Minimal gη-Open and Maximal gη-Closed sets

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Abstract:
In this paper a new class of sets namely minimal gη-closed set, maximal gη-open set, minimal gη-open set and maximal gη-closed set and their basic properties are studied.

Keywords: gη-closed set and minimal gη-closed set, maximal gη-open set, minimal gη-open set and maximal gη-closed set.

1. Introduction

In 1963, Levine [2] introduced the notion of semi-open sets in topological spaces. In 1984, Andrijevic [1] introduced some properties of the topology of α-sets. Norman Levine [3] introduced the concepts of generalized closed sets in topological spaces. The notion of gη-closed set and its different characterizations are discussed in [8].

Nakaoka and Oda [4,5,6] have introduced minimal open sets and maximal open sets, which are subclasses of open sets. Later on many authors concentrated in this direction and defined many different types of minimal and maximal open sets. Inspired with these developments we further study a new type of closed and open sets namely minimal gη-closed sets, maximal gη -open sets, minimal gη-open sets and maximal gη-closed sets.

2. Preliminaries

Definition : 2.1 A subset A of topological space (X,τ) is called

(i) α-open set [1] if A ⊆ int(cl(int(A))), α-closed set if cl(int(cl(A))) ⊆ A.

(ii) semi-open set [2] if A ⊆ cl(int(A)), semi-closed set if int(cl(A) ⊆ A.

(iii) η-open set [7] if A ⊆ int(cl(int(A))) ∪ cl (int (A)).

(iv) g-closed set [3] if cl(A) ⊆ U whenever A ⊆ U and U is open in (X,τ).

(v) gη-closed set [8] if ηcl(A)⊆ U whenever A ⊆ U and U is open in (X,τ).

3. Minimal Gη-Open Sets And Maximal Gη-Closed Sets

A new class of sets, called minimal gη-open sets and maximal gη-closed sets in topological spaces are introduced and some of their properties are proved in this section.

Definition 3.1: A proper nonempty gη-open subset U of X is said to be a Minimal gη-open set if any gη-open set contained in U is φ or U.

Example 3.2: Let X = {a,b,c,d}, τ = {φ, {a},{b},{a,b},{a,b,c}, X}. Then {a} is both Minimal open and Minimal gη-open but {c} is Minimal gη-open but not Minimal open.

Remark 3.3: Minimal open and minimal gη-open sets are independent of each other.

Example 3.4: Let X = {a,b,c}, τ = {φ,{a},{b},{a, b},X}. Then {a,b} is Minimal open but not Minimal gη-open and {a,c} is Minimal gη-open but not Minimal open.

Theorem 3.5: (i) Let U be a minimal gη-open set and W be a gη-open set. Then U ∩ W = φ or U⊂W.

(ii)Let U and V be minimal gη-open sets. Then U ∩ V = φ or U = V.

Proof: (i) Let U be a minimal gη-open set and W be a gη-open set. If U ∩ W = φ. Then there is nothing to prove. If U ∩ W ≠ φ. Then U ∩W ⊂U. Since U is a minimal gη-open set, we have U ∩ W = U. Therefore U ⊂W.

(ii) Let U and V be minimal gη-open sets. If U∩V ≠ φ, then U⊂ V and V⊂ U by (i). Therefore U = V.

Theorem 3.6: Let U be a minimal gη-open set. If x∈U, then U⊂W for any open neighborhood W of x.

Proof: Let U be a minimal gη-open set and x be an element of U. Suppose there exist a open neighborhood W of x such that U ⊄ W. Then U ∩ W is a gη-open set such that U ∩ W⊂ U and U ∩ W ≠ φ. Since U is a minimal gη-open set, we have U∩ W = U. That is U ⊂W, which is a contradiction for U ⊄ W. Therefore U⊂ W for any open neighborhood W of x.

Theorem 3.7: Let U be a minimal gη-open set. If x∈U, then U⊂W for some gη-open set W containing x.

Theorem 3.8: Let U be a minimal gη-open set. Then U = ∩{W: W∈ GηO(X, x)} for any element x of U.

Proof: By theorem[3.7] and U is gη-open set containing x, we have U⊂∩{ W: W∈ GηO(X, x)} ⊂U.

