Cyclic Coupled Fixed Point via Interpolative Kannan Type Contractions

Youssef El Bekri[1*], Mohamed Edraoui[2], Jamal Mouline[3], Abdelhafid Bassou[4]
[1*,2,3,4]Laboratory of Analysis, Modeling, and Simulation (LAMS), Department of Mathematics and Computer Sciences, Ben M'Sik Faculty of Sciences, Hassan II University of Casablanca, P.B 7955 Sidi Othmane, Casablanca, Morocco

Article Info
Page Number: 24 – 30
Publication Issue:
Vol 72 No. 2 (2023)

Article History
Article Received: *15 February 2023*
Revised: *20 April 2023*
Accepted: *10 May 2023*

Abstract: In this article, we introduce the notion of cyclic coupled Kannan type contraction via interpolation. Additionally, we prove the existence a strong coupled fixed point theorem for such mappings
Keywords: -Cyclic, Fixed Point, Interpolative Kannan Type Contraction, Metric space.

Introduction and Preliminaries

In the study of cyclic mappings, fixed point theorems provide conditions under which the existence and uniqueness of fixed points can be guaranteed. These conditions can vary depending on the specific class of cyclic mappings being considered.

For example, the Banach fixed point theorem guarantees the existence and uniqueness of a fixed point for contraction mappings on a complete metric space. On the other hand, the Brouwer fixed point theorem only guarantees the existence of at least one fixed point for continuous mappings on a compact, convex set.

There are also other fixed point theorems that guarantee the existence of a fixed point under weaker conditions or even multiple fixed points under certain conditions. For example, the Schauder fixed point theorem guarantees the existence of at least one fixed point for a continuous mapping on a closed, convex subset of a Banach space. And the Kakutani fixed point theorem guarantees the existence of a fixed point for a continuous mapping on a non-empty, convex, and compact subset of a locally convex topological vector space.

It is important to note that even if a fixed point theorem guarantees the existence of a fixed point for a given class of mappings, it does not necessarily provide a method for finding the fixed point. In practice, finding fixed points for a specific mapping may be difficult or even impossible.

Additionally, the uniqueness of fixed points may be very important in certain problems, in such cases, the fixed point theorems that guarantee the uniqueness of fixed points will be more useful.

In 2003, In their paper, Kirk et al. [4] introduced the concept of cyclical contractive mappings and extended the Banach fixed point result to the class of cyclic mappings. They generalized the notion of contractive mappings to cyclical contractive mappings and proved that these mappings also have a unique fixed point. This expanded the class of mappings for which a fixed point can be guaranteed and provided a new tool for studying the existence and uniqueness of fixed points in various mathematical contexts.

The concept of coupled fixed points was first introduced in the work of Guo et al. [13], Following the discovery of a coupled contraction mapping theorem by Bhaskar et al. [5], coupled fixed point results emerged in numerous subsequent studies, such as [5, 9, 10, 11, 12, 13]

Definition 1. *Let (E,d) be a metric space and let X and Y be two nonempty subsets of E.*

A mapping $T: X \cup Y \to X \cup Y$ is said to be a cyclic mapping provided that

$$T(X) \subseteq Y, T(Y) \subseteq X. \qquad (1)$$

A point $x \in X \cup Y$ is called a best proximity point if $d(x,Tx) = d(X,Y)$ where $d(X,Y) = \inf(d(x,y): x \in X, y \in Y)$.In 2011, Erdal Karapınar and Inci M. Erhan [1] proved.

The following fixed point theorem for a cyclic map.

Definition 2. *Let X and Y be non-empty subsets of a metric space (E,d). A cyclic map $T: X \cup Y \to X \cup Y$ is said to be a Kannan Type cyclic contraction if there exists $k \in \left(0, \frac{1}{2}\right)$ such that:*

$$d(Tx,Ty) \leq k[d(Tx,x) + d(Ty,y)], \forall x \in X, \forall y \in Y. \qquad (2)$$

Recently Karapinar [4] proposed a new Kannan-type contractive mapping using the concept of interpolation and proved a fixed point theorem in metric space. This new type of mapping, called "interpolative Kannan-type contractive mapping is a generalization of Kannan's fixed point theorem.

