Characterization of Unit Group of the Unitary Group Algebra

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Abstract

Let $U(2,2)$ and $U(2,3)$ denote the unitary group of 2×2 matrices over the finite field of order 4 and 9 respectively. In this paper, we determine the structure of the unit group of the semi simple group algebra of $U(2,2)$ and $U(2,3)$ respectively over an arbitrary field.

Keywords: Unitary group, Group algebra, Wedderburn decomposition, Unit group.

Introduction

Let $\mathcal{K}_q$ denotes the finite field of characteristic p, where p is a prime number. Let $\mathcal{K}_qG$ denotes the group algebra of the finite group G over $\mathcal{K}_q$. Let $\mathcal{U}\ (\mathcal{K}_qG)$ be the group of units of the group algebra $\mathcal{K}_qG$. It is a typical problem in group theory that regularly arises to ascertain the unit group of the group algebra of the finite group. Combinatorial number theory issues can also be resolved by the structure of the unit groups (See [14]). The structure of unit groups in various group algebras has been recently investigated and characterized.

This has been motivated many scientists to investigate the explicit structure of the unit group of $\mathcal{K}_qG$. The unit group of the group algebra over an abelian group have been discussed in [13]. In [4, 5, 9, 12], many research have been done to determine the unit group $\mathcal{U}\ (\mathcal{K}_qG)$ of group algebra for non -abelian groups. For dihedral groups, [1, 3, 6, 7] addressed the structure of the unit group $\mathcal{U}\ (\mathcal{K}_qG)$ of the group algebra $\mathcal{K}_qG$. Also, R.K. Sharma discussed the structure of the unit group of the group algebra of the special linear group in [10, 11].

The difficulty of solving the equation will grow as the size of the n increases. J.Z. Goncslves, et al described the group algebras for the unitary units in [2] and Neha Makhijani et al described the order of unitary subgroup of the modular group algebra $\mathcal{K}_{2^k}D_{2N}$ in [8]. In this paper, we defined the unit group $\mathcal{U}\mathcal{K}_q\big(U(2,2)\big)$ and $\mathcal{U}\ \mathcal{K}_q\big(U(2,3)\big)$ for the unitary group algebra $\mathcal{K}_qU(2,2)$and $\mathcal{K}_qU(2,3)$ respectively over the finite field $\mathcal{K}_q$ and the order of the unitary group is not divisible by the characteristic p to make the unitary group algebra semi simple and we provide a straightforward method for locating the $n_i's$.

Additionally, we provide the detailed characterization of the group algebra $\mathcal{K}_qU(2,2)$ and $\mathcal{K}_qU(2,3)$ in theorem 3.1 and 3.2. While part 3 contains the major outcome, section 2 deals with the preliminary information.

PRELIMINARIES

Definition 2.1. The unitary group $U(n,\mathcal{K}_t)$ is the set of $n \times n$ unitary matrices over $\mathcal{K}_{t^2}$ which is defined as,

$$U(n,\mathcal{K}_t) = \{S \in GL_n(\mathcal{K}_{t^2}) \text{ such that } SS^* = S^*S = I_n\}.$$

The order of the unitary group $U(n,\mathcal{K}_t)$ is given by,

$$t^{(n^2-n)/2} \prod_{i=1}^{n} \left(t^i - (-1)^i\right).$$

Definition 2.2. If $p \nmid |x|$, where $|x|$ denotes the order of x in G, then $x \in G$ is said to be the p-regular element.

Let s be the least common multiple of order of all the p-regular elements in G. The primitive s^{th} root of unity over $\mathcal{K}$ is represented by η. Therefore, $\mathcal{K}(\eta)$ is the splitting field over $\mathcal{K}$. Now, define the set $T_{G,\mathcal{K}} = \{t \mid \sigma(\eta) = \eta^t$, where $\sigma \in \text{Gal}(\mathcal{K}(\eta):\mathcal{K})\}$.

Definition 2.3. For any p -regular element $a \in G, \gamma_a = \sum_{h \in C_a} h$, the cyclotomic $\mathcal{K}$-class of γ_a is defined as,

$$S\mathcal{K}(\gamma_a) = \left\{\gamma_{a^t} \mid t \in T_{G,\mathcal{K}}\right\}.$$

Proposition 2.1. The number of non-isomorphic simple components of $\frac{\mathcal{K}G}{J(\mathcal{K}G)}$ is same as cyclotomic $\mathcal{K}$-classes in G.

