Dynamical System Induced by Tensor Product of Composition and Multiplication Operators on Tensor Product of Bloch Space of Continuous Functions

R. Gowri Manohari[1], P. Chandra Kala[2] and V. Anusuya[3]

[1]Research Scholar (Part Time - 18223152092016), Department of Mathematics, S.T. Hindu College, Nagercoil - 629 002, Tamil Nadu, India.

[2]Assistant Professor and Head, Sree Devi Kumari Women's College, Kuzhithurai, Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli - 627 012, Tamil Nadu, India.

[3]Assistant Professor, Department of Mathematics, S.T. Hindu College, Nagercoil - 629 002, Tamil Nadu, India.

Article Info
Page Number: 13460 - 13466
Publication Issue:
Vol 71 No. 4 (2022)

Abstract

Tensor product of composition and multiplication operators induce dynamical system on tensor product of Bloch space of continuous functions $C\mathcal{B}_0(X)$ or $C\mathcal{B}_b(X)$.

Article History
Article Received: *15 June2022*
Revised:*30 July2022*
Accepted: *25 August2022*

Key words- composition, Multiplication, operators, dynamical ,functions

1 Introduction

Let X and Y represents compact Hausdroff spaces and $C(X,Y)$ represents the space of all continuous complex valued functions from X and Y. If X equals to Y, we write $C(X)$ for $C(X,Y)$. Assume that $X \otimes Y$ is the tensor product of X and Y. Then each mapping $\tau{:}X{\to}X$ produces a linear transformation C_τ from $C(X,Y)$ itself denoted as $C_\tau f = f \circ \tau$ for every $f \in C(X,Y)$ and it is referred to as a composition operator on $C(X,Y)$ induced by τ. Consider $\vartheta{:}X{\to}C$ to be a mapping. The scalar multiplication then produces a linear transformation M_ϑ from $C(X,Y)$ itself defined as $M_\vartheta f = \vartheta f$ for any $f \in C(X,Y)$ where the product of functions is expressed point-wise and is known as a multiplication operator on $C(X,Y)$. Let $\vartheta_t{:}X{\to}\mathbb{R}$ defined by $\vartheta_t(x) = e^{th(x)}$ for all $t \in \mathbb{R}$ and $x \in X$ where $h \in C_b(X,\mathbb{R})$. Also $\tau_t{:}\mathbb{R}{\to}\mathbb{R}$ is defined by $\tau_t(\omega) = \tau + \omega$ the self map .

Let G be a topological group with e as the identity, let X be a topological space and $\vartheta{:}G \times X{\to}X$ be the continuous map such that

(i) $\vartheta(e,x) = x$ for every $x \in X$

(ii) $\vartheta(s + t,x) = \vartheta(s,\vartheta(t,x))$ for every $t,s \in G,x \in X$.

Then the triple (G,X,ϑ) is called a transformation group, X is a state space. If $G = (\mathbb{R}, +)$ the corresponding transformation group is called a dynamical system. The transformation group $(\mathbb{R},X,\vartheta)$ is known as continuous dynamical system. If X is a Banach space and $\vartheta(t,\alpha x + \beta y) = \alpha\vartheta(t,x) + \beta\vartheta(t,y)$, for $t \in \mathbb{R},\alpha,\beta \in C$, $x,y \in X$ the $(\mathbb{R}^+,X,\vartheta)$ is called a linear dynamical system .

[1] Corresponding author : kala.mathchandra@gmail.com

2000 AMS Subject Classification Code : 46MO5, 47A80, 47B37, 47B38.
Key words and Phrases : Tensor product operator, composition operators, multiplication operator, dynamical system and Bloch space.

The Bloch space, named for French mathematician Andre Bloch and abbreviated $\mathcal{B}$ in complex analysis, is the space of holomorphic functions f defined on the open unit disc $\mathcal{D}$ in the complex plane, such that the function $(1 - |z|^2)|f'(z)|$ is bounded. $\mathcal{B}$ is a Banach space, with the norm defined by $\parallel f \parallel_{\mathcal{B}} = |f(0)| + \sup_{z \in \mathcal{D}}(1 - |z|^2)|f'(z)|$. This is known as the Bloch norm and the Bloch space's constituents are known as Bloch functions. We define the set $\mathcal{B} = \{f \in H(D); \sup_{|z| < 1} \{(1 - |z|^2)|f'(z)| < \infty\}\}$.

