0-Conditions in the Lattice of Convex Sublattices

Dr.A.Vethamanickam[1] **Mrs .S.Christia Soniya**[2*]

[1].Former Associate Professor, Department of Mathematics,

Rani Anna government college for women. Tirunelveli.India.

[2*].Research scholar (Reg. No19221172092002.),

Rani Anna government college for women. Tirunelveli. India

Assistant Professor, Einstein College of Arts and Science, Tirunelveli.

(Affiliated to Manonmaniam Sundaranar University)

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Abstract .

In this paper, it is proved that if the lattice of all convex sublattices of a given lattice L is respectively 0-modular,0-distributive,0-supermodular,0-semi modular, super-0-distributive, pseudo-0-distributive, Eulerian, General disjointness condition, then L is also 0-modular,0-distributive,0-supermodular,0-semi modular, super-0-distributive, pseudo-0-distributive, Eulerian and satisfies General disjointness condition, then L also posses the same property.

Keywords : Lattice, Convex sublattice , New partial order, Ideal, Filter.

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Introduction

Let L be a lattice and $CS(L)$ be the set of all non empty convex sublattices of L. The lattice of all convex sublattices of `a lattice including empty set under set inclusion relation was first studied thoroughly by K.M.Koh in 1972 [5].There he investigated the interdependence of the lattices L and $((CS(L)\cup\{\varphi\}),\subseteq)$,from the lattice theoretical point of view.

In 1996, S.Lavanya and S.Parameshwara Bhatta [7] introduced another partial ordering on $CS(L)$. They have proved that both L and $CS(L)$ with respect to that ordering are in the same equational class. As a further study P.V. Ramana murty in 2002[9], had investigated the effect of that ordering on lattices which cannot be described by means of identities. Particularly he had looked into semi modular lattices. He obtained that for a lattice L, $CS(L)$ is semi modular if both $I(L)$ and $D(L)$ are semi modular. And if L is of finite length, he proved that the converse also holds.Recently R.Subbarayan [10] has proved that $CS(L)$ is 0-semi modular then L is 0-semi modular. This has motivated us to look into lattices which satisfy other 0-conditios. In this paper, we consider the lattices which are (i) 0-modular (ii) 0-distributive (iii) 0-super modular (iv) Super-0-distributive (v) Pseudo-0-distributive and(vi) Eulerian lattice and (vii) lattices satisfying the General disjointness condition.

Among these the converse is also proved to be true for the properties 0-distributivity and 0-semimodular. For other properties, we are able to prove only one way namely, $CS(L)$ satisfies the condition, then L also satisfies the condition. These results are analogous to corollary 8, page number 53[9].

2.Preliminaries

In this section, we give some basic definitions needed for the development of the paper.

2.1 Convex sublattice

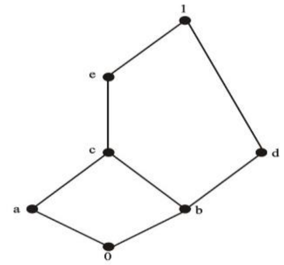
A sublattice K of a lattice L is called convex iff whenever $a,b \in K$, $c \in L$ and $a \le c \le b$ then $c \in K$.

For example, if $a,b \in L$, $a \le b$, the interval $[a,b] = \{x / a \le x \le b\}$ is an example of a convex sublattices of L.The collection of all convex sublattices of a lattice L is denoted by $CS(L)$.