Theorem 3. *Let us recall that an interpolative Kannan contraction on a metric space (E,d) is a self-mapping $T: E \to E$ such that there exist $k \in [0,1)$ and $\alpha \in (0,1)$ such that:*

$$d(Tx,Ty) \leq k[d(Tx,x)]^{\alpha}[d(Ty,y)]^{1-\alpha}, \ (x,y) \in E \times E \text{ with } x,y \notin Fix(T) \qquad (3)$$

Then T has a unique fixed point in E.

This theorem has been generalized in 2023 by Edraoui. M, El koufi [7].

Definition 4. [7]*Let (E,d) be a metric space and let X and Y be nonempty subsets of E. A cyclic map*

$T: X \cup Y \to X \cup Y$ is said to be a interpolative Kannan Type cyclic contraction if there exists $k \in [0,1)$ and $\alpha \in (0,1)$ such that:

$$d(Tx,Ty) \leq k[d(Tx,x)]^{\alpha}[d(Ty,y)]^{1-\alpha}, \text{ for all } (x,y) \in X \times Y \text{ with } x,y \notin Fix(T) \qquad (4)$$

Next, we give fixed point theorem for a interpolative Kannan Type cyclic contraction.

Theorem 5. [7] *Let (E, d) be a complete metric space and let X and Y be nonempty subsets of E and let $T: X \cup Y \to X \cup Y$ be interpolative Kannan Type cyclic contraction. Then T has a unique fixed point in*

$X \cap Y$.

Definition 6. *let E be a non-empty set and $F: E \times E \to E$ .An element $(x, y) \in E \times E$ is called a coupled fixed point of the mapping F if $F(x, y) = x$ and $F(y, x) = y$*

In 2014 [5], Binayak S. Choudhury and Pranati Maity. presented the idea of strong coupled fixed points and established certain cyclic coupled fixed point outcomes using Kannan type contractions. The definition of a strong coupled fixed point is as follows:

Definition 7. [5] *(Strong coupled fixed point). We call the coupled fixed point in the above definition to be strong coupled fixed point if $x = y$, that is, if $F(x, x) = x$*

Definition 8. *[5](Cyclic coupled Kannan type contraction). Let X and Y be two nonempty subsets of a metric space (E, d). We call a mapping $F: E \times E \to E$ a cyclic coupled Kannan type contraction with respect to X and Y if F is cyclic with respect to X and Y satisfying, for some $K \in (0,1/2)$, the inequality*

$$d\big(F(x, y), F(u, v)\big) \le k\big[d\big(x, F(x, y)\big) + d\big(u, F(u, v)\big)\big] \qquad (5)$$

where $x, u \in X,\ y, v \in Y$.

Theorem 9. [5] *Let X and Y be two nonempty closed subsets of a complete metric space (E, d). Let $F: E \times E \to E$ be a cyclic coupled Kannan type contraction with respect to X and Y and $X \cap Y \neq \varnothing$. Ten F has a strong coupled fixed point in $X \cap Y$.*

Main results

This paper introduces a coupled cyclic mapping, specifically the cyclic coupled interpolative Kannan type contraction, and demonstrates that such mappings possess strong coupled fixed points.

Definition 10. *Let (E, d) be a metric space and let X and Y be nonempty subsets of E. A cyclic map $F: E \times E \to E$ is said to be a cyclic coupled interpolative Kannan Type contraction with respect to X and Y if F is cyclic with respect for some $k \in \left(0, \frac{1}{2}\right)$ and $\alpha \in (0,1)$ such that:*

$$d\big(F(x, y), F(u, v)\big) \le k\big[d\big(x, F(x, y)\big)\big]^{1-\alpha}\big[d\big(u, F(u, v)\big)\big]^{\alpha},\ \textit{for all}\ x, u \in X,\ y, v \in Y \quad (6)$$

Theorem 11. *Let (E, d) be a complete metric space and let X and Y be nonempty subsets of E and let $F: E \times E \to E$ be cyclic coupled interpolative Kannan Type contraction, with respect to X and Y and $X \cap Y \neq \emptyset$. Ten F has a strong coupled fxed point in $X \cap Y$*