Theorem 3.9: Let U be a nonempty gη-open set. Then the following three conditions are equivalent.

(i) U is a minimal gη-open set

(ii) U⊂ gηcl(S) for any nonempty subset S of U

(iii) gη-cl(U) = gη-cl(S) for any nonempty subset S of U.

Proof: (i) ⇒ (ii) Let $x \in U$, U be minimal gη-open set and $S \neq \varphi \subset U$. By theorem[3.7], for any gη-open set W containing x, $S \subset U \subset W \Rightarrow S \subset W$. Now $S = S \cap U \subset S \cap W$. Since $S \neq \varphi$, $S \cap W \neq \varphi$. Since W is any gη-open set containing x, x∈gη-cl(S). That is x∈U ⇒ x∈gη-cl(S) ⇒ U⊂ gη-cl(S) for any nonempty subset S of U.

(ii) ⇒ (iii) Let S be a nonempty subset of U. That is S⊂U ⇒ gη-cl(S) ⊂ gη-cl(U) → (1).

Again from (ii) U⊂ gη-cl(S) for any S≠ φ ⊂U ⇒ gη-cl(U) ⊂ gη-cl(gη-cl(S)) = gη-cl (S).

That is gη-cl(U) ⊂ gη-cl(S) → (2).

From (1) and (2), we have gη-cl(U) = gη-cl(S) for any nonempty subset S of U.

(iii) ⇒ (i) From (iii) we have gη-cl(U) = gη-cl(S) for any nonempty subset S of U. Suppose U is not a minimal gη-open set. Then there exist a nonempty gη-open set V such that V⊂ U and V ≠ U. Now there exist an element a in U such that $a \notin V \Rightarrow a \in V^c$. That is gη-cl({a})⊂ gη-cl(V^c) = V^c, as V^c is gη-closed set in X. It follows that gη-cl({a}) ≠ gη-cl(U). This is a contradiction for gη-cl ({a}) = gη-cl(U) for any {a}≠ φ⊂U. Therefore U is minimal gη-open set.

Theorem 3.10: Let V be a nonempty finite gη-open set. Then there exist atleast one finite minimal gη-open set U such that U⊂V.

Proof: Let V be a nonempty finite gη-open set. If V is a minimal gη-open set, we may set U = V. If V is not a minimal gη-open set, then there exist a finite gη-open set V_1 such that $\varphi \neq V_1 \subset V$. If V_1 is a minimal gη-open set, we may set $U = V_1$. If V_1 is not a minimal gη-open set, then there exist finite gη-open set V_2 such that $\varphi \neq V_2 \subset V_1$. Continuing this process, we have a sequence of gη-open sets $V \supset V_1 \supset V_2 \supset V_3 \supset \ldots\ldots \supset V_k \supset \ldots\ldots$ Since V is a finite set, this process repeats only finitely. Then finally we get a minimal gη-open set $U = V_n$ for some positive integer n.

Definition 3.11: A topological space X is said to be locally finite space if each of its elements is contained in a finite open set.

Corollary 3.12: Let X be a locally finite space and V be a nonempty gη-open set. Then there exist at least one finite minimal gη-open set U such that U ⊂ V.

Proof: Let X be a locally finite space and V be a nonempty gη-open set. Let x in V. Since X is locally finite space, we have a finite open set V_x such that x in V_x. Then $V \cap V_x$ is a finite gη-open set. By Theorem 3.10 there exist an at least one finite minimal gη-open set U such that $U \subset V \cap V_x$. That is $U \subset V \cap V_x \subset V$. Hence there exist an at least one finite minimal gη-open set U such that U⊂V.

Corollary 3.13: Let V be a finite minimal open set. Then there exist an at least one finite minimal gη-open set U such that U⊂V.

Proof: Let V be a finite minimal open set. Then V is a nonempty finite gη-open set. By Theorem 3.10, there exist an at least one finite minimal gη-open set U such that U⊂V.

Definition 3.14: A proper nonempty gη-closed $F \subset X$ is said to be maximal gη-closed set if any gη-closed set containing F is either X or F.