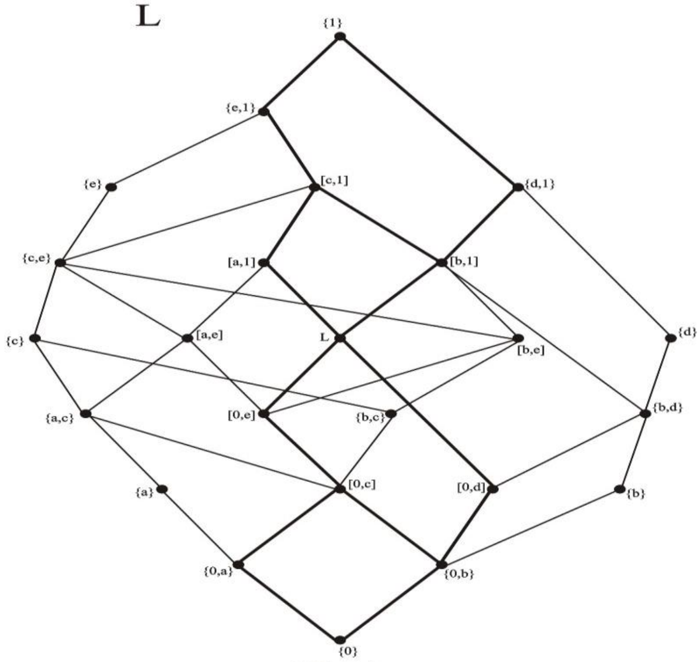
2.2 A new partial ordering on $CS(L)$

We define a binary relation $\le$ on $CS(L)$ by the following rule: for $A, B \in CS(L), A \le B$ if and only if ``for every $a \in A$ there exists a $b \in B$ such that $a \le b$ and for every $b \in B$ there exists an $a \in A$ such that $b \ge a$", clearly '$\le$ ' is a partial order on $CS(L)$.Moreover $\langle CS(L), \le \rangle$ forms a lattice (see[7]).

A simple structure of a 0-modular lattice L which is not modular and its $CS(L)$ are given in the following figure.



L



CS(L)

2.3 Ideal and Filter

A sublattice I of L is an ideal iff $i \in I$ and $a \in L$ imply that $a \wedge i \in I$.

A sublattice F of L is a Filter iff $f \in F$ and $a \in L$ imply that $a \vee f \in F$.

2.4 Remark

Let $I(L)$ denote the lattice of all ideals of L (ordered by $\subseteq$)

and $D(L)$ denote the lattice of all Filters of L (ordered by $\supseteq$)

Since L can be embedded in $I(L)$ and $I(L)$ is a sublattice of $CS(L)$, the mapping $f : L \to CS(L)$ defined by $f(a) = (a]$ for every $a \in L$ is an embedding.

2.5 Notations

Let L be a lattice. For a subset A of L we denote $(A], [A)$ and $\langle A \rangle$ respectively to represent the ideal, the filter and the convex sublattice of L generated by A.

2.6 Supermodular lattice

A lattice L is said to be supermodular if it satisfies the following identity

$(a\vee b)\wedge(a\vee c)\wedge(a\vee d)=a\vee[b\wedge c\wedge(a\vee d)]\vee[c\wedge d\wedge(a\vee b)]\vee[b\wedge d\wedge(a\vee c)]$ *for all* $a,b,c,d\in L.$

2.7 0-Supermodular lattice

A Lattice L is called 0-Supermodular, if whenever $b,c,d\in L$ satisfy

$b\wedge c=c\wedge d=b\wedge d=0,$ *then* $(a\vee b)\wedge(a\vee c)\wedge(a\vee d)=a$ *for every* $a\in L.$

2.8 0-modular lattice

A lattice L is said to be a 0-modular lattice if whenever $x\leq y$ *and* $y\wedge z=0,$

Then $x=(x\vee z)\wedge y$ *for all* $x,y,z\in L$.

Example M_3 is 0-modular.

2.9 0-Semimodular lattice

A lattice L is said to be a 0-semimodular lattice if whenever a is an atom of L and $x\in L$ such that $a\wedge x=0$, then $x\vee a$ covers x .

2.10 0-distributive lattice

A lattice L with 0 is said to be 0-distributive if for all $x,y,z\in L$,whenever $x\wedge y=0$ and

$x\wedge z=0$,then $x\wedge(y\vee z)=0.$

2.11 Pseudo-0-distributive

A lattice L is said to be pseudo-0-distributive if for all $a,b,c\in L$, $a\wedge b=0$ and $a\wedge c=0$ imply that $(a\vee b)\wedge c=b\wedge c$.

2.12 Super-0-distributive

A lattice L is said to be super-0-distributive if for all $a,b,c\in L$, $a\wedge b=0$ implies that

$(a\vee b)\wedge c=(a\wedge c)\vee(b\wedge c)$.