The seminorm $\parallel f \parallel_{\mathcal{B}} = \sup_{|z| < 1} \{(1 - |z|^2)|f'(z)|\}$ and the norm $\parallel f \parallel_{Bloch} = |f(0)| + \parallel f \parallel_{\mathcal{B}}$. The Bloch space is the set $\mathcal{B}$ with the norm $\parallel \cdot \parallel_{Bloch}$.

2 Dynamical system induced by tensor product of composition and multiplication operators on tensor product of Bloch space of continuous functions

Let X be a Hausdorff topological space and $C(X,E)$ be a collection of continuous functions from X into E. Consider E to be a collection of non-negative holomorphic functions on X. If $\mathcal{B}$ is a set of Bloch functions on X such that for each $x \in X$, there is some $b \in \mathcal{B}$ for which $b(x) > 0$ we write $\mathcal{B} > 0$. Let $hc(X,E)$ represents the set of all holomorphic functions from X into E. If $\mathcal{B}$ is a Bloch function system on X, then the pair $(X,\mathcal{B})$ is known as the Bloch topological system. Each Bloch topological system $(X,\mathcal{B})$ is associated with the Bloch space of continuous E-valued functions defined as $C\mathcal{B}_0(X,E) = \{f \otimes g \in C(X,E) \otimes C(X,E); b[f \otimes g]$ vanishes at infinity on X for each $b \in \mathcal{B}\}$.
$C\mathcal{B}_b(X,E) = \{f \otimes g \in C(X,E) \otimes C(X,E)$; $b[f(x) \otimes g(x)]$ is bounded in E for all $b \in \mathcal{B}\}$.

Let $b \in \mathcal{B}, q \in hc(X,E)$ and $f \otimes g \in C(X,E) \otimes C(X,E)$. If we define $\parallel f \otimes g \parallel_b = \sup_{|z|<1}\{(1 - |z|^2)|b'(z)||f(x)||g(y)|; x \in X, y \in Y, z \in D\}$ then $\parallel \cdot \parallel_b$ can be regarded as a seminorm on either $C\mathcal{B}_0(X,E)$ or $C\mathcal{B}_b(X,E)$ and the family $\{\parallel \cdot \parallel_{b,c}; b \in \mathcal{B}, c \in hc(X,E)\}$ of seminorms defines a Hausdorff locally convex topology on each of these spaces. This topology will be denoted by ω_b and the vector spaces $C\mathcal{B}_0(X,E)$ and $C\mathcal{B}_b(X,E)$ endowed with ω_b are called the Bloch space of vector-valued continuous functions. It has a basis of closed absolutely convex neighbourhoods of the origin of the form

$\mathcal{N}_{b,c} = \{f \otimes g; f,g \in \mathcal{CB}_0(X,Y) \ \ni \| f \otimes g \|_{b,c} \leq 1\}$. Also $\mathcal{CB}_0(X,E)$ is a closed subspace of $\mathcal{CB}_b(X,E)$.

Theorem 2.1. *Let* $\tau: X \to X$ *and* $\vartheta_t: X \to C$ *be a continuous functions then* $C_\tau f \otimes M_{\vartheta_t} g$ *is bounded for every* $t \in \mathbb{R}, f \otimes g \in \mathcal{CB}_0(X) \otimes \mathcal{CB}_0(X)$.

Proof. To prove that $C_\tau f \otimes M_{\vartheta_t} g$ is bounded. It is enough to prove that $C_\tau f \otimes M_{\vartheta_t} g$ is continuous at the origin.