Example 3.15: Let X = {a,b,c}, τ = {φ,{a},X}. Then {b,c} is both Maximal closed and Maximal gη-closed but {a,b} and {a,c} are Maximal gη-closed but not Maximal closed.

Remark 3.16: Maximal closed and maximal gη-closed sets are independent of each other:

Example 3.17: Let X = {a, b, c}, τ = {φ, {a}, {b, c}, X}. Then {a} is Maximal closed but not Maximal gη-closed and {a,b} is Maximal gη-closed but not Maximal closed.

Theorem 3.18: A proper nonempty subset F of X is maximal gη-closed set if and only if X-F is a minimal gη-open set.

Proof: Let F be a maximal gη-closed set. Suppose X-F is not a minimal gη-open set. Then there exist a gη-open set $U \neq X-F$ such that $\varphi \neq U \subset X-F$. That is $F \subset X-U$ and X-U is a gη-closed set which is a contradiction for F is a maximal gη-closed set.

Conversely let X-F be a minimal gη-open set. Suppose F is not a maximal gη-closed set, then there exist a gη-closed set $E \neq F$ such that $F \subset E \neq X$. That is $\varphi \neq X-E \subset X-F$ and X-E is a gη-open set which is a contradiction for X-F is a minimal gη-open set. Therefore F is a maximal gη-closed set.

Theorem 3.19: (i) Let F be a maximal gη-closed set and W be a gη-closed set. Then $F \cup W = X$ or $W \subset F$.

(ii) Let F and S be maximal gη-closed sets. Then $F \cup S = X$ or $F = S$.

Proof: (i) Let F be a maximal gη-closed set and W be a gη-closed set. If $F \cup W = X$. Then there is nothing to prove. Suppose $F \cup W \neq X$. Then $F \subset F \cup W$. Therefore $F \cup W = F \Rightarrow W \subset F$.

(ii) Let F and S be maximal gη-closed sets. If $F \cup S \neq X$, then we have $F \subset S$ and $S \subset F$ by (i). Therefore $F = S$.

Theorem 3.20: Let F be a maximal gη-closed set. If x is an element of F, then for any gη-closed set S containing x, $F \cup S = X$ or $S \subset F$.

Proof: Let F be a maximal gη-closed set and x is an element of F. Suppose there exist a gη-closed set S containing x such that $F \cup S \neq X$. Then $F \subset F \cup S$ and let $F \cup S$ is a gη-closed set. Since F is a gη-closed set, we have $F \cup S = F$. Therefore $S \subset F$.

Theorem 3.21: Let F_α, F_β, F_δ be maximal gη-closed sets such that $F\alpha \neq F_\beta$. If $F_\alpha \cap F_\beta \subset F_\delta$, then either $F_\alpha = F_\delta$ or $F_\beta = F_\delta$

Proof: Given that $F_\alpha \cap F_\beta \subset F_\delta$. If $F_\alpha = F_\delta$. Then there is nothing to prove.

If $F_\alpha \neq F_\delta$ then we have to prove $F_\beta = F_\delta$. Now $F_\beta \cap F_\delta = F_\beta \cap (F_\delta \cap X) = F_\beta \cap (F_\delta \cap (F_\alpha \cup F_\beta)$(by thm. 3.19 (ii)) $= F_\beta \cap ((F_\delta \cap F_\alpha) \cup (F_\delta \cap F_\beta)) = (F_\beta \cap F_\delta \cap F_\alpha) \cup (F_\beta \cap F_\delta \cap F_\beta)$

= ($F_\alpha \cap F_\beta$) ∪ ($F_\delta \cap F_\beta$) (by $F_\alpha \cap F_\beta \subset F_\delta$) = ($F_\alpha \cup F_\delta$) ∩ F_β = X ∩ F_β (Since F_α and F_δ are maximal gη-closed sets by theorem[3.19] (ii), $F_\alpha \cup F_\delta = X = F_\beta$. That is $F_\beta \cap F_\delta = F_\beta \Rightarrow F_\beta \subset F_\delta$ Since F_β and F_δ are maximal gη-closed sets, we have $F_\beta = F_\delta$ Therefore $F_\beta = F_\delta$.

Theorem 3.22: Let F_α, F_β and F_δ be different maximal gη-closed sets to each other. Then ($F_\alpha \cap F_\beta$) ⊄ ($F_\alpha \cap F_\delta$).

