Investigating Solutions of Volterra Integral Equations Using the Successive Approximations

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Abstract: This paper explores the subject of integral equations with an emphasis on Volterra integral equations (VIEs). Through the use of the method of successive approximations (MOSA), the main goal of this work is to investigate and analyze the solutions to (VIEs). Furthermore, we show how to transform a normal differential equation's initial value problem into a (VIEs) and a Volterra integral differential equation. We seek to provide a greater knowledge of the behavior and solutions of (VIEs) by a thorough investigation of the proposed method.

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1. Introduction

Integral equations are the foundation of mathematical and scientific inquiry. They permeate a wide range of fields and produce useful discoveries that influence how we perceive the natural world. The pervasiveness of integral equations highlights their crucial role in explaining complicated phenomena, from the intricate worlds of physics and engineering to the intricate interplay of economic systems [1-7]. By describing relationships between functions as integrals, these equations go beyond ordinary differential equations and provide a potent method for modeling situations where interactions between variables are inextricably linked.

Volterra integral equations (VIEs) stand out as a particularly fascinating subset among the various integral equation forms because they may describe the dynamic evolution of systems through time [8]. Researchers can capture complex temporal correlations that evade traditional differential equations using these equations, which go right to the heart of dynamic phenomena. The modeling of complex processes like population dynamics [9], chemical reactions [10], and signal processing [11], where the past interactions between entities have a significant impact, is where (VIEs) find their strongest application. They are excellent instruments for recording the complex behaviors of systems that develop in response to a history of impacts because of their adaptability in tolerating memory effects and previous interactions.

The purpose of this work is to investigate the solutions of (VIEs) using the method of successive approximations, as well as to demonstrate the process of converting a normal differential equation to a Volterra integral equation.

2. Integral equations

Integral equations are a branch of mathematics that deals with equations involving unknown functions within integrals. Unlike ordinary differential equations (ODEs) that involve derivatives of unknown functions, integral equations involve integrals of unknown functions. They have applications in various fields such as physics, engineering, economics, and biology, where the problem can be naturally described in terms of a relationship between a function and an integral of that function.

2.1. Definition

Integral equation is an equation that shows the function of an unknown $u(\tau)$under the integral logo. A typical form of integral equation in the $u(\tau)$ is the model

$$u(\tau) = f(\tau) + \int_{\alpha(\tau)}^{\beta(\tau)} W(\tau, t)u(t)dt, \tag{1}$$

where $W(\tau, t)$ is called the nucleus of the integral equation, and $\alpha(\tau)$ and $\beta(\tau)$ are the limits of the integral. In (1), it is easily observed that the unrecognized $u(\tau)$ function appears under the combined mark as mentioned above, and exits the combined mark in most other cases as discussed later. It is important to note that the kernel $W(\tau, t)$ and the function $f(\tau)$ in equation (1) are predefined. Our goal is to identify $u(\tau)$ that satisfies equation (1) and this can be achieved using different techniques that will be discussed in our paper.

Integral equations are classified into two main types: Fredholm equations and Volterra equations. The distinction between these types is based on the limits of integration in the integral equation. Fredholm equations have fixed limits of integration, while Volterra equations have variable limits.

There are two broad categories of integral equations:

2.2. Linear Integral Equations: In these equations, the unknown function appears linearly within the integral. Linear integral equations are further divided into two types:

I. Fredholm Integral Equations: These involve the unknown function in the integral equation itself. They are commonly encountered in problems involving boundary value problems, scattering phenomena, and eigenvalue problems.

Given the Fredholm most standard form for the model of linear equations

$$\varphi(\tau)u(\tau) = f(\tau) + \lambda \int_a^b K(\tau, t)u(t)dt, \tag{2}$$

when the integral limits a and b are constants, the unknown function $u(\tau)$ appears linearly under the integration sign. If the function $\varphi(\tau) = 1$, equation (2) simply becomes

$$u(\tau) = f(\tau) + \lambda \int_a^b K(\tau, t)u(t)dt, \tag{3}$$

and this equation is called FIE of type II; whereas if $\varphi(\tau) = 0$, then (2) yields

$$f(\tau) + \lambda \int_a^\tau K(\tau, t)u(t)dt = 0, \tag{4}$$

which is called FIE of the first kind [12].