Proof. Let $x_0 \in X$ and $y_0 \in Y$ be any two elements and let the sequences $\{x_n\}$ and $\{y_n\}$ be defined as

$$x_1 = F(y_0, x_0);\ x_{n+1} = F(y_n, x_n) \text{ and } y_1 = F(x_0, y_0)\ y_{n+1} = F(x_n, y_n) \text{ for all } n \in \mathbb{N}$$

Then by (6) we have

$$\begin{aligned} d(x_1, y_2) &= d(x_1, F(x_1, y_1)) \\ &= d(F(y_0, x_0), F(x_1, y_1)) \\ &\le k[d(y_0, F(y_0, x_0))]^{1-\alpha}[d(x_1, F(x_1, y_1))]^{\alpha} \\ &= k[d(y_0, x_1)]^{1-\alpha}[d(x_1, y_2)]^{\alpha} \end{aligned}$$

which implies that,

$$[d(x_1, y_2)]^{1-\alpha} \le k[d(y_0, x_1)]^{1-\alpha} \qquad (7)$$

And so $d(x_1, y_2) \le td(y_0, x_1)$, where $t = k^{\frac{1}{1-\alpha}}$ and clearly $t \in (0,1)$.

and

$$\begin{aligned} d(y_2, x_1) &= d(y_1, F(y_1, x_1)) \\ &= d(F(x_0, x_0), F(y_1, x_1)) \\ &\le k[d(x_0, F(x_0, y_0))]^{1-\alpha}[d(y_1, F(y_1, x_1))]^{\alpha} \\ &= k[d(x_0, y_1)]^{1-\alpha}[d(y_1, x_2)]^{\alpha} \end{aligned}$$

which implies that,

$$[d(y_2, x_1)]^{1-\alpha} \le k[d(x_0, y_1)]^{1-\alpha} \qquad (8)$$

And so $d(y_2, x_1) \le td(x_0, y_1)$, where $t = k^{\frac{1}{1-\alpha}}$ and clearly $t \in (0,1)$.

Inductively, using this process for all $n \in \mathbb{N}$ we have

$$d(x_n, y_{n+1}) \le t^n d(y_0, x_1) \text{ and } d(y_{n+1,} x_n) \le t^n d(x_0, y_1) \qquad (9)$$

for all $n \le m$ where n is even

Let m be even.

$$\begin{aligned} d(x_{m+1,} y_{m+2}) &= d(F(y_m, x_m), F(x_{m+1}, y_{m+1})) \\ &\le k[d(y_m, F(y_m, x_m))]^{1-\alpha}[d(x_{m+1}, F(x_{m+1}, y_{m+1}))]^{\alpha} \\ &\le k[d(y_m, x_{m+1})]^{1-\alpha}[d(x_{m+1}, y_{m+2})]^{\alpha} \end{aligned}$$

So, we have

$$[d(x_{m+1}, y_{m+2})]^{1-\alpha} \le k[d(y_m, x_{m+1})]^{1-\alpha}$$

$$\begin{aligned} d(x_{m+1}, y_{m+2}) &\leq k^{\frac{1}{1-\alpha}} d(y_m, x_{m+1}) \\ &\leq t d(y_m, x_{m+1}) \\ &\leq t[t^n d(y_0, x_1)] = t^{n+1} d(y_0, x_1) \end{aligned}$$

then

$$d(x_{m+1}, y_{m+2}) \leq t^{n+1} d(y_0, x_1) \quad (10)$$

Similarly, we have

$$d(y_{m+2}, x_{m+1}) \leq t^{n+1}(x_0, y_1) \quad (11)$$

From the above we conclude that, for all odd integer n, we have:

$$\begin{aligned} d(x_n, y_{n+1}) &\leq t^n d(y_0, x_1) \quad (12) \\ d(y_n, x_{n+1}) &\leq t^n d(x_0, y_1) \end{aligned}$$