Proof: Let ($F_\alpha \cap F_\beta$) ⊂ ($F_\alpha \cap F_\delta$) ⇒ ($F_\alpha \cap F_\beta$) ∪ ($F_\delta \cap F_\beta$) ⊂ ($F_\alpha \cap F_\delta$) ∪ ($F_\delta \cap F_\beta$) ⇒ ($F_\alpha \cup F_\delta$) ∩ $F_\beta \subset F_\delta$ ∩ ($F_\alpha \cup F_\beta$). Since by theorem 3.19 (ii), $F_\alpha \cup F_\delta = X$ and $F_\alpha \cup F_\beta = X \Rightarrow X \cap F_\beta \subset F_\delta \cap X \Rightarrow F_\beta \subset F_\delta$ From the definition of maximal gη-closed set it follows that $F_\beta = F_\delta$, which is a contradiction to the fact that F_α, F_β and F_δ are different to each other. Therefore ($F_\alpha \cap F_\beta$) ⊄ ($F_\alpha \cap F_\delta$).

Theorem 3.23: Let F be a maximal gη-closed set and x be an element of F. Then F = ∪ { S: S is a gη-closed set containing x such that F ∪ S ≠ X}.

Proof: By theorem 3.21 and fact that F is a gη-closed set containing x, we have F⊂∪{ S: S is a gη-closed set containing x such that F ∪ S ≠ X} – F.

Theorem 3.24: Let F be a proper nonempty cofinite gη-closed set. Then there exist a cofinite maximal gη-closed set E such that F⊂ E.

Proof: If F is maximal gη-closed set, we may set E = F. If F is not a maximal gη-closed set, then there exist a cofinite gη-closed set F_1 such that $F \subset F_1 \neq X$. If F_1 is a maximal gη-closed set, we may set E = F_1. If F_1 is not a maximal gη-closed set, then there exist a cofinite gη-closed set F_2 such that $F \subset F_1 \subset F_2 \neq X$. Continuing this process, we have a sequence of gη-closed, $F \subset F_1 \subset F_2 \subset - - - \subset F_k$ - - - - - - . Since F is a cofinite set, this process repeats only finitely. Then finally we get a maximal gη-closed set E = E_n for some positive integer n.

Theorem 3.25: Let F be a maximal gη-closed set. If x is an element of X-F. Then X-F⊂ E for any gη-closed set E containing x.

Proof: Let F be a maximal gη-closed set and x in X-F. E ⊄ F for any gη-closed set E containing x. Then E ∪ F = X by theorem 3.19(ii). Therefore X-F ⊂ E.

4. Minimal Gη-Closed Sets And Maximal Gη-Open Sets

The notion of Minimal gη-closed sets and Maximal gη-open sets in topological spaces are studied in this section.

Definition 4.1: A proper nonempty gη-closed subset F of X is said to be a Minimal gη-closed set if any gη-closed set contained in F is φ or F.

Example 4.2: Let X = {a,b,c}, τ = {φ,{a},{b,c},X}. Then {a} is both Minimal closed and Minimal gη-closed set.

Remark 4.3: Minimal closed and minimal gη-closed sets are independent to each other.

Example 4.4: Let X = {a,b,c}, τ = {φ,{a},{b},{a, b}, X}. Then {c} is Minimal closed but not Minimal gη-closed set and {a} and {b} are Minimal gη-closed but not Minimal closed.

Definition 4.5: A proper nonempty gη-open U⊂X is said to be a Maximal gη-open set if any gη-open set containing U is either X or U.

Example 4.6: Let X = {a,b,c,d}, τ = {φ,{a},{b},{a,b},{a,b,c},X}. Then {a,b,c} is both maximal open and maximal gη-open.

Remark 4.7: Maximal open set and maximal gη-open set are independent to each other.

Example 4.8: Let X = {a,b,c}, τ = {φ,{a},{b},{a,b},X}. Then {a,b} is Maximal open but not maximal gη-open and {b,c} is Maximal gη-open but not maximal open.

Theorem 4.9: A proper nonempty subset U of X is maximal gη-open set if and only if f X-U is a minimal gη -closed set.