II. Volterra Integral Equations (VIEs): These involve the unknown function in the kernel (the function being integrated) of the integral equation. They often appear in problems related to population dynamics, chemical reactions, and certain physical processes.

The most standard form of linear VIEs is the model

$$\varphi(\tau)u(\tau) = f(\tau) + \lambda \int_a^\tau K(\tau, t)u(t)dt, \tag{5}$$

when the integral limits are τ, the unknown function $u(\tau)$ appears linearly under the integration sign. If the function $u(\tau) = 1$, equation (5) simply becomes

$$u(\tau) = f(\tau) + \lambda \int_a^\tau K(\tau, t)u(t)dt, \tag{6}$$

this is known as the (VIEs) of the second type; whereas if $\varphi(\tau) = 0$, equation (5) becomes

$$f(\tau) + \lambda \int_a^\tau K(\tau, t)u(t)dt = 0, \tag{7}$$

this is known as Volterra's first type equation [13].

2.2. Nonlinear Integral Equations: In these equations, the unknown function appears nonlinearly within the integral. Solving nonlinear integral equations can be more complex and challenging, often requiring numerical methods or approximations.

It is important to note that integral equations arise in engineering, physics, chemistry, and biological problems. Many of the initial value problems and limitations associated with normal and partial differential equations can be cast in integral equations of Volterra and Fredholm types, respectively. If the unknown function $u(\tau)$ that appears under the integration sign is listed in the functional model $F(u(\tau))$ as the power of $u(\tau)$ is no longer the unit, for example $F(u(\tau))=u^n(x)$, $n \neq 1$, or $sinu(\tau)$etc., then the integral equations of Volterra and Fredholm are classified as integral parts of nonlinear equations [14]. As for the examples, the integral of the following equations is an integral part of the non-linear equations:

$$u(\tau) = f(\tau) + \lambda \int_a^\tau K(\tau, t)u^2(t)dt,$$

$$u(\tau) = f(\tau) + \lambda \int_a^\tau K(\tau, t)\sin\big(u(t)\big)\,dt,$$

$$u(\tau) = f(\tau) + \lambda \int_a^\tau K(\tau, t)\,\mathrm{In}\big(u(t)\big)\,dt.$$

Then, if we set $f(\tau) = 0$, the resulting equations is called homogeneous equation complementary, otherwise called the integration of heterogeneous equation.

3. The Method of Successive Approximations

In this section, we describe the Method of Successive Approximations as a potent method for resolving (VIEs). We outline a step-by-step methodology for this approach and demonstrate its convergence characteristics. We demonstrate the effectiveness of the Method of

Successive Approximations in obtaining approximations to the Volterra integral equations through examples.

In this way, the replacement of the function as $u(\tau)$ within the merged brand equation Volterra (1) is not known as any selective function with a real value of selective $u_0(\tau)$, called zeroth approximation [15]. This replacement will give the first approximation $u_1(\tau)$ by

$$u_1(\tau) = f(\tau) + \lambda \int_0^\tau K(\tau, t) u_0(t) dt. \quad (8)$$

Obviously $u_1(\tau)$ is continuous if $f(\tau), K(\tau, t)$ and $u_0(\tau)$ are continuous. The second approximation $u_2(\tau)$ can be obtained similarly by replacing $u_0(\tau)$ in equation (8) with $u_1(\tau)$ obtained above. And we found

$$u_2(\tau) = f(\tau) + \lambda \int_0^\tau K(\tau, t) u_1(t) dt. \quad (9)$$

Continuing this way, we can get an infinite sequence of functions
$u_0(\tau), u_1(\tau), u_2(\tau), \ldots, u_n(\tau), \ldots$
the relationship will satisfy repetition

$$u_n(\tau) = f(\tau) + \lambda \int_0^\tau K(\tau, t) u_{n-1}(t) dt, \quad (10)$$

