$\theta-\phi$ −Contraction Mapping in Complete b-Metric Spaces

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Abstract
In this paper, we have presented some fixed point results for generalized θ – Φ - contractive mappings in complete b-metric space. Further, we establish some fixed point theorems for this type of mapping defined on such spaces. Hence, our results unify, generalize and complement the comparable results from the current literature. We have concluded examples to support our main results.

Keywords— Fixed point; b-metric space; contractive mappings;

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1. Introduction

A very popular tool to solve existence of fixed point problems is the Banach Contraction Theorem [2] which plays an important role in several branches of mathematics. Bakhtin [2] the concept of b-metric spaces and by generalizing the famous Banach contraction principle in metric spaces proved the contraction mapping principle in b-metric spaces. Recently, the fixed point in non-convex analysis, especially in an ordered normed space, occupies a prominent place in many aspects $(see\ [6-11])$, the author defines an ordering by using a cone, which naturally induces a partial ordering in Banach spaces. Moreover, some fixed point theorems were proved for contractive mappings expanding certain results of fixed points in metric spaces $(see\ [4,5,12,14,15])$. Later some fixed point theorems for b −metric spaces were given by Xie and Wang [12]. Throughout this paper, we have proved a generalization of fixed point theorem for results for $\theta-\phi-$ contractaction mappings in complete b −metric space by using triangular inequality.

2. Definitions Preliminaries

Definition 2. 1. ([12]) Let (X,d) be a nonempty set and $s\geq 1$ be a given real number. A function $d: X\times X\to [0,\infty)$ is a b −metric if, for all $x,y,z\in X$, the following conditions are satisfied:
$(b1)\ \ d(x,y)=0\ if\ and\ only\ if\ x=y,$
$(b2)\ \ d(x,y)=d(y,x),$
$(b3)\ \ d(x,z)\leq s[d(x,y)+d(y,z)]$
In this case, the pair (X,d) is called a b −metric space.
It should be noted that, the class of b −metric spaces is effectively larger than that of metric spaces; every metric is a b −metric with $s=1$.
Definition 2. 2. ([12]) Let $\{x_n\}$ be a sequence in a b-metric space (X,d).

a. $\{x_n\}$ is called b −convergent if and only if there is $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$.
b. $\{x_n\}$ is a b −Cauchy sequence if and only if $d(x_n, x_m) \to 0\ as\ n, m \to \infty$.
c. A b-metric space (X, d) is said to be complete if and only if each b −cauchy sequence in this space is b −convergent.

Lemma 2.3. ([11]) Let (X, d) be a b −metric space with $s \geq 1$.
i. If a sequence $\{x_n\} \subset X$ is a b −convergent sequence, then it admits a unique limit.
ii. Every b-convergent sequence in X is b −cauchy.

Definition 2.4. ([10]) Let (X, d) be a b-metric space. A subset $Y \subset X$ is called closed if and only if for each sequence $\{x_n\}$ in Y which b-converges to an element x, we have $x \in Y$.

Zheng $et\ al$ [13] introduced a new type of contractions called $\theta - \phi$ −contractions in metric spaces and proved a new fixed point theorem for such mapping.

Definition 2.5. ([10]) We denote by Θ the set of function $\theta : [0, \infty] \to [1, \infty]$ satisfying the following conditions:
(θ_1) θ Is increasing;
(θ_2) For each sequence $(x_n) \in [0, \infty]$, $\lim_{n\to\infty}, \theta(x_n) = 1 \Leftrightarrow \lim_{n\to\infty} x_n = 0$;
(θ_3) θ Is continuous on $[0, \infty]$.

Definition 2.6. ([10]) We denote by Φ the set of function $\phi : [1, \infty] \to [1, \infty]$ satisfying the following conditions:
(ϕ_1) $\phi: [1, \infty] \to [1, \infty]$ Is non-decreasing;
(ϕ_2) For each sequence $(x_n) \in [0, \infty]$, $\lim_{n\to\infty}, \phi(x_n) = 1$;
(ϕ_3) θ Is continuous on $[1, \infty]$.