Proof: Let U be a maximal gη-open set. Suppose X-U is not a minimal gη-closed set. Then there exist an gη-closed set V ≠ X-U such that φ ≠ V ⊂X-U. That is U⊂X-V and X-V is a gη-open set which is a contradiction for U is a minimal gη-closed set.

Conversely let X-U be a minimal gη-closed set. Suppose U is not a maximal gη-open set. Then there exist an gη-open set E ≠ U such that U⊂ E ≠ X. That is φ ≠ X-E ⊂X-U and X-E is a gη-closed set which is a contradiction for X-U is a minimal gη-closed set. Therefore U is a maximal gη-closed set.

Lemma 4.10: (i) Let U be a minimal gη-closed set and W be a gη-closed set. Then U ∩ W = φ or U ⊂ W.

(ii) Let U and V be minimal gη-closed sets. Then U ∩ V = φ or U = V.

Proof: (i) Let U be a minimal gη-closed set and W be a gη-closed set. If U ∩ W = φ. Then there is nothing to prove. If U ∩ W ≠ φ. Then U ∩W ⊂U. Since U is a minimal gη-closed set, we have U ∩ W = U. Therefore U ⊂W.

(ii) Let U and V be minimal gη-closed sets. If U∩V ≠ φ, then U⊂V and V⊂ U by (i). Therefore U = V.

Theorem 4.11: Let U be a minimal gη-closed set. If x∈U, then U⊂W for any open neighborhood W of x.

Proof: Let U be a minimal gη-closed set and x be an element of U. Suppose there exist an open neighborhood W of x such that U ⊄ W. Then U ∩ W is a gη-closed set such that U ∩ W⊂ U and U ∩ W ≠ φ. Since U is a minimal gη-closed set, we have U∩ W = U. That is U ⊂W, which is a contradiction for U ⊄ W. Therefore U⊂W for any open neighborhood W of x.

Theorem 4.12: Let U be a minimal gη-closed set. If x∈U, then U⊂W for some gη-closed set W containing x.

Theorem 4.13: Let U be a minimal gη-closed set. Then U = ∩{ W: W∈GηC(X, x)}for any element x of U

Proof: By theorem[4.12] and U is gη-closed set containing x, we have U⊂∩{ W: W∈ GηC(X, x)} ⊂U.

Theorem 4.14: Let U be a nonempty gη-closed set. Then the following three conditions are equivalent.

(i) U is a minimal gη-closed set

(ii) U⊂ gη-cl(S) for any nonempty subset S of U

(iii) gη-cl(U) = gη-cl(S) for any nonempty subset S of U.

Proof: (i) ⇒ (ii) Let x∈ U. U be minimal gη-closed set and S ≠ φ ⊂U. By theorem 4.12, for any gη-closed set W containing x, S⊂U⊂W ⇒ S⊂W. Now S = S∩U⊂S∩W. Since S≠ φ, S∩W ≠ φ. Since W is any gη-closed set containing x, by theorem 4.12, x∈ gη-cl(S). That is x∈U ⇒ x∈ gη-cl(S) ⇒ U ⊂ gη-cl(S) for any nonempty subset S of U.

(ii) ⇒ (iii) Let S be a nonempty subset of U. That is S⊂U ⇒ gη-cl(S) ⊂ gη-cl(U) → (1).

Again from (ii) U⊂ gη-cl(S) for any S ≠ φ ⊂U ⇒ gη-cl(U) ⊂ gη-cl(gη-cl(S)) = gη-cl (S). That is gη-cl(U) ⊂ gη-cl(S) → (2).

From (1) and (2), we have gη-cl(U) = gη-cl(S) for any nonempty subset S of U.

(iii) ⇒ (i) From (iii) we have gη-cl(U) = gη-cl(S) for any nonempty subset S of U. Suppose U is not a minimal gη-closed set. Then there exist a nonempty gη-closed set V such that V ⊂U and V ≠ U. Now there exist an element a in U such that a∉ V ⇒ a∈V^c. That is gη-cl({a})⊂ gη-cl(V^c) = V^c, as V^c is gη-closed set in X. It follows that gη-cl({a}) ≠ gη-cl(U). This is a contradiction for gη-cl({a}) = gη-cl(U) for any {a}≠ φ ⊂U. Therefore U is a minimal gη-closed set.

