Crossing CR-fuzzy Q –ideals and its Correlation Coefficients

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Abstract
As an extension of bipolar fuzzy sets, we consider the concept of crossing CR-fuzzy Q -ideals(sub-algebras) of a Q -algebra. Relationships between the crossing CR-fuzzy BCK- ideal (sub-algebras) and the crossing CR-fuzzy Q -ideals of a Q -algebra are investigated. In addition, the homomorphic images and the inverse images of the crossing CR-fuzzy Q -ideals are also studied. Related properties of crossing CR-fuzzy Q -ideals are explored and discussed. Moreover the Cartesian crossing CR-fuzzy Q -ideals are investigated. Finally, novel correlation coefficient between two crossing CR-fuzzy sets are also studied...
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Introduction

Iseki et al.[10] studied the concept of BCK-algebras in 1966 .As an extension of set theoretic difference and propositional calculus, Iseki [9] introduced the notion of a BCI-algebra, which is a generalization of the BCK-algebra. Since then, numerous mathematical papers have been written investigating the algebraic properties of the BCK/BCI-algebras and their relationships with other structures, including lattices and Boolean algebras. In particular, there is a great deal of literature which has been produced on the theory of BCK/BCI-algebras. In particular, the emphasis seems to have been put on the ideal theory of BCK/BCI-algebras. J. Neggers, S. S. Ahn, and H. S. Kim [22] introduced a new notion, called Q-algebra, which is a generalization of BCH/BCI/BCK-algebras and generalizes some theorems discussed in BCI-algebras. Fuzzy set theory is the concept and technique that applies a form of mathematical precision to human thought processes that in many ways are imprecise and ambiguous by the standards of classical mathematics. Fuzzy sets, intuitionistic fuzzy sets, interval-valued fuzzy sets, bipolar fuzzy sets, and other mathematical tools are often useful approaches to dealing with uncertainties. In 1956, Zadeh [29] introduced the notion of fuzzy sets. At present, this concept has been applied to many mathematical branches. In fuzzy set theory, there are several types of fuzzy set extensions, such as intuitionistic fuzzy sets, interval valued fuzzy sets, vague sets, and so on. In 1991, Xi [25] applied the concept of fuzzy sets to BCI, BCK, and MV-algebras. The intuitionistic fuzzy set theory is useful in various application areas, such as algebraic structures, control systems, and various engineering fields. Many researchers have explored various applications of intuitionistic fuzzy sets such as medical application, real life situations and education . Recently, Lee [19] introduced an extension of fuzzy sets named bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval [0, 1] to [-1, 1]. Recently Jun et al. [15,16] introduced a new function which is called negative-valued function, and constructed N-structures. They applied N-structures to BCK/BCI-algebras, and discussed N-subalgebras and N-ideals in BCK/BCI-algebras. Jun et al. [17, 18] established an extension of a bipolar-valued fuzzy set, which is introduced by Lee [19]. They called it a crossing cubic structure, and investigated several properties. They They applied crossing cubic structures to BCK/BCI-algebras, and studied crossing cubic sub algebras. Yager [27,28] launched a nonstandard fuzzy set referred to as Pythagorean fuzzy set which is the generalization of intuitionistic fuzzy sets. The construct of Pythagorean fuzzy sets can be used to characterize uncertain information more sufficiently and accurately than intuitionistic fuzzy set. [7] defined a new generalized Pythagorean fuzzy set is called (3, 2)-Fuzzy sets. In 2020. Fermatean fuzzy sets proposed by Senapati and Yager [24], can handle uncertain information more easily in the process of decision making. They also defined basic operations over the Fermatean fuzzy sets. The main advantage of Fermatean fuzzy sets is that it can describe more uncertainties than Pythagorean fuzzy sets, which can be applied in many decision-making problems. The relevant research can be referred to [2], SR-Fuzzy sets).Pythagorean fuzzy set is one of the successful extensions of the fuzzy set for handling uncertainties in information. Under this environment, Salih et al, [23], introduce a new type of generalized fuzzy sets is called CR-fuzzy sets and compare CR-fuzzy sets with Pythagorean fuzzy sets and Fermatean fuzzy sets. The set operations, score function and accuracy function of CR-fuzzy sets will study along with their several properties. Recently Jun et al [11] introduced the concept of the (m; n)-fuzzy set which is the subclass of intuitionistic fuzzy set, Pythagorean fuzzy set, (3; 2)-fuzzy set, Fermatean fuzzy set , n-Pythagorean fuzzy set and compared with them. They introduced some operations for the (m; n)-fuzzy set, investigate their properties and applied the (m; n)-fuzzy set to BCK-algebras and BCI-algebras. They introduced the (m; n)-fuzzy subalgebra in BCK-algebras and BCI-algebras and investigate their properties. Ahn etal [1] apply the concept of (2, 3)-fuzzy sets to BCK-algebras and BCI-algebas.,Ibrahim et al. [7]introduced (3; 2)-fuzzy sets and applied it to topological spaces.