Let, for some integer m, then by applying (6) and (12) we get

$$\begin{aligned} d(x_{m+1}, y_{m+1}) &= d\big(F(y_m, x_m), F(x_m, y_m)\big) \\ &\leq k\big[d\big(y_m, F(y_m, x_m)\big)\big]^{1-\alpha}\big[d\big(x_m, F(x_m, y_m)\big)\big]^{\alpha} \\ &\leq k[d(y_m, x_{m+1})]^{1-\alpha}[d(x_m, y_{m+1})]^{\alpha} \\ &\leq k[t^m d(x_0, y_1)]^{1-\alpha}[t^m d(y_0, x_1)]^{\alpha} \\ &\leq t^{1-\alpha}[t^m d(x_0, y_1)]^{1-\alpha}[t^m d(y_0, x_1)]^{\alpha} \\ &\leq t^m[t d(x_0, y_1)]^{1-\alpha}[d(y_0, x_1)]^{\alpha} \end{aligned}$$

Similarly, we have

$$d\big(y_{m+1,} x_{m+1}\big) \leq t^m[t d(y_0, x_1)]^{1-\alpha}[d(x_0, y_1)]^{\alpha}$$

Let, for some integer n we have:

$$\begin{aligned} d(x_n, y_n) &\leq \frac{t^n}{t^\alpha}[d(x_0, y_1)]^{1-\alpha}[d(y_0, x_1)]^{\alpha} \quad (13) \\ d\big(y_{n,} x_n\big) &\leq \frac{t^n}{t^\alpha}[t d(y_0, x_1)]^{1-\alpha}[d(x_0, y_1)]^{\alpha} \end{aligned}$$

Now, by (6), (12) and (13), for all $n \geq 1$, we have

$$\begin{aligned} d(x_n, x_{n+1}) + d(y_n, y_{n+1}) &\leq d(x_n, y_{n+1}) + d(y_{n+1}, x_{n+1}) + d(y_n, x_{n+1}) + d(x_{n+1}, y_{n+1}) \\ &\leq t^n d(y_0, x_1) + t^n[t d(y_0, x_1)]^{1-\alpha}[d(x_0, y_1)]^{\alpha} + t^n d(x_0, y_1) \\ &\qquad + t^n[t d(x_0, y_1)]^{1-\alpha}[d(y_0, x_1)]^{\alpha} \\ &\leq t^n \begin{pmatrix} d(y_0, x_1) + [t d(y_0, x_1)]^{1-\alpha}[d(x_0, y_1)]^{\alpha} + d(x_0, y_1) + \\ [t d(x_0, y_1)]^{1-\alpha}[d(y_0, x_1)]^{\alpha} \end{pmatrix} \end{aligned}$$

Since $t \in (0,1)$.Consequently,

$$\sum_{n=1}^{\infty} d\left(x_n, x_{n+1}\right)+d(y_n, y_{n+1}) \leq \sum_{n=1}^{\infty} t^n\begin{pmatrix} d(y_0, x_1)+[td(y_0, x_1)]^{1-\alpha}[d(x_0, y_1)]^{\alpha} \\ +d(x_0, y_1)+[td(x_0, y_1)]^{1-\alpha}[d(y_0, x_1)]^{\alpha}\end{pmatrix}$$

Therefore, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in (E, d) by the completeness of E, then they are convergent to x and y respectively.

Since X and Y are closed subsets, $\{x_n\} \subset X$, and $\{y_n\} \subset Y$, it follows that:

$$\lim_{n\to\infty} x_n = x \in X and \lim_{n\to\infty} y_n = y \in Y \qquad (14)$$

Again, from (13), $d(x_n, y_n) \to 0$ as $n \to \infty$

then

$$d(x, y) = 0 \text{ whichi mplies that } x = y \qquad (15)$$

we conclude $x \in X \cap Y$ and hence $X \cap Y \neq \emptyset$

Now, by (6), for all $n \geq 1$, we get:

$$\begin{aligned} d\big(x, F(x,y)\big) &\leq d(x, x_{n+1}) + d\big(x_{n+1}, F(x,y)\big) \quad (16) \\ &= d\left(x, F\left(y_{n,} x_n\right)\right) + d\left(F\left(y_{n,} x_n\right), F(x,y)\right) \\ &\leq d(x, x_{n+1}) + k\big[d\big(y_n, F(y_n, x_n)\big)\big]^{1-\alpha}\big[d\big(x, F(x,y)\big)\big]^{\alpha} \\ &= d(x, x_{n+1}) + k[d(y_n, x_{n+1})]^{1-\alpha}\big[d\big(x, F(x,y)\big)\big]^{\alpha} \end{aligned}$$

By evaluating the limit as $n \to \infty$ in the preceding inequality, and employing equations (14) and (15), we ascertain that $d\big(x, F(x,y)\big) = 0$. Consequently, we can deduce that $x = F(x,x)$, meaning x represents a strong coupled fixed point of F.