Theorem 4.15: Let V be a nonempty finite gη-closed set. Then there exist an at least one finite minimal gη-closed set U such that U⊂V.

Proof: Let V be a nonempty finite gη-closed set. If V is a minimal gη-closed set, we may set U = V. If V is not a minimal gη-closed set, then there exist an finite gη-closed set V_1 such that φ ≠ V_1⊂V. If V_1 is a minimal gη-closed set, we may set U = V_1. If V_1 is not a minimal gη-closed set, then there exist a finite gη-closed set V_2 such that φ ≠ V_2 ⊂ V_1. Continuing this process, we have a sequence of gη-closed sets V ⊃ V_1 ⊃ V_2 ⊃ V_3⊃ ..... ⊃ V_k ⊃ ...... Since V is a finite set, this process repeats only finitely. Then finally we get a minimal gη-closed set U = V_n for some positive integer n.

Corollary 4.16: Let X be a locally finite space and V be a nonempty gη-closed set. Then there exist an at least one finite minimal gη-closed set U such that U ⊂V.

Proof: Let X be a locally finite space and V be a nonempty gη-closed set. Let x in V. Since X is locally finite space, we have a finite open set V_x such that x in V_x. Then $V \cap V_x$ is a finite gη-closed set. By theorem 4.15 there exist an at least one finite minimal gη-closed set U such that $U \subset V \cap V_x$. That is $U \subset V \cap V_x \subset V$. Hence there exist an at least one finite minimal gη-closed set U such that $U \subset V$.

Corollary 4.17: Let V be a finite minimal open set. Then there exist an at least one finite minimal gη-closed set U such that $U \subset V$.

Proof: Let V be a finite minimal open set. Then V is a nonempty finite gη-closed set.By theorem 4.15, there exist an at least one finite minimal gη-closed set U such that $U \subset V$.

Theorem 4.18 : A proper nonempty subset F of X is maximal gη-open set if and only if X-F is a minimal gη-closed set.

Proof: Let F be a maximal gη-open set. Suppose X-F is not a minimal gη-closed set. Then there exist a gη-closed set $U \neq X-F$ such that $\varphi \neq U \subset X-F$. That is $F \subset X-U$ and X-U is a gη-open set which is a contradiction for F is a maximal gη-open set.

Conversely, let X-F be a minimal gη-closed set. Suppose F is not a maximal gη-open set. Then there exist a gη-open set $E \neq F$ such that $F \subset E \neq X$. That is $\varphi \neq X-E \subset X-F$ and X-E is a gη-closed set which is a contradiction for X-F is a minimal gη-closed set. Therefore F is a maximal gη-open set.

Theorem 4.19: (i) Let F be a maximal gη-open set and W be a gη-open set. Then $F \cup W = X$ or $W \subset F$.

(ii) Let F and S be maximal gη-open sets. Then $F \cup S = X$ or $F = S$.

Proof: (i) Let F be a maximal gη-open set and W be a gη-open set. If $F \cup W = X$. Then there is nothing to prove. Suppose $F \cup W \neq X$. Then $F \subset F \cup W$. Therefore $F \cup W = F \Rightarrow W \subset F$.

(ii) Let F and S be maximal gη-open sets. If $F \cup S \neq X$, then we have $F \subset S$ and $S \subset F$ by (i). Therefore $F = S$.

Theorem 4.20: Let F be a maximal gη-open set. If x is an element of F, then for any gη-open set S containing x, $F \cup S = X$ or $S \subset F$.

Proof: Let F be a maximal gη-open set and x is an element of F. Suppose there exist a gη-open set S containing x such that $F \cup S \neq X$. Then $F \subset F \cup S$ and let $F \cup S$ is a gη-open set. Since F is a gη-open set, we have $F \cup S = F$. Therefore $S \subset F$.

Theorem 4.21: Let F_α, F_β, F_δ be maximal gη-open sets such that $F_\alpha \neq F_\beta$. If $F_\alpha \cap F_\beta \subset F_\delta$, then either $F_\alpha = F_\delta$ or $F_\beta = F_\delta$

Proof: Given that $F_\alpha \cap F_\beta \subset F_\delta$. If $F_\alpha = F_\delta$. Then there is nothing to prove.