Definition 2.7. ([12]) Let X be a non empty set and $\theta: X \times X \to [1, \infty)$. A function $d_\theta : X \times X \to [0, \infty)$ is called b −metric if for all $x, y, z \in X$ it satisfies:
$d_\theta 1$ $d_\theta(x, y) = 0\ iff\ x = y$;
$d_\theta 2$ $d_\theta(x, y) = d_\theta(y, x)$;
$d_\theta 3$ $d_\theta(x, z) \leq \theta(x, z)[d_\theta(x, y) + d_\theta(y, z)]$.
The pair (X, d_θ) is called b-metric.

Remark 2.8. If $\theta(x, y) = s\ for\ s \geq 1$ then we obtain the definition of a b-metric space

Theorem 2.9. Let(Z, d) be a complete b-metric space with $k \geq 1$ and let $R : Z \to Z$ be a continuous mapping satisfying the contractive condition

$$\theta\big(d(Ru, Rv)\big) \leq \phi\left[\psi_1 \frac{d(u, Ru)d(u, Rv) + d(v, Rv)d(v, Ru)}{d(u, Rv) + d(v, Ru)}\right]$$

For all $u, v \in Z$ and $\psi_1 \in [0,1]$. Then R has a unique fixed point in Z.

3. Main Results

Theorem 3.1. Let (Z, d) be a complete b −metric space with $k \geq 1$ and let $R : Z \to Z$ be a continuous mapping satisfying the contractive condition

$$\theta\big(d(Ru, Rv)\big) \leq \psi_1 \frac{d(u, Ru)d(u, Rv) + d(v, Rv)d(v, Ru)}{d(u, Rv) + d(v, Ru)} + \psi_2 \frac{[d(Ru, v) + d(u, v)][1 + d(v, Rv)]}{1 + d(u, v)} + \psi_3 d(u, v) \quad (1)$$

For all $u, v \in Z$ and $\psi_1, \psi_2, \psi_3 \in [0,1]$ $with\ k(2\psi_1 + \psi_2 + \psi_3) < 1.$ then R has a unique fixed point in Z.

Proof. Let z_0 be an arbitrary point in Z. Define a sequence $\{z_n\}$ in Z such that $z_1 = R(z_0),\ z_1 = R(z_1), \dots \dots$
Replace u by z_{n-1} and v by z_n in (1), we have

$$\theta\big(d(z_n, z_{n+1})\big) = \phi\big(d(Rz_{n-1}, Rz_n)\big)$$

$$\leq \psi_1 \frac{d(z_{n-1}, Rz_{n-1})d(z_{n-1}, Rz_n) + d(z_n, Rz_n)d(z_n, Rz_{n-1})}{d(z_{n-1}, Rz_n) + d(z_n, Rz_{n-1})}$$

$$+ \psi_2 \frac{[d(Rz_{n-1}, z_n) + d(z_{n-1}, z_n)][1 + d(z_n, Rz_n)]}{1 + d(z_{n-1}, z_n)}$$

$$+\psi_3 d(z_{n-1}, z_n)$$

$$\leq \psi_1 \frac{d(z_{n-1}, z_n)d(z_{n-1}, z_{n+1}) + d(z_n, z_{n+1})d(z_n, z_n)}{d(z_{n-1}, z_{n+1}) + d(z_n, z_n)} + \psi_2 \frac{[d(z_n, z_n) + d(z_{n-1}, z_n)][1 + d(z_n, z_{n+1})]}{1 + d(z_{n-1}, z_n)}$$

$$+\psi_3 d(z_{n-1}, z_n)$$

Using triangular inequality

$$\leq \psi_1 \frac{d(z_{n-1}, z_n)k(d(z_{n-1}, z_n) + d(z_n, z_{n+1}) + d(z_n, z_{n+1})k\big(d(z_{n-1}, z_n) + d(z_n, z_{n+1})\big)}{k\big(d(z_{n-1}, z_n) + d(z_n, z_{n+1})\big)}$$

$$+ \psi_2 \frac{[d(z_n, z_n) + d(z_{n-1}, z_n)][1 + d(z_n, z_{n+1})]}{1 + k\big(d(z_{n-1}, z_n)\big)}$$

$$+\psi_3 d(z_{n-1}, z_n)$$

$$\leq \psi_1\big(d(z_{n-1}, z_n) + d(z_n, z_{n+1})\big) + \psi_2\big(d(z_n, z_{n-1}) + d(z_{n+1}, z_n)\big) + \psi_3 d(z_{n-1}, z_n)$$