We claim that $(C_\tau \otimes M_{\vartheta_t})(\mathcal{N}_{b,c}) \subseteq \mathcal{N}_{b,c}$. For, $f \otimes g \in \mathcal{N}_{b,c}$ we have

$$\begin{aligned}
\| C_\tau f \otimes M_{\vartheta_t} g \|_b &= \sup_{|z|<1} \{(1-|z|^2)|b'(z)||C_\tau f(x)||M_{\vartheta_t} g(y)|; x \in X, y \in Y, z \in \mathcal{D}\} \\
&= \sup_{|z|<1} \{(1-|z|^2)|b'(z)||(f \circ \tau)(x)||\vartheta_t(y) g(y)| x \in X, y \in Y, z \in D\} \\
&= \sup_{|z|<1} \{(1-|z|^2)|b'(z)||f(\tau(x))||e^{th(y)} g(y)|\} \\
&\leq \sup_{|z|<1} \{(1-|z|^2)|b'(z)||f(x)||e^{|t|}|h(y)| \ g(y)|\} \\
&= \sup_{|z|<1} \{(1-|z|^2)|b'(z)||f(x)||g(y)| \text{ as } t \to 0\} \\
&= \| f \otimes g \|_b \leq 1
\end{aligned}$$

Therefore $\| C_\tau f \otimes M_{\vartheta_t} g \|_b \leq 1$.

Therefore $C_\tau f \otimes M_{\vartheta_t} g$ is continuous at the origin.

Therefore $C_\tau f \otimes M_{\vartheta_t} g$ is bounded.

Theorem 2.2. *Let* $B(E)$ *be the Banach algebra of all bounded linear operators on* E. *Let* $C^\infty(X, B(E))$ *be the space of bounded functions. Let* $c_\alpha(\tau_{t_\alpha})$ *converges to* $c(\tau_t)$ *in* $C^\alpha(X, B(E))$ *and let* b_α *be a sequence converging to* b *in* $\mathcal{CB}_0(X,E)$. *Then the product* $b_\alpha c_\alpha(\tau_{t_\alpha})$ *converges to* $bc(\tau_t)$ *in* $\mathcal{CB}_0(X,E) \otimes \mathcal{CB}_0(X,E)$.

Proof. Let $c_\alpha(\tau_{t_\alpha})$ converges to $c(\tau_t)$ in $C^\infty(X, B(E))$ and let b_α be a sequence converging to b in $\mathcal{CB}_0(X,E)$.

To prove that $b_\alpha c_\alpha(\tau_{t_\alpha})$ converges to $bc(\tau_t)$ in $\mathcal{CB}_0(X,E) \otimes \mathcal{CB}_0(X,E)$. Now,

$$
\begin{aligned}
\| b_n c_n(\tau_{t_n}) - bc(\tau_t) \|_B &= \sup_{|z|<1} \{(1-|z|^2)\left|[b_n c_n(\tau_{t_n} - bc(\tau_t)]'(z)\right|; z \in \mathcal{D}\} \\
&= \sup_{|z|<1} \{(1-|z|^2)|[b_n c_n(\tau_{t_n})]'(z) - [bc(\tau_t)]'(z)\} \\
&= \sup_{|z|<1} \{(1-|z|^2)|b_n'(z) c_n(\tau_{t_n}(z)) + b_n(z)(c_n \tau_{t_n})'(z) \\
&\quad - b'(z)c(\tau_t(z)) - b(z)[c(\tau_t)]'(z))\} \\
&= \sup_{|z|<1} \{(1-|z|^2)|b_n'(z) c_n(\tau_{t_n}(z)) + b_n(z)(c_n \tau_{t_n})'(z) \\
&\quad - b_n(z)(c\tau_t)'(z) + b_n(z)(c\tau_t)'(z) - b_n'(z)c(\tau_t(z)) \\
&\quad + b_n'(z)c(\tau_t(z)) - b'(z)c(\tau_t(z)) - b(z)[c(\tau_t)]'(z)|; z \in \mathcal{D}\} \\
&= \sup_{|z|<1} \{(1-|z|^2)|b_n'(z) c_n(\tau_{t_n}(z)) - b_n'(z)c(\tau_t(z)) \\
&\quad + b_n(z)(c_n \tau_{t_n})'(z) - b_n(z)(c\tau_t)'(z) + b_n(z)(c\tau_t)'(z) \\
&\quad - b(z)[c(\tau_t)]'(z) + b_n'(z)c(\tau_t(z)) - b'(z)c(\tau_t(z))|; z \in \mathcal{D}\} \\
&= \sup_{|z|<1} \left[(1-|z|^2)|b_n'(z)|\left|c_n(\tau_{t_n}(z)) - c(\tau_t(z))\right|\right] \\
&\quad + \sup_{|z|<1} \left[(1-|z|^2)|b_n(z)|\left|(c_n(\tau_{t_n}) - c(\tau_t))'(z)\right|\right] \\
&\quad + \sup\left[(1-|z|^2)\left|(c\tau_t)'(z)\right|\left|b_n(z) - b(z)\right|\right] \\
&\quad + \sup_{|z|<1} \left[(1-|z|^2)\left|c(\tau_t(z))\right|\left|(b_n - b)'(z)\right|\right] \\
&= \| b_n \|_B \left|c_n(\tau_{t_n}(z)) - c(\tau_t(z))\right| + |b_n(z)| \, \| c_n(\tau_{t_n}) - c(\tau_t) \|_B \\
&\quad + \| c(\tau_t) \|_B |b_n(z) - b(z)| + \left|c(\tau_t(z))\right| \, \| b_n - b \|_B \to 0
\end{aligned}
$$

