successor of P, then $ePQ \subset eP \subset eR$ holds for all indecomposable idempotents $e \notin P$.

Proof. We are given one e with $ePQ \subset ep \subset eR$; clearly $e \notin P$. We know that $xP = eP$ holds for all $x \in eR - eP$. We have $xP \subseteq eP \subset xR \subseteq eR$; so $xP \subset eP$ leads to $eP^2 \subseteq xP \subset eP$, Which together with $ePQ \subset eP$ contradicts.

Consider any other $e' \notin P$. Then $eRe'$ P, and there is $x_0 \in eRe' - P$. The first consideration shows $x_0P = eP$, and this implies $eRe' = eP$. Now We have $e' PQ \subset e'P$ [ since $e'PQ = e'P$ leads to the contradiction $eP = eRe' P = eRe' pQ = ePQ \subset eP$], and therefore $e' PQ \subset e'P \subset e'R$

Lemma 3.6 Let $P \subseteq Q$ be prime ideals of a serial ring R. Then Q is a successor of P is and only if $PQ \subset P$. In particular, this cannot happen if P is Goldie and $P \subset Q$.

Proof. Q is a successor of P if $PQ \subset P \cap Q = P$. If P is Goldie, hence (almost) nonsingular, it is shows that eAP = eP for all e and all ideal $A \supset P$. We conclude AP = P, and symmetrically PA = P. In particular, if $Q \supset P$, then Q cannot be a successor of P.

The next corollary shows that the successor relation between incomparable primes is symmetric, and occurs only rarely.

Corollary 3.7 Let P and Q be two incomparable prime ideals of a serial ring.

(i) Q is a successor of P is and only if P is a left successor of Q.

(ii) If so, then Q and P contain the same prime ideals properly (and are in particular the minimal members of their respective towers);

(iii) Moreover, Q and P are Goldie, and determine each other uniquely.

Proof.

(i) Trivial.

(ii) Consider any prime $Q' \subset P$. We have $ePQ \subset eP \subset eR$ for some e. If eP = ePQ', we deduce $e \in Q' \subset P$ hence eP = eR, a contradiction. Thus $ePQ' \subset eP$. applied to M = eP, shows that Q and Q' are comparable. As $Q \subseteq Q'$ leads to the contradiction $Q \subset P$, we conclude $Q' \subset Q$.

(iii) To see that P is Goldie, consider any $e \notin P$. says that $ePQ \subset eP \subset eR$, and in particular $eP \neq 0$. The first claim in the proof of (3.5) establishes xP = eP for all $x \in eR - eR$. Then the implication from (iii) to (i) in the proof of (2.4) shows that eR/eP is a nonsingular R/P-module. Thus R/P is a right nonsingular ring, and P is Goldie.

Le $Q_1$, $Q_2$ be successors of P, both incomparable with P. Then $ePQ_i \subset eP \subset eR$ holds for all $e \notin P$ and i = 1, 2 by (3.5). Corollary (3.2) shows that $Q_1$ and $Q_2$ are comparable. If $Q_1 \subset Q_2$ , (ii) yields $Q_1 \subset P$, contrary to the incomparability assumption. Similarly $Q_2 \subset Q_1$ is impossible; hence $Q_1 = Q_2$.

**References :**

1. A. W. CHATTERS, Serial rings with Krull dimension, Glasgow Math. J.32 (1990), 71-78.
2. A.W. CHATTERS and C.R. HAJARNAVIS, Rings with chain conditions, in "Research Notes in Mathematics," Vol. 44, Pitman, New York, 1980.
3. L.FUCHS and L.SALCE, Modules Over valuation domains, in "Lecture Notes in Pure and applied Mathematics," Vo. 97, DEKKER, New York, 1985.
4. C. LANSKI, Nil sub rings of Goldie rings are nilpotent, Canad. J.Math. 21 (1969), 904-907.
5. M.H. UPHAM, Serial rings with right Krull dimension one, J.Algebra 109 (1987).
6. G. Krause and T.H. Lenagan, Transfinite powers of the Jacobson radical, *Comm. Algebra* 7 (1979), 1-8.

❑❑❑