In this paper, we modify the ideas of Jun et al. [17,18] and Salih et al, [16], to introduce the notion of crossing CR-fuzzy Q -ideals of a Q -algebra and investigate its properties. Furthermore we study the homomorphic image and inverse image of crossing CR-fuzzy Q -ideals of a Q -algebra under homomorphism of Q -algebras. Moreover, the Cartesian product of crossing CR-fuzzy Q -ideals in Cartesian product Q-algebras is given. Finally, novel correlation coefficient between two crossing CR-fuzzy sets are also studied.

2. Preliminaries

Definition 2.1 [9] Let X be a set with a binary operation " $*$ " and a constant 0,then $(X,*,0)$ is called a BCI -algebra ,if it satisfies the following axioms:

(BCI -1) $((x*y)*(x*z))*(z*y)=0$

(BCI -2) $(x*(x*y))*y=0$

(BCI -3) $x*x=0$

(BCI -4) $x*y=0$ and $y*x=0$ implies $x=y$

for all $x,y,z\in X$

If a BCI -algebra X satisfies the identity $0*x=0$, for all $x\in X$, then X is called a BCK algebra. It is known that the class of BCK -algebras is a proper subclass of the class of BCI -algebras.

Definition 2.2 [9] Let $(X,*,0)$ be a BCK – algebra , and let S be a non – empty subset of X, then S is called a sub - algebra of X, if for all $x,y\in S$, $x*y\in S$, i.e S is closed under the binary operation $*$ of X.

Definition 2.3 [22] An algebraic system $(X,*,0)$ of type (2, 0) is called a Q-algebra if it satisfying the following axioms:

(1) $x*x=0$,

(2) $x*0=x$,

(3) $(x*y)*z=(x*z)*y$ for all x, y and $z\in X$.

For brevity we also call X a Q-algebra. In X we can define a binary relation $\leq$ by $x\leq y$ if and only if $x*y=0$.

Example 2.4. [22] Let $X = \{0, 1, 2\}$ in which $*$ is defined by the following table:

Table (1)

*	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then $(X;*,0)$ is a Q-algebra.

Theorem 2.5[22].Every BCK-algebra is a Q-algebra , but the converse is not true

Example 2.6[22]**.**Let X={0,1,2,3}and the binary operation ”$*$” on X is defined as follows.

Table (2)

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	0	0	0
3	3	3	3	0

Clearly (X,$*$,0) is a Q-algebra but it is not BCK-algebra.

Proposition 2.7 [22]. If $(X;*,0)$ is a Q-algebra, then

(4) $(x*(x*y))*y=0$ for any $x,y \in X$.

The following theorems give relations between Q-algebras and the different algebras (BCK / BCI/ BCH-algebras)

Theorem 2.8 [22]. Every BCH-algebra X is a Q-algebra. Every Q-algebra X satisfying the condition (4) is a BCH-algebra.

Theorem 2.9[22]. Every Q-algebra satisfying the condition (BCI -4) and ((BCI -1)), is a BCI-algebra.

Theorem 2.10. [22]. Every Q-algebra X satisfying the condition ((BCI -1)), (BCI -4) and

(6) $(x*y)*x=0$

for any $x, y \in X$, is a BCK-algebra.

Remark 2.11.**[22]** Every QS-algebra X is a Q-algebra. Every Q-algebra X satisfying the condition

(5) $(x*y)*(x*z)=z*y$

for any $x, y \in X$, is QS-algebra.

Definition 2.12 [10]. A non empty subset I of a BCK-algebra X is called an ideal of X if it satisfies

(I_1) $0 \in I$,

(I_2) $x \in I$ *and* $y * x \in I$ implies $y \in I$ *for all* $x , y \in X$.

Definition 2.13. **[21]** A non empty subset I of a Q-algebra X is called a Q-ideal of X if

(Q_1) $0 \in I$,

(Q_2) $(x*y)*z \in I$ and $y \in I$ imply $x*z \in I$ for all x, y and $z \in X$.

Definition 2.14**[29].** Let X be non-empty set , a fuzzy set μ in X is a function μ : X → [0,1] .