as $\left|c_n(\tau_{t_n}(z)) - c(\tau_t(z))\right| \to 0$, $\| c_n(\tau_{t_n}) - c(\tau_t) \|_B \to 0$, $\| b_n - b \|_B \to 0$ and $|b_n(z) - b(z)| \to 0$. Therefore $\| b_n c_n(\tau_{t_n}) - bc(\tau_r) \|_B \to 0$. Therefore $b_\alpha c_\alpha(\tau_{t_\alpha})$ converges to $bc(\tau_t)$ in $C\mathcal{B}_0(X,E) \otimes C\mathcal{B}_0(X,E)$.

Theorem 2.3. *Let* $\nabla: \mathbb{R} \times C\mathcal{B}_0(\mathbb{R}) \otimes C\mathcal{B}_0(\mathbb{R}) \to C(\mathbb{R}) \otimes C(\mathbb{R})$ *be the function defined by* $\nabla(t, f \otimes g) = (C_{\tau_t} \otimes M_{\vartheta_t})(f \otimes g)$ *for all* $t \in \mathbb{R}$ *and* $f \otimes g \in C\mathcal{B}_0(\mathbb{R}) \otimes C\mathcal{B}_0(\mathbb{R})$ *then* ∇ *is a linear dynamical system on* $C\mathcal{B}_0(\mathbb{R}) \otimes C\mathcal{B}_0(\mathbb{R})$.

Proof. Since $C_{\tau_t} \otimes M_{\vartheta_t}$ is a tensor product on $C\mathcal{B}_0(\mathbb{R}) \otimes C\mathcal{B}_0(\mathbb{R})$ for all $t \in \mathbb{R}$ and $f \otimes g \in C\mathcal{B}_0(\mathbb{R}) \otimes C\mathcal{B}_0(\mathbb{R})$. It can be easily seen that $\nabla(0, f \otimes g) = f \otimes g$ and $\nabla(s+t, f \otimes g) = \nabla(s, \nabla(t, f \otimes g))$.

In order to show that $\nabla(t, f \otimes g)$ is a dynamical system on $C\mathcal{B}_0(\mathbb{R}) \otimes C\mathcal{B}_0(\mathbb{R})$. It is enough to prove that ∇ is continuous.