Proposition 2.2. Let G' be the commutator subgroup of G and $\mathcal{K}G$ be a semi simple group algebra then,

$$\mathcal{K}G \simeq \mathcal{K}\left(\frac{G}{G'}\right) \oplus \triangle\,(G,G').$$

Theorem 2.1. Assume that G has t cyclotomic $\mathcal{K}$-classes and $\text{Gal}(\mathcal{K}(\eta):\mathcal{K})$ is a cyclic group, then $|S_i| = [K_i:\mathcal{K}]$ with appropriate index ordering if $S_1, S_2, \cdots, S_t$ are the cyclotomic $\mathcal{K}$-classes of G and $K_1, K_2, \cdots, K_t$ are the simple components of $Z\left(\frac{\mathcal{K}G}{J(\mathcal{K}G)}\right)$.

Main Result

In this section, let G_1 denotes $U(2,2)$ and G_2 denotes $U(2,3)$ and we define the structure of unit group of the group algebra $\mathcal{K}_qG_1$ and $\mathcal{K}_qG_2$ for suitable prime number p, where $q = p^k$. By Maschke's theorem, for $p > 3$ the group algebra $\mathcal{K}_qG_1$ and $\mathcal{K}_qG_2$ are semi simple. Now, we discuss the Wedderburn decomposition of $\mathcal{K}_qG_1$ and $\mathcal{K}_qG_2$ for $p > 3$.

Theorem 3.1. The Wedderburn decomposition of $\mathcal{K}_qG_1$, where G_1 is the unitary group defined above is given by,

(i) $\mathcal{K}_q G_1 \cong (\mathcal{K}_q)^6 \oplus \mathbf{M}(2, \mathcal{K}_q)^3$, when $p^k \equiv 1 \bmod 6$

(ii) $\mathcal{K}_q G_1 \cong (\mathcal{K}_q)^2 \oplus (\mathcal{K}_{q^2})^2 \oplus \mathbf{M}(2, \mathcal{K}_{q^2}) \oplus \mathbf{M}(2, \mathcal{K}_q)$, when $p^k \equiv 5 \bmod 6$.

Proof. The order of the group G_1 is 18 and it has 9 conjugacy classes. The representatives, size and the order of representatives are tabulated below,

Representative	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9
Size	1	2	3	2	3	2	3	1	1
Order	1	3	6	3	6	2	2	3	3

where,

$$\xi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \xi_4 = \begin{pmatrix} 0 & x \\ x & x \end{pmatrix}, \quad \xi_7 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\xi_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \xi_5 = \begin{pmatrix} 0 & x+1 \\ x+1 & 0 \end{pmatrix}, \quad \xi_8 = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$$
$$\xi_3 = \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}, \quad \xi_6 = \begin{pmatrix} 0 & x+1 \\ x+1 & x+1 \end{pmatrix}, \quad \xi_9 = \begin{pmatrix} x+1 & 0 \\ 0 & x+1 \end{pmatrix}$$

Clearly, the exponent of G_1 is 6. Since $\mathcal{K}_q G_1$ is semi-simple, by Wedderburn decomposition theorem,

$$\mathcal{K}_q G_1 \cong \bigoplus_{i=1}^{r} \mathbf{M}(n_i, \mathcal{K}_i).$$

In the above equation, $\mathcal{K}_i$ is a finite extension of $\mathcal{K}_q$. The derived subgroup of G_1 is C_3 and its factor group $G_1/G_1' \simeq C_6$. Since $T_{G_1,\mathcal{K}_q} = \{1,5\} \bmod 6$, we proceed the proof in two cases.

Case(i): For $p^k \equiv 1 \bmod 6$ and by proposition 2.2,

$$\mathcal{K}_q G_1 \cong (\mathcal{K}_q)^6 \bigoplus_{i=1}^{3} \mathbf{M}(n_i, \mathcal{K}_i).$$

The cardinality of cyclotomic $\mathcal{K}_q$-class of γ_ξ is 1, for all ξ in G_1 (i.e., $|S\mathcal{K}_q(\gamma_\xi)| = 1, \forall \xi \in G_1$). Using proposition 2.1 and theorem 2.1,

$$\mathcal{K}_q G_1 \cong (\mathcal{K}_q)^6 \bigoplus_{i=1}^{3} \mathbf{M}(n_i, \mathcal{K}_q) \Rightarrow 12 = n_1^2 + n_2^2 + n_3^2.$$