2.13 Graded poset

A Poset P is graded if all maximal chains in P have the same length.

2.14 Eulerian Poset

A finite graded poset P is said to be Eulerian if its mobius function assumes the value $\mu(x,y)=(-1)^{l(x,y)}$ for all $x\leq y$ in P , where $l(x,y)=r(y)-r(x)$ and r is the rank function on P .

3. On the preservability of 0-conditions in CS(L)

In this section we prove the following theorems.

3.1 Theorem

If $CS(L)$ is 0-Supermodular, then L is 0-Supermodular.

Proof

Let $CS(L)$ is 0-Supermodular

Let $b,c,d \in L$ such that $b \wedge c = c \wedge d = b \wedge d = 0 -- \rightarrow (1)$

we have to prove that $(a \vee b) \wedge (a \vee c) \wedge (a \vee d) = a$ for every $a \in L$.

From (1) In $CS(L)$,we have $(b \wedge c] = \{0\}, (c \wedge d] = \{0\}, (b \wedge d] = \{0\}$.

that is, $(b] \wedge (c] = \{0\}, (c] \wedge (d] = \{0\}, (b] \wedge (d] = \{0\}$.

Now $a \in CS(L)$

Therefore, in $CS(L)$, $((a] \vee (b]) \wedge ((a] \vee (c]) \wedge ((a] \vee (d]) = (a]$ as $CS(L)$ is 0-supermodular.

which implies $(a \vee b] \wedge (a \vee c] \wedge (a \vee d] = (a]$.

which implies that $((a \vee b) \wedge (a \vee c) \wedge (a \vee d)] = (a]$.

Therefore ,$(a \vee b) \wedge (a \vee c) \wedge (a \vee d) = a$.

Hence, L is 0-Supermodular.

3.2 Theorem

$CS(L)$ is 0-semimodular, if and only if L is 0-semimodular.

Proof

The proof of the part "If $CS(L)$ is 0-semimodular then L is also 0-semimodular" can be found in[10].

Conversely,

Suppose that L is 0-semimodular

we claim that $CS(L)$ is 0-semimodular

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Take an atom{0,a} in $CS(L)$,where a is an atom in L .

let X be any element in CS(L) such that $\{0,a\} \wedge X = \{0\}$

That is $\langle \{0\} \cup \{a \wedge x / x \in X\} \rangle = \{0\}$

which implies that $a \wedge x = 0$ for every $x \in X$.

which implies that $a \vee x \succ x$ for every $x \in X$ (Since L is 0-semimodular)-----(1)

To prove that $\{0,a\} \vee X \succ X \ in \ CS(L)$

we have $\{0,a\} \vee X = \langle X \cup \{a \vee x / x \in X\} \rangle ---\rightarrow (*)$

suppose there exists a $Y \in CS(L)$ such that $\langle X \cup \{a \vee x / x \in X\} \rangle > Y > X$

Therefore, for every $y \in Y$,there exists a $t \in \langle X \cup \{a \vee x / x \in X\} \rangle$ such that $y \le t ---\rightarrow (2)$

And for every $s \in \langle X \cup \{a \vee x / x \in X\} \rangle$,there exists a $y_1 \in Y$ such that $s \ge y_1 ---\rightarrow (3)$

Also there exists a $x_1 \in X$ such that $y_1 \ge x_1 ---\rightarrow (4)$

by (1) , $a \vee x \succ x$ for every $x \in X$.

By (3), if s is of the form $x \in X$, then $x \ge y_1 \ge x_1$ implies $y_1 \in X$ (Since X is convex)

Now $\{0,a\} \vee X = \left\{ \begin{array}{l} t \in L / s_1 \vee x_1 \le t \le s_2 \vee x_2 \\ where\, s_1, s_2 \in \{0,a\} \ and \ x_1, x_2 \in X \end{array} \right\}$

$$Then \ \{0,a\} \vee X = \left\{ \begin{array}{l} t \in L / x_1 \le t \le x_2 \ or \\ a \vee x_3 \le t \le a \vee x_4 \ or \\ x_1 \le t \le a \vee x_2 \ or \\ a \vee x_1 \le t \le x_2 \\ where\, s_1, s_2 \in \{0,a\} \ and \ x_1, x_2 \in X \end{array} \right\}$$