Let $g_\alpha \to g$ and let $(t_\alpha, f_\alpha \otimes g_\alpha)$ be a set in $\mathbb{R} \times C\mathcal{B}_0(\mathbb{R}) \otimes C\mathcal{B}_0(\mathbb{R})$ such that $(t_\alpha, f_\alpha \otimes g_\alpha) \to (t, f \otimes g)$. We shall prove that, $\nabla(t_\alpha, f_\alpha \otimes g_\alpha) \to \nabla(t, f \otimes g)$. Now

$$\begin{aligned}
\| \nabla(t_\alpha, f_\alpha \otimes g_\alpha) \quad & -\nabla(t, f \otimes g) \|_{\mathcal{B}} = \| C_{\tau_{t_\alpha}} \otimes M_{\vartheta_{t_\alpha}})(f_\alpha \otimes g_\alpha) - (C_{\tau_t} \otimes M_{\vartheta_t})(f \otimes g) \|_{\mathcal{B}} \\
& = \sup_{|z|<1} \{(1-|z|^2)\left|\left[(C_{\tau_{t_\alpha}} f_\alpha)(M_{\vartheta_{t_\alpha}} g_\alpha) - (C_{\tau_t} f)(M_{\vartheta_t} g)\right]'(z)\right|\} \\
& = \sup_{|z|<1} \{(1-|z|^2)\left|\left[(f_\alpha \circ \tau_{t_\alpha})(\vartheta_{t_\alpha} g_\alpha)\right]'(z) - \left[(f \circ \tau_t)(\vartheta_t g)\right]'(z)\right|\} \\
& = \sup_{|z|<1} \{(1-|z|^2)|f_\alpha(\tau_{t_\alpha})(z)\vartheta_{t_\alpha}(z)g'_\alpha(z) + f_\alpha(\tau_{t_\alpha})(z)\vartheta'_{t_\alpha}(z)g_\alpha(z) \\
& + \vartheta_{t_\alpha}(z)g_\alpha(z)f'_\alpha(\tau_{t_\alpha})\tau'_{t_\alpha}(z) - f(\tau_t(z))\vartheta_t(z)g'(z) - f(\tau_t(z))g(z)\vartheta'_t(z) \\
& - \vartheta_t(z)g(z)f'(\tau_t(z))\tau'_t(z)|\} \\
& = \sup_{|z|<1} \{(1-|z|^2)|f_\alpha(\tau_{t_\alpha})(z)\vartheta_{t_\alpha}(z)g'_\alpha(z) - f_\alpha(\tau_{t_\alpha})(z)g'(z)\vartheta_{t_\alpha}(z) \\
& + f_\alpha(\tau_{t_\alpha})(z)\vartheta_{t_\alpha}(z)g'(z) + f_\alpha(\tau_{t_\alpha})(z)\vartheta'_{t_\alpha}(z)g_\alpha(z) - f_\alpha(\tau_{t_\alpha}(z))\vartheta'_{t_\alpha}(z)g(z) \\
& + f_\alpha(\tau_{t_\alpha}(z))\vartheta'_{t_\alpha}(z)g(z) + \vartheta_{t_\alpha}(z)g_\alpha(z)f'_\alpha(\tau_{t_\alpha})\tau'_{t_\alpha}(z) \\
& - \vartheta_{t_\alpha}(z)g_\alpha(z)f'(\tau_t(z))\tau'_t(z) + \vartheta_{t_\alpha}(z)g_\alpha(z)f'(\tau_t(z))\tau'_t(z) \\
& - f(\tau_t(z))\vartheta_t(z)g'(z) - f(\tau_t(z))g(z)\vartheta'_t(z) - \vartheta_t(z)g(z)f'(\tau_t(z))\tau'_t(z)|\} \\
& = \| g_\alpha - g \|_B |f_\alpha(\tau_{t_\alpha}(z))||\vartheta_{t_\alpha}(z)| + \| g \|_B |\vartheta_{t_\alpha}(z)||[f_\alpha(\tau_{t_\alpha}) - f(\tau_t)](z)| \\
& + \| \vartheta_{t_\alpha} \|_B |f_\alpha(\tau_{t_\alpha}(z))||(g_\alpha - g)(z)| + \| \vartheta_{t_\alpha} \|_B |g(z)||[f_\alpha(\tau_{t_\alpha}) - f(\tau_t)](z)| \\
& + \| f_\alpha(\tau_{t_\alpha}) - f(\tau_t) \|_B |\vartheta_{t_\alpha}(z)||g_\alpha(z)| + \| f(\tau_t) \|_B |\vartheta_{t_\alpha}(z)||(g_\alpha - g)(z)| \to 0
\end{aligned}$$

as $g_\alpha \to g$ and $f_\alpha(\tau_{t_\alpha}) \to f(\tau_t)$. Therefore $\nabla(t, f \otimes g)$ is a dynamical system on $\mathcal{CB}_0(\mathbb{R}) \otimes \mathcal{CB}_0(\mathbb{R})$.