for $n = 1, 2, 3, \ldots$ and $u_0(\tau)$ the equivalent of any job has its real value specified. More commonly selected function for $u_0(\tau)$ are 0, 1, and τ. Thus, at the limit, the solution $u(\tau)$ of the equation (8) is obtained as

$$u(\tau) = \lim_{n \to \infty} u_n(\tau), \quad (11)$$

so that the resulting solution $u(\tau)$ is independent of the selection of zeros rounded $u_0(\tau)$. This process is very simple approximation. However, if we follow the method of successive approximation for Picard, we need to setup $u_0(\tau) = f(\tau)$ and specify $u_1(\tau)$ and the other rounding as follows [16]:

$$u_1(\tau) = f(\tau) + \lambda \int_0^\tau K(\tau, t) f(t) dt,$$

$$u_2(\tau) = f(\tau) + \lambda \int_0^\tau K(\tau, t) u_1(t) dt,$$

$$\vdots$$

$$u_{n-1}(\tau) = f(\tau) + \lambda \int_0^\tau K(\tau, t) u_{n-2}(t) dt,$$

$$u_n(\tau) = f(\tau) + \lambda \int_0^\tau K(\tau, t) u_{n-1}(t) dt. \quad (12)$$

4. Application and Examples

In order to support the ideas covered throughout the study, we provide examples and real-world applications of Volterra integral equations in this section.

Example 4.1. Solve the following Volterra integral equation:

$$u(\tau) = \tau + \lambda \int_0^\tau x u(t) dt, \lambda > 0.$$

Solution. To solve the given Volterra integral equation using the method of successive approximations, we will follow these steps:

Step 1: Initial Approximation

Let's start with the initial approximation: $u_0(\tau) = \tau$,

Step 2: Successive Approximations

We will use the formula for successive approximations:

$u_{n+1}(\tau) = x + \lambda \int_0^\tau \tau u_n(t)dt$,

Let's calculate the first few successive approximations:

Initial Approximation: $u_0(\tau) = \tau$, and $\lambda = 1$

$u_1(\tau) = \tau + \int_a^\tau \tau u_0(t)dt = \tau + \int_0^\tau \tau^2 dt = \tau + \frac{\tau^3}{3}$,

$u_2(\tau) = \tau + \int_a^\tau \tau u_1(t)dt = \tau + \int_0^\tau \left(\tau^2 + \frac{\tau^4}{3}\right) dt = \tau + \frac{\tau^3}{3} + \frac{\tau^5}{15}$,

$u_3(\tau) = \tau + \int_0^\tau \tau u_2(t)dt = \tau + \int_0^\tau \left(\tau^2 + \frac{\tau^4}{3} + \frac{\tau^6}{15}\right) dt = \tau + \frac{\tau^3}{3} + \frac{\tau^5}{15} + \frac{\tau^7}{105}$.

In general, the n-th approximation can be written as:

$u_n(\tau) = \tau + \sum_{k=1}^{n} \frac{\tau^{2k+1}}{(2k+1)!!}$.

Where $(2k+1)!!$ represents the double factorial of $(2k+1)$, which is the product of all odd integers up to $(2k+1)!!$.

So, the closed form series solution for the given Volterra integral equation is:

$u(\tau) = \lim_{n\to\infty} u_n(\tau) = \tau + \sum_{k=1}^{n} \frac{\tau^{2k+1}}{(2k+1)!!}$.

Example 4.2. Solve the following VIE

$$u(\tau) = -1 + e^\tau + \frac{1}{2}\tau^2 e^\tau - \frac{1}{2}\int_0^\tau tu(t)dt,$$

Solution. Consider the recurrence relation to round zeros $u(\tau)$ we choose $u(0) = 0$.