Therefore

$$\theta\big(d(z_n, z_{n+1})\big) \leq \frac{\psi_1 + \psi_2 + \psi_3}{1 - \psi_1} d(z_{n-1}, z_n) = hd(z_{n-1}, z_n) \qquad (2)$$

Where $h = \frac{\psi_1+\psi_2+\psi_3}{1-\psi_1} < 1\ as\ k(2\psi_1 + \psi_2 + \psi_3) < 1.$

we have

$$\theta\big(d(z_{n-1}, z_n)\big) \leq hd(z_{n-2}, z_{n-1})$$

By (2) we get,

$$\theta\big(d(z_n, z_{n+1})\big) \leq h^2 d(z_{n-2}, z_{n-1})$$

Continue this process, we get

$$\theta\big(d(z_n, z_{n+1})\big) \leq h^n d(z_n, z_0)$$

Since $0 \leq h < 1\ as\ n \to \infty, h^n \to 0. Thus$ $\{z_n\}$ is complete b −metric space in Z such that $T(u = limT(z_n) = limz_{n+1} = u$. Thus u is a fixed point of R.

Uniqueness:

Let $u \in U$ is a fixed point of R. then by (1),

$$\theta\big(d(u, u)\big) = \phi\big(d(Ru, Ru)\big)$$

$$\leq \psi_1 \frac{d(u, u)d(u, u) + d(u, u)d(u, u)}{d(u, u) + d(u, u)} + \psi_2 \frac{[d(u, u) + d(u, u)][1 + d(u, u)]}{1 + d(u, u)} + \psi_3 d(u, u)$$

$$\leq (\psi_1 + \psi_2 + \psi_3)d(u, u).$$

Which is true only if $d(u, u) = 0$, since $0 \leq k(2\psi_1 + \psi_2 + \psi_3) < 1\ and\ d(u, u) \geq 0$. Thus $d(u, u) \geq 0$, if Z is a fixed point of R. then we have,

$$d(u,v) = d(Ru,Rv) \leq \psi_2 d(u,v)$$

Which gives $d(u,v) = 0$, since $0 \leq \psi_2 < 1$ *and* $d(u,v) \geq 0$. Thus fixed point of R is unique.

Example:3.2. Let $Z = [0,1]$. Define $d: Z \times Z \to \mathbb{R}^+$ by

$$d(u,v) = |u+v|^2 + |u-v|^2 + \frac{|u+v| + |u-v|}{2}$$

For all $u, v \in Z$. Define $F(u,v) = \frac{uv}{8}$

Example:3.3. Let $Z = \{p.q, r, s, t, u, v, w\}, E = \mathbb{R}^2$ and $p = \{(u,v): u, v \geq 0\}$ is a b-metric in E. Define $d: Z \times X \to E$ as follows:

$$d(u,u) = 0, \forall\, z \in Z$$
$$d(p,q) = d(q,p) = (8,64)$$
$$d(p,r) = d(r,p) = d(r,s) = d(s,r) = d(q,r) = d(r,q) = d(q,s) = d(s,q) = d(p,t) = d(t,p) = d(p,u) = d(u,p) = (1,8)$$
$$d(p,u) = d(u,p) = d(q,t) = d(t,q) = d(r,u) = d(u,r) = d(s,u) = d(u,s) = d(t,u) = d(u,w) = (10,70)$$

Then $d(Z,d)$ is a complete b −metric space,

4. Conclusion

Hence in this paper we have proved a fixed point theorem for $\theta - \phi -$ contraction mapping in complete b −metric space by using triangular inequality, which is generalization and extension of the results due to Mitiku et al [10], Rossafi et al [11].

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