If $F_\alpha \neq F_\delta$ then we have to prove $F_\beta = F_\delta$. Now $F_\beta \cap F_\delta = F_\beta \cap (F_\delta \cap X) = F_\beta \cap (F_\delta \cap (F_\alpha \cup F_\beta)$ (by thm. 4.19 (ii)) $= F_\beta \cap ((F_\delta \cap F_\alpha) \cup (F_\delta \cap F_\beta)) = (F_\beta \cap F_\delta \cap F_\alpha) \cup (F_\beta \cap F_\delta \cap F_\beta) = (F_\alpha$

$\cap F_\beta) \cup (F_\delta \cap F_\beta)$ (by $F_\alpha \cap F_\beta \subset F_\delta$) = $(F_\alpha \cup F_\delta) \cap F_\beta = X \cap F_\beta$ (Since F_α and F_δ are maximal gη-open sets by theorem 4.19 (ii), $F_\alpha \cup F_\delta = X = F_\beta$. That is $F_\beta \cap F_\delta = F_\beta \Rightarrow F_\beta \subset F_\delta$ Since F_β and F_δ are maximal gη-open sets, we have $F_\beta = F_\delta$ Therefore $F_\beta = F_\delta$

Theorem 4.22: Let F_α, F_β and F_δ be different maximal gη-open sets to each other. Then $(F_\alpha \cap F_\beta) \not\subset (F_\alpha \cap F_\delta)$.

Proof: Let $(F_\alpha \cap F_\beta) \subset (F_\alpha \cap F_\delta) \Rightarrow (F_\alpha \cap F_\beta) \cup (F_\delta \cap F_\beta) \subset (F_\alpha \cap F_\delta) \cup (F_\delta \cap F_\beta) \Rightarrow (F_\alpha \cup F_\delta) \cap F_\beta \subset F_\delta \cap (F_\alpha \cup F_\beta)$. Since by theorem 4.19 (ii), $F_\alpha \cup F_\delta = X$ and $F_\alpha \cup F_\beta = X \Rightarrow X \cap F_\beta \subset F_\delta \cap X \Rightarrow F_\beta \subset F_\delta$ From the definition of maximal gη-open set it follows that $F_\beta = F_\delta$, which is a contradiction to the fact that F_α, F_β and F_δ are different to each other. Therefore $(F_\alpha \cap F_\beta) \not\subset (F_\alpha \cap F_\delta)$.

Theorem 4.23: Let F be a maximal gη-open set and x be an element of F. Then $F = \cup$ { S: S is a gη-open set containing x such that $F \cup S \neq X$}.

Proof: By theorem 4.21 and fact that F is a gη-open set containing x, we have $F \subset$ { S: S is a gη-open set containing x such that $F \cup S \neq X$} $\subset F$.

Theorem 4.24: Let F be a proper nonempty cofinite gη-open set. Then there exist a cofinite maximal gη-open set E such that $F \subset E$.

Proof: If F is maximal gη-open set, we may set E = F. If F is not a maximal gη-open set, then there exist a cofinite gη-open set F_1 such that $F \subset F_1 \neq X$. If F_1 is a maximal gη-open set, we may set $E = F_1$. If F_1 is not a maximal gη-open set, then there exist a cofinite gη-open set F_2 such that $F \subset F_1 \subset F_2 \neq X$. Continuing this process, we have a sequence of gη-open, $F \subset F_1 \subset F_2 \subset \ldots F_k \subset \ldots\ldots$ Since F is a cofinite set, this process repeats only finitely. Then, finally we get a maximal gη-open set $E = E_n$ for some positive integer n.

Theorem 4.25: Let F be a maximal gη-open set. If x is an element of X-F. Then $X\text{-}F \subset E$ for any gη-open set E containing x.

Proof: Let F be a maximal gη-open set and x in X-F. $E \not\subset F$ for any gη-open set E containing x. Then $E \cup F = X$ by theorem 4.19 (ii).Therefore $X\text{-}F \subset E$.

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