We next use the iteration formula

$$u_{n+1}(\tau) = -1 + e^\tau + \tau^2 e^\tau - \frac{1}{2}\int_0^\tau tu_n(t)(t)dt, n \geq 0$$

Substituting $u_0(\tau)$, we get

$u_1(\tau) = -1 + e^\tau + \tau^2 e^\tau$

$u_2(\tau) = -3 + \frac{1}{4}\tau^2 + e^\tau\left(3 - 2\tau + \frac{5}{4}\tau^2 - \frac{1}{4}\tau^3\right)$,

$u_3(\tau) = \tau\left(1 - \tau + \frac{\tau^2}{2!} - \frac{\tau^3}{3!} + \frac{\tau^4}{4!} + \cdots\right)$,

$u_{n+1}(\tau) = \tau\left(1 - \tau + \frac{\tau^2}{2!} - \frac{\tau^3}{3!} + \frac{\tau^4}{4!} + \cdots\right) = \tau e^{-\tau}$.

5. Converting initial value problem to volterra integral equation

Conversion from Normal Differential Equation to Volterra Integral Equation: One intriguing aspect of integral equations is their connection to ordinary differential equations. We demonstrate the systematic process of converting an initial value problem of a normal differential equation into a Volterra integral equation. Through a series of mathematical transformations, we establish a bridge between these two seemingly distinct mathematical formulations.

In this section, the problem will change from the initial value of the common differential equation to the combined VIE and VIE problems. For simplicity, we will use this process to issue the initial value of the second introductory class

$$y''(\tau)+p(\tau)y'(\tau)+q(\tau)y(\tau)=g(\tau),$$
$$y(0)=\alpha, y'(0)=\beta,$$

where α and β are constants. The $p(\tau)$, and $q(\tau)$, functions are analytical, and $g(\tau)$ continues during the discussion period. To achieve our goal we set it first

$$y''(\tau)=u(\tau),$$

where $u(\tau)$ is a continues function. Integrating both sides of $y''(\tau)=u(\tau)$ from 0 to τ yields

$$y''(\tau)-y'(0)=\int_0^\tau u(t)dt,$$

or equivalently

$$y'(\tau)=\beta+\int_0^\tau u(t)dt.$$

Integrating both sides of the last equation from 0 to τ, we get

$$y(\tau)-y(0)=\beta\,\tau+\iint_0^\tau u(t)dtdt$$

Remark 5.1. If

$$G(\tau)=\iint_0^\tau F(t)dtd\,\tau=\int_0^\tau(\tau-t)F(t)dt,$$
$$\frac{d}{d\,\tau}G(\tau)=\int_0^\tau F(t)dt.$$

Then, $y(\tau)$ is equivalently

$$y(\tau)=\alpha+\beta\,\tau+\int_0^\tau(\tau-t)u(t)dt.$$

Substituting $y''(\tau)$, $y'(\tau)$, and $y(\tau)$ into the initial value problem above yields the Voltera integral equation:

$$u(\tau)+p(\tau)\left[\beta+\int_0^\tau u(t)dt\right]+q(\tau)\left[\alpha+\beta\,\tau+\int_0^\tau(\tau-t)u(t)dt\right]=g(\,\tau).$$

Thus, the equation above can be written in the form of a standard integral equation Voltera:

$$u(\tau)=f(\tau)-\int_0^\tau W(\tau,t)u(t)dt,$$

where

$$W(\tau,t)=p(\tau)+q(\tau)(\tau-t),$$

and

$$f(\tau)=g(\tau)-[\beta p(\tau)+\alpha q(\tau)+\beta\tau q(\tau)].$$

Example 5.1. Convert the following initial value problem to the Voltera integral problem Equivalent:

$$y'(\tau)-2\tau y(\tau)=e^{\tau^2},\qquad y(0)=1.$$

We first set

$$y'(\tau) = u(\tau).$$

Integrating both side, using the initial condition $y(0) = 1$. we have

$$y(\tau) - y(0) = \int_0^\tau u(t)dt,$$

or equivalently

$$y(\tau) = 1 + \int_0^\tau u(t)dt.$$

Substituting $y'(\tau)$ and $y(\tau)$ into a given initial value problem gives the equivalent Voltera integral equation:

Conclusion

In conclusion, this study has presented a thorough investigation into the solutions of Volterra integral equations utilizing the method of successive approximations. Furthermore, we demonstrated the procedure of converting initial value problems of normal differential equations to Volterra integral equations and Volterra integral differential equations. We hope that by combining theoretical analysis, numerical examples, and practical applications, we have improved the reader's knowledge of this mathematical technique and their consequences in diverse scientific disciplines.