Definition 2.15. [25] Let X be a BCK-algebra. a fuzzy set μ in X is called a fuzzy BCK-ideal of X if it satisfies:

(FI_1) $\mu(0) \geq \mu(x)$,

(FI_2) $\mu(x) \geq \min\{\mu(x*y), \mu(y)\}$, for all x, y and $z \in X$.

Definition 2.16**[21].** Let X be a Q-algebra. A fuzzy set μ in X is called a fuzzy Q-ideal of X if it satisfies:

(FI_1) $\mu(0) \geq \mu(x)$,

(FQ) $\mu(x*z) \geq \min\{\mu((x*y)*z), \mu(y)\}$, for all x, y and $z \in X$.

Lemma 2.17.**[21]** Any fuzzy Q-ideal of Q-algebra is a fuzzy BCK-ideal of *X*.

Remark 2.18:Let $\mu_A : X \to [0,1]$ and $\lambda_A : X \to [0,1]$ be fuzzy sets in a set X. The structure $A = \{(x, \mu_A(x), \lambda(x)) \mid x \in X\}$, is called

(1) an intuitionistic fuzzy set in X (See [3]), if it satisfies: $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$

(2) an a Pythgorean fuzzy set in X (See [27]), if it satisfies : $0 \leq \mu_A^2(x) + \lambda_A^2(x) \leq 1$

3) an a n- Pythgorean fuzzy set in X (See [4]), if it satisfies: $0 \leq \mu_A^n(x) + \lambda_A^n(x) \leq 1$

(4) an a Fermatean fuzzy set in X (See [24]), if it satisfies: $0 \leq \mu_A^3(x) + \lambda_A^3(x) \leq 1$

(5) (3; 2)- fuzzy set in X (See [1,7]), if it satisfies : $0 \leq \mu_A^3(x) + \lambda_A^2(x) \leq 1$

6) (n; m)- fuzzy set in X (See [11]), if it satisfies : $0 \le \mu_A^n(x) + \lambda_A^m(x) \le 1$

SR-Fuzzy set fuzzy set in X (See [2]), if it satisfies: $0 \le \mu_A^2(x) + \sqrt{\lambda_A(x)} \le 1$

CR-FS - fuzzy set in X (See [23]), if it satisfies: $0 \le \mu_A^3(x) + \sqrt[3]{\lambda_A(x)} \le 1$

Definition 2.19. [15,16] Let X be a non-empty set. A function from X → [−1,0] is called a negative-valued function (*N*-function) from X to [−1,0]. Denote by F(X, [-1, 0]) the collection of functions from a set X to [-1, 0]. We say that, an element of F(X, [-1, 0]) is a negative-valued function from X to [-1, 0] (briefly , *N*-function on X). By an *N*-structure we mean an ordered pair (X, τ_A^N), where τ_A^N is an *N*-function on X.
Let the set of all *N*-function on X denoted by $NF(X)$.

Definition 2.20 ([15,16]). Let X be a nonempty set and let $A, B \in NF(X)$.
(i) We say that the *N*-function $A = (x, \mu_A^N)$ on X is subset of the *N*-function $B = (x, \mu_B^N)$, denoted by $A \subset B$, if for each $x \in X$, $\mu_A^N(x) \ge \mu_B^N(x)$.
(ii) The complement of N-function $A = (x, \mu^N)$, denoted by $A^C = (x, (\mu^N)^C)$ is a N-function in X defined as: for each $x \in X$, $A^C(x) = (x, (\mu^N)^C(x)) = (x, -1 - \mu^N(x))$
i.e., $(\mu^N(x))^C = -1 - \mu^N(x)$
(iii) The intersection of two *N*-functions $A = (x, \mu_A^N)$ and $B = (x, \mu_B^N)$, denoted by $A \cap B$, is a N-function in X defined as: for each $x \in X$,

$$(A \cap B)(x) := \{(\mu_A^N(x) \vee \mu_B^N(x))\}$$

(iv) The union of two *N*-functions $A = (x, \mu_A^N)$ and $B = (x, \mu_B^N)$, denoted by $A \cup B$, is a N-function in X defined as: for each $x \in X$,
$(A \cup B)(x) := \{((\mu_A^N(x) \wedge \mu_B^N(x))\}$.

C*rossing CR-fuzzy* ***Q*** *- (sub-algebra) ideal on* ***Q*** *- algebras*

In this section, we list some concepts related to crossing CR -fuzzy sets .