Claim : $\{0,a\} \vee X \le Y$

We prove that for every $t \in \{0,a\} \vee X$,there exists a $y \in Y$ such that $t \le y$ and

for every $y_{11} \in Y$,there exists $t_{11} \in \langle X \cup \{a \vee x / x \in X\} \rangle$ such that $t_{11} \le y_{11}$

Consider a $t \in \{0,a\} \vee X$

(i)Now take the case when for some $x_2, x_3 \in X$, $x_2 \le t \le x_3 ---\rightarrow (5)$

$t \ge y_1$ is true as $Y < \{0,a\} \vee X ---\rightarrow (6)$

Now $x_3 \in X$,there exists a $y_3 \in Y$ such that $x_3 \le y_3 ---\rightarrow (7)$

Equation (5) and (7) implies $t \le x_3 \le y_3$

Now consider an element $y_{11} \in Y$,there exists a $x_{11} \in X$ such that $y_{11} \ge x_{11}$ (since $Y \ge X$)

As x_{11} can be considered as an element of $\{0,a\} \vee X$,

We have arrived at an element x_{11} of $\{0,a\} \vee X$,below y_{11}

Therefore, $\{0,a\} \vee X \le Y$

(ii) Take the case when $a \vee x_3 \le t \le a \vee x_4$ for some $x_3, x_4 \in X$

$t \ge y_1$ is clear for some $y_1 \in Y$ by (6)

Now $x_4 \in X$ and $X \le Y$ implies that there exists an element $y_4 \in Y$ such that $x_4 \le y_4$.

Therefore, $t \le a \vee x_4 \le a \vee y_4$

Now $a \vee x_4 \succ x_4$ since L is 0-semimodular.

Therefore, $t \le x_4 \le y_4$.

Hence, $\{0,a\} \vee X \le Y$ in this case also.

(iii)Now consider the case when $x_5 \le t \le a \vee x_6$

As in the case (ii), we can argue that $t \le y_6$ for some $y_6 \in Y$

Therefore, in this case also $\{0,a\} \vee X \le Y$

(iv)Finally ,when $a \vee x_7 \le t \le x_8$ for some $x_7, x_8 \in X$,

Then as in the first case, we get $\{0,a\} \vee X \le Y$.

Hence, in all the cases we have $\{0,a\} \vee X \le Y$.

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So, $\{0,a\} \vee X = Y$.

Therefore, $\{0,a\} \vee X \succ X$.

Hence, we conclude that $CS(L)$ is 0-semimodular.

3.3 Theorem

If $CS(L)$ is 0-modular, then L is 0-modular.

Proof

Suppose that $CS(L)$ is 0-modular.

we have to prove that L is 0-modular.

that is to prove that for every $x, y, z \in L$ such that $x \leq y$ *and* $y \wedge z = 0$,we have $(x \vee z) \wedge y = x$.

Let $x, y, z \in L$ and $x \leq y$ *and* $y \wedge z = 0$.

since $x \leq y$ we have (x] $\subseteq$ (y]

and $y \wedge z = 0$ implies that $(y \wedge z] = \{0\}$

Therefore, $(y] \wedge (z] = \{0\}$.

If $t \in (y] \wedge (z]$, then $t \leq y$ and $t \leq z$.

Which implies that $t \leq y \wedge z = 0$

As $CS(L)$ is 0-modular, we have $((x] \vee (z]) \wedge (y] = (x]$.

Which implies that $(x \vee z] \wedge (y] = (x]$.

Which implies that $((x \vee z) \wedge y] = (x]$.

Hence ,$(x \vee z) \wedge y = x$.

Hence, L is 0-modular.

3.4 Theorem

If $CS(L)$ is Eulerian, then L is Eulerian.

Proof

Let $CS(L)$ be Eulerian.

we have to prove that L is Eulerian.

that is to prove that $\mu(x,y)=(-1)^{\{r(y)-r(x)\}}$ for all $x \le y$ in L.