Next we have to prove that ∇ is linear. That is to prove that

$\nabla[t, \alpha(f \otimes g) + \beta(i \otimes j)] = \alpha\nabla(t, f \otimes g) + \beta\nabla(t, i \otimes j)$

for all $t \in \mathbb{R}, \alpha, \beta \in C$, $f \otimes g$ and $i \otimes j \in \mathcal{CB}_0(\mathbb{R}) \otimes \mathcal{CB}_0(\mathbb{R})$. Therefore $\nabla(t, f \otimes g)$ is a linear dynamical system on $\mathcal{CB}_0(\mathbb{R}) \otimes \mathcal{CB}_0(\mathbb{R})$.

Theorem 2.4. *Let* $\tau: X \to X$ *and* $\vartheta_t: X \to C$ *be a continuous functions then* $M_{\vartheta_t} f \otimes C_\tau g$ *is bounded for every* $t \in \mathbb{R}$, $f \otimes g \in \mathcal{CB}_0(X) \otimes \mathcal{CB}_0(X)$.

Proof. To prove that $M_{\vartheta_t} \otimes C_\tau g$ is bounded. It is enough to prove that $M_{\vartheta_t} f \otimes C_\tau g$ is continuous at the origin.

We claim that $(M_{\vartheta_t} \otimes C_\tau)(\mathcal{N}_{b,c}) \subseteq \mathcal{N}_{b,c}$. For, $f \otimes g \in \mathcal{N}_{b,c}$ we have

$$\begin{aligned}
\| M_{\vartheta_t} f \otimes C_\tau g \|_b \quad & = \sup_{|z|<1} \{(1-|z|^2)|b'(z)||M_{\vartheta_t} f(x)||C_\tau g(y)|; x \in X, y \in Y, z \in \mathcal{D}\} \\
& = \sup_{|z|<1} \{(1-|z|^2)|b'(z)||f(x)|g(y)| \text{ as } t \to 0\} \\
& = \| f \otimes g \|_b \le 1
\end{aligned}$$

Therefore $\| M_{\vartheta_t} f \otimes C_t g \|_b \le 1$. Therefore $M_{\vartheta_t} f \otimes C_\tau g$ is continuous at the origin. Therefore $M_{\vartheta_t} \otimes C_\tau g$ is bounded.

Theorem 2.5. *Let* $\nabla: \mathbb{R} \times \mathcal{CB}_0(\mathbb{R}) \otimes \mathcal{CB}_0(\mathbb{R}) \to C(\mathbb{R}) \otimes C(\mathbb{R})$ *be the function defined by* $\nabla(t, f \otimes g) = M_{\vartheta_t} \otimes C_{\tau_t}(f \otimes g)$ *for all* $t \in \mathbb{R}$ *and* $f \otimes g \in \mathcal{CB}_0(\mathbb{R}) \otimes \mathcal{CB}_0(\mathbb{R})$ *then* ∇ *is a linear dynamical system on* $\mathcal{CB}_0(\mathbb{R}) \otimes \mathcal{CB}_0(\mathbb{R})$.

Proof. $M_{\vartheta_t} \otimes C_{\tau_t}$ is a tensor product on $C(\mathbb{R}) \otimes C(\mathbb{R})$ for every $t \in \mathbb{R}$ and $f \otimes g \in C\mathcal{B}_0(\mathbb{R}) \otimes C\mathcal{B}_0(\mathbb{R})$. We have $\nabla(0,f \otimes g) = f \otimes g$ and $\nabla(s+t,f \otimes g) = \nabla(s,\nabla(t,f \otimes g))$.

In order to show that $\nabla(t,f \otimes g)$ is a dynamical system on $C\mathcal{B}_0(\mathbb{R}) \otimes C\mathcal{B}_0(\mathbb{R})$.