The values of $n_1 = n_2 = n_3 = 2$. Hence, the Wedderburn decomposition of $\mathcal{K}_q G_1$ is,

$$\mathcal{K}_q G_1 \cong (\mathcal{K}_q)^6 \oplus \mathbf{M}(2, \mathcal{K}_q)^3.$$

Case(ii): For $p^k \equiv 5$ mod 6 and by proposition 2.2,

$$\mathcal{K}_q G_1 \cong \left(\mathcal{K}_q\right)^2 \oplus \left(\mathcal{K}_{q^2}\right)^2 \bigoplus_{i=1}^{3} \mathbf{M}(n_i, \mathcal{K}_i).$$

The cyclotomic $\mathcal{K}_q$-classes of γ_{ξ_i} are,

$$S\mathcal{K}_q(\gamma_{\xi_1}) = \{\gamma_{\xi_1}\}, S\mathcal{K}_q(\gamma_{\xi_2}) = \{\gamma_{\xi_2}\}, S\mathcal{K}_q(\gamma_{\xi_7}) = \{\gamma_{\xi_7}\},$$
$$S\mathcal{K}_q(\gamma_{\xi_3}) = \{\gamma_{\xi_3}, \gamma_{\xi_5}\}, S\mathcal{K}_q(\gamma_{\xi_4}) = \{\gamma_{\xi_4}, \gamma_{\xi_6}\}, S\mathcal{K}_q(\gamma_{\xi_8}) = \{\gamma_{\xi_8}, \gamma_{\xi_9}\}.$$

Using proposition 2.1 and theorem2.1,

$$\mathcal{K}_q G_1 \cong \left(\mathcal{K}_q\right)^2 \oplus \left(\mathcal{K}_{q^2}\right)^2 \oplus \mathbf{M}(n_1, \mathcal{K}_{q^2}) \oplus \mathbf{M}(n_2, \mathcal{K}_q) \Rightarrow 12 = 2n_1^2 + n_2^2.$$

The values of $n_1 = n_2 = 2$. Hence, the Wedderburn decomposition of $\mathcal{K}_q G_1$ is,

$$\mathcal{K}_q G_1 \cong \left(\mathcal{K}_q\right)^2 \oplus \left(\mathcal{K}_{q^2}\right)^2 \oplus \mathbf{M}(2, \mathcal{K}_{q^2}) \oplus \mathbf{M}(2, \mathcal{K}_q).$$

Corollary 3.1. Notations as above, the unit group $\mathcal{K}_q G_1$ is,

Conditions on p^k	$\mathcal{U}(\mathcal{K}_q(U(2,2)))$
$p^k \equiv 1$ mod 6	$\left(\mathcal{K}_q^*\right)^6 \oplus GL(2, \mathcal{K}_q)^3$
$p^k \equiv 5$ mod 6	$\left(\mathcal{K}_q^*\right)^2 \oplus \left(\mathcal{K}_{q^2}^*\right)^2 \oplus GL(2, \mathcal{K}_{q^2}) \oplus GL(2, \mathcal{K}_q)$

Theorem 3.2. The Wedderburn decomposition of $\mathcal{K}_q G_2$, where G_2 is the unitary group defined above, is given by,

(i) $\mathcal{K}_q G_2 \cong \left(\mathcal{K}_q\right)^4 \oplus \mathbf{M}(2, \mathcal{K}_q)^6 \oplus \mathbf{M}(3, \mathcal{K}_q)^4 \oplus \mathbf{M}(4, \mathcal{K}_q)^2$, when $p^k \equiv \{1,5,13,17\}$ mod 24.

(ii) $\mathcal{K}_q G_2 \cong \left(\mathcal{K}_q\right)^2 \oplus \mathcal{K}_{q^2} \oplus \mathbf{M}(2, \mathcal{K}_q)^2 \oplus \mathbf{M}(2, \mathcal{K}_{q^2}) \oplus \mathbf{M}(2, \mathcal{K}_{q^2}) \oplus \mathbf{M}(3, \mathcal{K}_q)^2 \oplus \mathbf{M}(3, \mathcal{K}_{q^2}) \oplus \mathbf{M}(4, \mathcal{K}_{q^2})$, when $p^k \equiv \{7,11,19,23\}$ mod 24.