Let $x, y \in L$ and $x \le y$.

Therefore, $(x] \subseteq (y]$ *in* $CS(L)$.

Now $\mu((x],(y])=(-1)^{(r(y]-r(x])}$ as $CS(L)$ is Eulerian.

Since it is easily seen that $r((y])=r(y)$ for all $y \in L$.

And $\mu((x],(y])=\mu(x,y)$ for all $x, y \in L$ as $[\{0\}, L] \cong I(L)$.

Hence $\mu(x,y)=(-1)^{r(y)-r(x)}$.

Therefore, L is Eulerian.

The converse is not true for $|L|>1$.

For example, The two element chain is Eulerian, but its lattice of convex sublattices is a 3 element chain which is not Eulerian.

3.5 Theorem

$CS(L)$ is 0-distributive if and only if L is 0-distributive.

Proof

Suppose $CS(L)$ is 0-distributive.

we have to prove that L is 0-distributive

Let $x, y, z \in L$ such that $x \wedge y = 0$ and $x \wedge z = 0$.

To prove $x \wedge (y \vee z) = 0$.

Now $(x \wedge y] = \{0\}$ and $(x \wedge z] = \{0\}$.

which implies $(x] \wedge (y] = \{0\}$ and $(x] \wedge (z] = \{0\}$.

Therefore, $(x] \wedge ((y] \vee (z]) = \{0\}$.

That is, $\left(x\right]\wedge\left(y\vee z\right]=\{0\}$.

That is, $\left(x\wedge\left(y\vee z\right)\right]=\{0\}$.

Hence, $x\wedge\left(y\vee z\right)=0$.

Therefore, L is 0-distributive.

Conversely,

Suppose that L is 0-distributive.

then for every $x,y,z\in L$, whenever $x\wedge y=0$ and $x\wedge z=0$, then $x\wedge\left(y\vee z\right)=0$.

We claim that $CS\left(L\right)$ is 0-distributive

Let $X\wedge Y=\{0\}$, $X\wedge Z=\{0\}$ where $X,Y,Z\in CS\left(L\right)$.

To prove that $X\wedge\left(Y\vee Z\right)=\{0\}$.

We know that $X\wedge\left(Y\vee Z\right)=\left\{t\in L/x_1\wedge s_1\leq t\leq x_2\wedge s_2\right\} where\, x_1,x_2\in X\, and\, s_1,s_2\in Y\vee Z$

Now $s_1,s_2\in Y\vee Z$ implies that $y_{11}\vee z_{11}\leq s_1\leq y_{21}\vee z_{21}, y_{12}\vee z_{12}\leq s_2\leq y_{22}\vee z_{22}$

for some $y_{11},y_{21},y_{12},y_{22}\in Y\, and\, z_{11},z_{21},z_{12},z_{22}\in Z$.

Hence, $x_1\wedge s_1\leq t\leq x_2\wedge s_2$ implies $x_1\wedge\left(y_{11}\vee z_{11}\right)\leq t\leq x_2\wedge\left(y_{22}\vee z_{22}\right)$

Therefore, $0\leq \mathrm{t}\leq 0$ (since L is 0-distributive)

as $x_1\wedge y_{11}=0$, $x_1\wedge z_{11}=0$, $x_2\wedge y_{22}=0, x_2\wedge z_{22}=0.$

Which implies that $\mathrm{t}=0$

Therefore, $X\wedge\left(Y\vee Z\right)=\{0\}$.

Hence, $CS\left(L\right)$ is 0-distributive.

3.6 Theorem

If $CS\left(L\right)$ is super-0-distributive, then L is super -0-distributive

Proof

Suppose that $CS\left(L\right)$ is super 0-distributive.

we have to prove that L is super 0-distributive.

Let $x, y, z \in L$ such that $x \wedge y = 0$.

To prove that $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ for every $x, y, z \in L$.

Take an element $z \in L$,therefore $(z] \in CS(L)$.

we have $(x \wedge y] = \{0\}$ So, $(x] \wedge (y] = \{0\}$.