It is enough to prove that ∇ is Continuous. Now

$$
\begin{aligned}
\| \nabla(t_\alpha,f_\alpha \otimes g_\alpha) \quad & -\nabla(t,f \otimes g) \|_B = \| \left(M_{\vartheta_{t_\alpha}} \otimes C_{\tau_{t_\alpha}}\right)(f_\alpha \otimes g_\alpha) - \left(M_{\vartheta_t} \otimes C_{\tau_t}\right)(f \otimes g) \|_B \\
&= \sup_{|z|<1} \{(1-|z|^2)\left|\left[M_{\vartheta_{t_\alpha}} f_\alpha)\left(C_{\tau_{t_\alpha}} g_\alpha\right) - \left(M_{\vartheta_t} f\right)\left(C_{\tau_t} g\right)\right]'(z)\right|\} \\
&= \sup_{|z|<1} \{(1-|z|^2)\left|\left[(\vartheta_{t_\alpha} f_\alpha)(g_\alpha \circ \tau_{t_\alpha})\right]'(z) - \left[(\vartheta_t f)(g \circ \tau_t)\right]'(z)\right|\} \\
&= \sup_{|z|<1} [(1-|z|^2)|(\vartheta_{t_\alpha} f_\alpha)(z)\{g'_\alpha(\tau_{t_\alpha})\tau'_{t_\alpha}(z)\} + g_\alpha(\tau_t)(z)\{\vartheta_{t_\alpha}(z) f'_\alpha(z) \\
&+ f_\alpha(z)\vartheta'_{t_\alpha}(z)\} - (\vartheta_t f)(z)\{g'(\tau_t)\tau'_t(z)\} - g(\tau_t(z))\{\vartheta_t(z) f'(z) + f(z)\vartheta'_t(z)\}|] \\
&= \sup_{|z|<1} [(1-|z|^2)|\vartheta_{t_\alpha}(z) f_\alpha(z) g'_\alpha(\tau_{t_\alpha}(z))\tau'_{t_\alpha}(z) + g_\alpha(\tau_{t_\alpha}(z))\vartheta_{t_\alpha}(z) f'_\alpha(z) \\
&+ g_\alpha(\tau_{t_\alpha})(z) f_\alpha(z)\vartheta'_{t_\alpha}(z) - \vartheta_t(z) f(z) g'(\tau_t(z))\tau'_t(z) - g(\tau_t(z))\vartheta_t(z) f'(z) \\
&- g(\tau_t(z)) f(z)\vartheta'_t(z)|] \\
&= \sup_{|z|<1} [(1-|z|^2)|\vartheta_{t_\alpha}(z) f_\alpha(z) g'_\alpha(\tau_{t_\alpha}(z))\tau'_{t_\alpha}(z) - \vartheta_{t_\alpha}(z) f(z) g'_\alpha(\tau_{\tau_t}(z))\tau'_{t_\alpha}(z) \\
&+ \vartheta_{t_\alpha}(z) f_{(z)} g'_\alpha(\tau_{t_\alpha}(z))\tau'_{t_\alpha}(z) + g_\alpha(\tau_{t_\alpha}(z))\vartheta_{t_\alpha}(z) f'_\alpha(z) - g_\alpha((\tau_{t_\alpha}(z))\vartheta_{t_\alpha}(z) f'(z) \\
&+ g_\alpha(\tau_{t_\alpha}(z))\vartheta_{t_\alpha}(z) f'(z) + g_\alpha(\tau_{t_\alpha}(z)) f_\alpha(z)\vartheta'_{t_\alpha}(z) - g_\alpha(\tau_{t_\alpha}(z)) f(z)\vartheta'_{t_\alpha}(z) \\
&+ g_\alpha(\tau_{t_\alpha}(z)) f(z)\vartheta'_{t_\alpha}(z) - \vartheta_t(z) f(z) g'(\tau_t(z))\tau'_t(z) \\
&- g(\tau_t(z))\vartheta_t(z) f'(z) - g(\tau_t(z)) f(z)\vartheta'_t(z)|\} \\
&= |\vartheta_{t_\alpha}(z)||f_\alpha(z) - f(z)| \| g_\alpha(\tau_{t_\alpha}) \|_B + |g_\alpha(\tau_{t_\alpha}(z))||\vartheta_{t_\alpha}(z)| \| f_\alpha - f \|_B \\
&+ |g_\alpha(\tau_{t_\alpha}(z)||f_\alpha(z) - f(z)| \| \vartheta_{t_\alpha} \|_B + |\vartheta_{t_\alpha}(z)||f(z)| \| g_\alpha(\tau_{t_\alpha}) - g(\tau_t) \|_B \\
&+ |\vartheta_{t_\alpha}(z)| \| f \|_B |g_\alpha(\tau_{t_\alpha}(z)) - g(\tau_t(z))| + |f(z)| \| \vartheta_{t_\alpha} \|_B |g_\alpha(\tau_{t_\alpha}(z)) - g(\tau_t(z))| \\
&\to 0
\end{aligned}
$$