Proof. The order of the group G_2 is 96 and it has 16 conjugacy classes. The representatives (Rep), size and the order of representatives are tabulated below,

Rep	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}	ξ_{15}	ξ_{16}
Size	1	8	8	12	6	6	6	8	8	12	6	6	6	1	1	1
Order	1	12	12	8	4	4	4	6	3	8	4	4	2	2	4	4

where

$$\xi_1=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \xi_5=\begin{pmatrix}0&x\\2x&2x+1\end{pmatrix},\ \xi_9=\begin{pmatrix}0&x+1\\x+1&2\end{pmatrix},\ \xi_{13}=\begin{pmatrix}0&1\\1&0\end{pmatrix},$$
$$\xi_2=\begin{pmatrix}0&1\\1&x+1\end{pmatrix},\ \xi_6=\begin{pmatrix}0&x\\2x&x+2\end{pmatrix},\xi_{10}=\begin{pmatrix}0&2x+1\\x+2&0\end{pmatrix},\xi_{14}=\begin{pmatrix}2&0\\0&2\end{pmatrix},$$
$$\xi_3=\begin{pmatrix}0&1\\1&2x+2\end{pmatrix},\xi_7=\begin{pmatrix}0&x+1\\x+1&0\end{pmatrix},\ \xi_{11}=\begin{pmatrix}0&2x+1\\2+x&2x\end{pmatrix},\xi_{15}=\begin{pmatrix}x+1&0\\0&x+1\end{pmatrix},$$
$$\xi_4=\begin{pmatrix}0&x\\2x&0\end{pmatrix},\xi_8=\begin{pmatrix}0&x+1\\x+1&1\end{pmatrix},\xi_{12}=\begin{pmatrix}0&2x+1\\2+x&0\end{pmatrix},\xi_{16}=\begin{pmatrix}2x+2&0\\0&2x+2\end{pmatrix}.$$

Clearly, the exponent of G_2 is 24. Since $\mathcal{K}_q G_2$ is semi-simple, by Wedderburn decomposition theorem,

$$\mathcal{K}_q G_2 \cong \bigoplus_{i=1}^{r} \mathbf{M}(n_i, \mathcal{K}_i).$$

Here, $\mathcal{K}_i$ is a finite extension of $\mathcal{K}_q$. Observe that the derived subgroup of the group G_2 is $SL(2,3)$ and the factor group $G_2/G_2' \simeq C_4$. Since $T_{G_2,\mathcal{K}_q}=\{1,5,7,11,13,17,19,23\}$ mod 24, we proceed the proof in 8 cases and consolidated them into two.

Case (1): For $p^k \equiv \{1,5,13,17\}$ mod 24 and by proposition 2.2,

$$\mathcal{K}_q G_2 \cong (\mathcal{K}_q)^4 \bigoplus_{i=1}^{12} \mathbf{M}(n_i, \mathcal{K}_i).$$

The cardinality of cyclotomic $\mathcal{K}_q$-class of γ_ξ is 1, for all ξ in G_2 (i.e., $|S\mathcal{K}_q(\gamma_\xi)|=1, \forall \xi \in G_2$).

By proposition 2.1 and theorem 2.1,

$$\mathcal{K}_q G_2 \cong (\mathcal{K}_q)^4 \bigoplus_{i=1}^{12} \mathbf{M}(n_i, \mathcal{K}_q) \Rightarrow 92=\sum_{i=1}^{12} n_i^2, n_i \geq 2.$$

There are 2 possible choices for $n_i's$,

(2,2,2,2,2,2,2,2,2,2,4,6) and (2,2,2,2,2,2,3,3,3,3,4,4).

To find it uniquely, take the normal subgroup $N=C_2$ of G_2 and the factor group $G_2/N \simeq A_4 \rtimes C_4$ and $|G_2/N|=48$.

$$\Rightarrow 44=n_1^2+n_2^2+\cdots+n_6^2.$$

The values of n_i 's are (2,2,3,3,3,3) and the Wedderburn decomposition is $\mathcal{K}_q(G_2/N) \cong (\mathcal{K}_q)^4 \oplus \mathbf{M}(2,\mathcal{K}_q)^2 \oplus \mathbf{M}(3,\mathcal{K}_q)^4$. Therefore, the choices are reduced uniquely to

(2,2,2,2,2,2,3,3,3,3,4,4).