This completes the proof of the theorem.

References

[1] Erdal Karapınar and Inci M. Erhan, Best Proximity Point on Different Type Contractions Applied Mathematics & Information Sciences 5(3) (2011), 558-569J.

[2] E. Karapinar, Revisiting the Kannan type contractions via interpolation. Adv. Theory Nonlinear Anal. Appl. 2, no. 2 (2018), 85–87.

[3] R. Kannan, Some results on fixed points. Bull. Calcutta Math. Soc. 60, 71-76 (1968).

[4] Kirk, W.A.; Srinivasan, P.S.; Veeramani, P. Fixed point fo mappings satisfyaing cyclical contractive conditions. Fixed Point Theory2003, 4, 79–89.

[5] B. S. Choudhury and P. Maity, “Cyclic coupled fxed point result using Kannan type contractions,” Journal of Operators, vol. 2014, Article ID 876749, 5 pages, 2014.

[6] Karapinar, E.; Alqahtani, O.; Aydi, H. On interpolative Hardy-Rogers type contractions. Symmetry 2019, 11,8M. Young, The Technical Writer’s Handbook. Mill Valley, CA: University Science, 1989.

[7] Mohamed Edraoui , Amine El koufi and Soukaina Semami Fixed points results for various types of interpolative cyclic contraction Appl. Gen. Topol doi:10.4995/agt.2023.19515D. Kornack and P. Rakic, “Cell Proliferation without Neurogenesis in Adult Primate Neocortex,” Science, vol. 294, Dec. 2001, pp. 2127-2130, doi:10.1126/science.1065467.

[8] N. Taş, “Interpolative contractions and discontinuity at fixed point”, Appl. Gen. Topol., vol. 24, no. 1, pp. 145–156, Apr. 2023.

[9] N. Bilgili, I. M. Erhan, E. Karapinar, D. Turkoglu, A note on Coupled fixed point theorems for mixed g-monotone mappings in partially ordered metric spaces, Fixed Point Theory Appl. Vol. 2014, 2014:120, 6 pp.

[10] Mohamed Edraoui, Mohamed Aamri and Samih Lazaiz Fixed Point Theorem in Locally K-Convex Space International Journal of Mathematical Analysis Vol. 12, 2018, no. 10, 485 - 490

[11] B. S. Choudhury, A. Kundu, Two coupled weak contraction theorems in partially ordered metric spaces, Revista de la Real Academia de Ciencias Exactas., Fisicas y Naturales. Serie A. Matematicas. 108(2014), 335-351.

[12] B. S. Choudhury, P. Maity, Coupled fixed point results in generalized metric spaces, Math. Comput. Modelling. 54(2011) 73-79

[13] Mohamed, E.; Mohamed, A.; Samih, L. Relatively Cyclic and Noncyclic P-Contractions in Locally K-Convex Space. Axioms 2019, 8, 96. https://doi.org/10.3390/axioms8030096

[14] V. Lakshmikantham and L. Ciric, “Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces,” Nonlinear Analysis: Theory, Methods & Applications, vol. 70, no. 12 pp. 4341–4349, 2009

[15] M. Edraoui, M. Aamri, S. Lazaiz, Fixed Point Theorems For Set Valued Caristi Type Mappings In Locally Convex Space, Adv. Fixed Point Theory, 7 (2017), no. 4, 500-511

[16] N. Bilgili, I. M. Erhan, E. Karapinar, D. Turkoglu, A note on Coupled fixed point theorems for mixed g-monotone mappings in partially ordered metric spaces, Fixed Point Theory Appl. Vol. 2014, 2014:120 6 pp