Future Research: This study offers up new directions for investigation, such as the investigation of more sophisticated ways for solving integral equations, the application of these techniques to more intricate and specialized integral equations, and the creation of algorithms for effective numerical implementation.

References

[1] Roy, K., Panja, S., Saha, M., & Parvaneh, V. (2020). An extended b-metric-type space and related fixed point theorems with an application to nonlinear integral equations. *Advances in Mathematical Physics*, *2020*, 1-7.

[2] Epelbaum, E., Gasparyan, A. M., Gegelia, J., Meißner, U. G., & Ren, X. L. (2020). How to renormalize integral equations with singular potentials in effective field theory. *The European Physical Journal A*, *56*, 1-8.

[3] Mundewadi, R. A., & Kumbinarasaiah, S. (2019). Numerical solution of Abel′ s integral equations using Hermite Wavelet. *Applied Mathematics and Nonlinear Sciences*, *4*(1), 169-180.

[4] Zheng, X., Kupresak, M., Verellen, N., Moshchalkov, V. V., & Vandenbosch, G. A. (2019). A Review on the Application of Integral Equation-Based Computational Methods to Scattering Problems in Plasmonics. *Advanced Theory and Simulations*, *2*(9), 1900087.

[5] Guo, R., Lin, Z., Shan, T., Li, M., Yang, F., Xu, S., & Abubakar, A. (2021). Solving combined field integral equation with deep neural network for 2-D conducting object. *IEEE Antennas and Wireless Propagation Letters*, *20*(4), 538-542.

[6] Solodusha, S., & Bulatov, M. (2021). Integral equations related to volterra series and inverse problems: Elements of theory and applications in heat power engineering. *Mathematics*, *9*(16), 1905.

[7] Deep, A., Abbas, S., Singh, B., Alharthi, M. R., & Nisar, K. S. (2021). Solvability of functional stochastic integral equations via Darbo's fixed point theorem. *Alexandria Engineering Journal*, *60*(6), 5631-5636.
[8] Ramirez, M. C. H. (2021). *Me, Myself and I: Time-Inconsistent Stochastic Control, Contract Theory and Backward Stochastic Volterra Integral Equations*. Columbia University.
[9] De Bonis, M. C., Laurita, C., & Sagaria, V. (2022). A numerical method for linear Volterra integral equations on infinite intervals and its application to the resolution of metastatic tumor growth models. *Applied Numerical Mathematics*, *172*, 475-496.
[10] Attary, M. (2018). On the numerical solution of nonlinear integral equation arising in conductor like screening model for realistic solvents. *Mathematical Sciences*, *12*, 177-183.
[11] Conte, D., & Paternoster, B. (2017). Parallel methods for weakly singular Volterra Integral Equations on GPUs. *Applied Numerical Mathematics*, *114*, 30-37.
[12] Wazwaz, A. M. (2015). *First Course In Integral Equations, A*. World Scientific Publishing Company.
[13] Burton, T. A. (2005). Volterra integral and differential equations. Elsevier.
[14] Moradi, S., Mohammadi Anjedani, M., & Analoei, E. (2015). On existence and uniqueness of solutions of a nonlinear Volterra-Fredholm integral equation. *International Journal of Nonlinear Analysis and Applications*, *6*(1), 62-68.
[15] Wazwaz, A. M., & Rach, R. (2016). Two reliable methods for solving the Volterra integral equation with a weakly singular kernel. *Journal of computational and Applied Mathematics*, *302*, 71-80.
[16] Manaa, S. A., Easif, F. H., & Mahmood, B. A. (2013). Successive Approximation Method for Solving Nonlinear Diffusion Equation with Convection Term. *IOSR Journal of Engineering*, *3*, 28-31.