Hence, the Wedderburn decomposition is,

$$\mathcal{K}_q G_2 \cong (\mathcal{K}_q)^4 \oplus \mathbf{M}(2,\mathcal{K}_q)^6 \oplus \mathbf{M}(3,\mathcal{K}_q)^4 \oplus \mathbf{M}(4,\mathcal{K}_q)^2.$$

Case (2): For $p^k \equiv \{7,11,19,23\}$ mod 24 and by proposition 2.2,

$$\mathcal{K}_q G_2 \cong (\mathcal{K}_q)^2 \oplus (\mathcal{K}_{q^2}) \bigoplus_{i=1}^{12} \mathbf{M}(n_i,\mathcal{K}_i).$$

The cyclotomic $\mathcal{K}_q$ classes of γ_{ξ_i} are,

$$S\mathcal{K}_q(\gamma_{\xi_1}) = \{\gamma_{\xi_1}\}, S\mathcal{K}_q(\gamma_{\xi_7}) = \{\gamma_{\xi_7}\}, S\mathcal{K}_q(\gamma_{\xi_8}) = \{\gamma_{\xi_8}\}, S\mathcal{K}_q(\gamma_{\xi_9}) = \{\gamma_{\xi_9}\},$$
$$S\mathcal{K}_q(\gamma_{\xi_{13}}) = \{\gamma_{\xi_{13}}\}, S\mathcal{K}_q(\gamma_{\xi_{14}}) = \{\gamma_{\xi_{14}}\},$$
$$S\mathcal{K}_q(\gamma_{\xi_2}) = \{\gamma_{\xi_2},\gamma_{\xi_3}\}, S\mathcal{K}_q(\gamma_{\xi_4}) = \{\gamma_{\xi_4},\gamma_{\xi_{10}}\}, S\mathcal{K}_q(\gamma_{\xi_5}) = \{\gamma_{\xi_5},\gamma_{\xi_{11}}\},$$
$$S\mathcal{K}_q(\gamma_{\xi_6}) = \{\gamma_{\xi_6},\gamma_{\xi_{12}}\}, S\mathcal{K}_q(\gamma_{\xi_{15}}) = \{\gamma_{\xi_{15}},\gamma_{\xi_{16}}\}.$$

By proposition 2.1 and theorem 2.1,

$\mathcal{K}_q G_2 \cong (\mathcal{K}_q)^2 \oplus (\mathcal{K}_{q^2}) \oplus_{i=1}^{4} \mathbf{M}(n_i,\mathcal{K}_{q^2}) \oplus_{i=5}^{8} \mathbf{M}(n_i,\mathcal{K}_q) \Rightarrow 92 = \sum_{i=1}^{4} 2n_i^2 + \sum_{i=5}^{8} n_i^2, n_i \geq 2.$

To get it uniquely, repeat the process same as above and observe that the Wedderburn decomposition of $\mathcal{K}_q(G_2/N) \cong (\mathcal{K}_q)^2 \oplus \mathcal{K}_{q^2} \oplus \mathbf{M}(2,\mathcal{K}_q)^2 \oplus \mathbf{M}(3,\mathcal{K}_q)^2 \oplus \mathbf{M}(3,\mathcal{K}_{q^2})$.

Therefore, we get the unique $n_i's$ values of G_2, (2,2,3,4,2,2,3,3). Hence, the Wedderburn decomposition is,

$$\mathcal{K}_q G_2 \cong (\mathcal{K}_q)^2 \oplus \mathcal{K}_{q^2} \oplus \mathbf{M}(2,\mathcal{K}_q)^2 \oplus \mathbf{M}(2,\mathcal{K}_{q^2})^2 \oplus \mathbf{M}(3,\mathcal{K}_q)^2 \oplus \mathbf{M}(3,\mathcal{K}_{q^2}) \oplus \mathbf{M}(4,\mathcal{K}_{q^2}).$$

Corollary 3.2. Notations as above, the unit group of $\mathcal{K}_q G_2$,

Conditions on p^k	$\mathcal{U}(\mathcal{K}_q(U(2,3)))$
$p^k \equiv \{1,5,13,17\}$mod24	$(\mathcal{K}_q^*)^4 \oplus GL(2,\mathcal{K}_q)^6 \oplus GL(3,\mathcal{K}_q)^4 \oplus GL(4,\mathcal{K}_q)^2$
$p^k \equiv \{7,11,19,23\}$mod24	$(\mathcal{K}_q^*)^2 \oplus (\mathcal{K}_{q^2}^*) \oplus GL(2,\mathcal{K}_q)^2 \oplus GL(2,\mathcal{K}_{q^2})^2$ $\oplus GL(3,\mathcal{K}_q^*)^2 \oplus GL(3,\mathcal{K}_{q^2}) \oplus GL(4,\mathcal{K}_{q^2})$

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