Therefore, ((x] ∨ (y]) ∧ (z] = ((x] ∧ (z]) ∨ ((y] ∧ (z]) as $CS(L)$ is super 0-distributive.

which implies that $((x \vee y) \wedge z] = (x \wedge z] \vee (y \wedge z] = ((x \wedge z) \vee (y \wedge z)]$.

which implies $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$.

Hence, L is super 0-distributive.

3.7 Theorem

If $CS(L)$ is pseudo-0-distributive, then L is pseudo-0-distributive.

Proof

Suppose that $CS(L)$ is pseudo-0-distributive .

we have to prove that L is pseudo-0-distributive.

Let $x, y, z \in L$ such that $x \wedge y = 0$ and $x \wedge z = 0$.

To prove $(x \vee y) \wedge z = y \wedge z$.

Therefore, we have $(x \wedge y] = \{0\}$ and $(x \wedge z] = \{0\}$.

Therefore, $(x] \wedge (y] = \{0\}, (x] \wedge (z] = \{0\}$.

Therefore, $((x] \vee (y]) \wedge (z] = (y] \wedge (z]$ as $CS(L)$ is pseudo-0-distributive

which implies that $(x \vee y] \wedge (z] = (y] \wedge (z]$.

which implies that $((x \vee y) \wedge z] = (y \wedge z]$.

which implies $(x \vee y) \wedge z = y \wedge z$

Hence, L is pseudo 0-distributive.

3.8 Theorem

If $CS(L)$ satisfies general disjointness condition ,then L also satisfies the general disjointness condition.

Proof

Suppose that $CS(L)$ satisfies general disjointness condition.

we have to prove that L satisfies general disjointness condition.

That is, to prove that $x \wedge y = 0$ and $(x \vee y) \wedge z = 0$ implies that $x \wedge (y \vee z) = 0$ for every $x, y, z \in L$.

Let $x, y, z \in L$ such that $x \wedge y = 0$ and $(x \vee y) \wedge z = 0$.

which implies that $(x \wedge y] = \{0\}$ and $((x \vee y) \wedge z] = \{0\}$.

That is, $(x] \wedge (y] = \{0\}$ and $((x] \vee (y]) \wedge (z] = \{0\}$.

Therefore, $(x \wedge ((y] \vee (z])) = \{0\}$ as $CS(L)$ satisfies general disjointness condition.

which implies $(x \wedge (y \vee z)] = \{0\}$.

which implies $x \wedge (y \vee z) = 0$.

Therefore, L satisfies the general disjointness condition.

Remark

Proving the converse of theorem 3.1,3.3,3.6,3.7,3.8 remains open.

References

1. Arivukkarasu.J and Vethamanickam.A.,On 0-supermodular lattices, Mathematical Sciences International Research Journal, Volume, Vol-3,Issue-2,748-754 (2014).
2. Chen, C.C and Koh, K.M., On the lattice of convex sublattices of a finite lattice, Nantha Math.,5, 92-95(1972)
3. Gratzer,G., General lattice theory, Birkhauser Verlag, Basel,(1978).
4. Iqbalunnisa,W.B. Vasantha Kandasamy, Florentine smarandache,Supermodular lattices,Educational PublisherInc.Ohio,(2012).
5. Koh, K.M.,On the lattice of convex sublattices of a lattice, Nantha Math., 6,18-37(1972).
6. Koh, K.M., On the complementation of the CS(L) of a lattice L, Tamkang J. Math., 7,145-150(1976).

7. Lavanya, S., Parameshwara Bhatta, S., A New approach to the lattice of convex sublattices of a lattice, Algebra Univ,35,63-71(1996).
8. Marmazeev, V.I., The lattice of convex sublattices of a lattice (Russian), Ordered sets and lattices,9,50-58,110-111, (1986) Saratov Gos Univ., Saratov.
9. Ramana Murty, P.V., On the lattice of convex sublattices of a lattice, Southeast Asian Bulletin of Mathematics26,51-55 (2003).
10. Subbarayan,R., Equational class-like properties of 0-distributive lattices, $J\overline{nana}bha$, vol.52(2)(2022),73-76.