as $f_\alpha \to f$ and $g_\alpha(\tau_{t_\alpha}) \to g(\tau_t)$. Therefore $\nabla(t,f \otimes g)$ is a dynamical system on $C\mathcal{B}_0(\mathbb{R}) \otimes C\mathcal{B}_0(\mathbb{R})$.

Next we have to prove that ∇ is linear. That is to prove that $\nabla(t,\alpha(f \otimes g) + \beta(i \otimes j)) = \alpha\nabla(t,f \otimes g) + \beta\nabla(t,i \otimes j)$ for all $t \in \mathbb{R}$, $\alpha,\beta \in C$, $f \otimes g$ and $i \otimes j \in C\mathcal{B}_0(\mathbb{R}) \otimes C\mathcal{B}_0(\mathbb{R})$.

Therefore $\nabla(t,f \otimes g)$ is a linear dynamical system on $C\mathcal{B}_0(\mathbb{R}) \otimes C\mathcal{B}_0(\mathbb{R})$.

Note 2.6. *All results in this section hold if we replace* $C\mathcal{B}_b(X,E)$ *as a substitute of* $C\mathcal{B}_0(X,E)$.

References

[1] Chandrakala,P., Senthil Kumar, D., and Panayarp, S. "Weighted composition operator and dynamical pan, system on waighted function spaces." FJMS, Vol 42, No. 2, (2010), 281-288.

[2] Manhas, J. S. "Weighted Composition Operators on Weighted Locally Convex Spaces of Analytic Functions." Southeast Asian Bulletin of Mathematics 29.1 (2005) : 127-140.

[3] Manhas, J. S. "Weighted composition operators and dynamical systems on weighted spaces of holomorphic functions on Banach spaces." Ann. Funct. Anal 4.2 (2013): 58-71.

[4] Rubel, Lee A., and Richard M. Timoney. “An extremal property of the Bloch space.” Proceedings of the American Mathematical Society 75.1 (1979): 45-49.
[5] Ryan, Raymond A., and R. a Ryan. Introduction to tensor products of Banach spaces. Vol. 73. London: Springer, 2002.
[6] Senthilkumar, D., and P. Chandra Kala. “Operators and dynamical system in measurable function spaces.” Global journal of mathematical sciences: Theory and practical 4.2 (2012): 107-114.
[7] SenthilKumar, D., P. Chandra Kala, and T. Prasad. “Tensor product operators induce Dynamical systems on weighted locally convex space.” Jordan Journal of Mathematics and statistics (JJMs) 6 (2013): 169-181.
[8] SenthilKumar, D., P. Chandra Kala, and T. Prasad. “Tensor product operators induce Dynamical systems on weighted locally convex space.” Jordan Journal of Mathematics and statistics (JJMs) 6 (2013): 169-181.
[9] Singh, Raj Kishor, and Jasbir Singh Manhas. Composition operators on function spaces. Elsevier, 1993.
[10] Singh, R. K., and J. S. Manhas. “Multiplication operators and dynamical systems on weighted spaces of cross-sections.” Proceedings of the American Mathematical Society 119.2 (1993): 547-554.