***Definition*3.1 Definition 3.1**. Let X be a nonempty set. By a crossing CR -fuzzy set in X we mean a structure $A = \{(x, \mu_A(x), \lambda_A(x), \nu_A^N(x)) \mid x \in X\}$,, where the function μ_A: $X \to [0,1]$ and λ_A: $X \to [0,1]$ denote the degree of membership and degree of non membership, respectively such that $0 \le \mu_A^3(x) + \sqrt[3]{\lambda_A}(x) \le 1$ and $\nu_A^N : X \to [-1,0]$ is *N*-function on X, such that $-1 \le (\nu_A^N)^3(x) \le 0$ for all $x \in X$
The Set of crossing- CR -fuzzy set over X is denoted by $A(X), B(X), \ldots$

Example3.2
Define $A = \{(x, \mu_A(x), \lambda_A(x), \nu_A^N) \mid x \in X\}$, as follows:
$A(0) = \{0.5, 0.3, -0.6\}$, $A(a) = \{0.4, 0.3, -0.5\}$, $A(b) = \{(0.3, 0.2, -0.4)\}$, $A(c) = \{0.2, 0.1, -0.3\}$
It is easy to check that $A = \{(x, \mu_A(x), \lambda_A(x), \nu_A^N(x)) \mid x \in X\}$, is crossing CR - fuzzy set

Definition3.3. Let $A=\{(x,\mu_A(x),\lambda_A(x),\nu_A^N(x))\,|\,x\in X\}$, and $B=\{(x,\mu_B(x),\lambda_B(x),\nu_B^N)\,|\,x\in X\}$, be two crossing CR - fuzzy sets of *X*, then we say:

1. $A\subseteq B$ if and only if

$(1)\,\mu_A^3(x)\le\mu_B^3(x),\sqrt[3]{\lambda_A}(x)\le\sqrt[3]{\lambda_B}(x)$ and $(\nu_A^N)^3(x)\ge(\nu_B^N)^3(x)$

Example3.4. Define $A=\{0.2,0.1,\text{-}0.1\}$an $B=\{0.3,0.2,\text{-}0.2\}$,since

$\mu_A^3(0.2)=0.008\le\mu_B^3(0.3)=0.027,and$

$\sqrt[3]{\lambda_A}(0.1)=0.464158883612779\le\sqrt[3]{\lambda_B}(0.2)=0.5848035476425732$

$(\nu_A^N)^3=(-0.1)^3=\text{-}0.001>(\nu_B^N)^3=(-0.2)^3=\text{-}0.008$ then $A\subseteq B$.

2. $A=B$ if and only if $\mu_A^3(x)=\mu_B^3(x),\sqrt[3]{\lambda_A}(x)=\sqrt[3]{\lambda_B}(x)$ and $(\nu_A^N)^3=(\nu_B^N)^3$.

Definition 3.5. Let $X\ne\Phi$ be Q- algebras , then a crossing - CR – fuzzy set $A=\{(x,\mu_A(x),\lambda_A(x),\nu_A^N(x))\,|\,x\in X\}$,over a set X is called crossing CR –fuzzy Q-sub algebras if the following are holed

(S1) $\mu_A^3(x*z)\ge min\{\mu_A^3(x),\mu_A^3(z)\}$, $\sqrt[3]{\lambda_A}(x*z)\le max\{\sqrt[3]{\lambda_A}(x),\sqrt[3]{\lambda_A}(z)\}$ and

(S2) $(\nu_A^N)^3(x*z)\le max\{(\nu_A^N)^3(x),(\nu_A^N)^3(z)\}$,where $\mu_A(x),\lambda_A(x):X\to[0,1]$ such that $0\le\mu_A^3(x)+\sqrt[3]{\lambda_A}(x)\le 1$, and $\nu_A^N:X\to[-1,0]$ such that $-1\le(\nu_A^N)^3\le 0$.

Definition 3.6. Let $X\ne\Phi$ be Q- algebras , then the crossing - CR – fuzzy set $A=\{(x,\mu_A(x),\lambda_A(x),\nu_A^N(x))\,|\,x\in X\}$,over a set X is called crossing - CR –fuzzy BCK-ideal if the following are holed

(Cbck0) $\mu_A^3(0)\ge\mu_A^3(x),\sqrt[3]{\lambda_A}(0)\le\sqrt[3]{\lambda_A}(x)$and $(\nu_A^N)^3(0)\le(\nu_A^N)^3(x)$

(Cbck1) $\mu_A^3(x)\ge min\{\mu_A^3(x*y),\mu_A^3(y)\},\sqrt[3]{\lambda_A}(x)\le max\{\sqrt[3]{\lambda_A}(x*y),\sqrt[3]{\lambda_A}(y)\}$,and

(Cbck2) $(\nu_A^N)^3(x)\le max\{(\nu_A^N)^3(x*y),(\nu_A^N)^3(y)\}$ where $\mu_A(x),\lambda_A(x):X\to[0,1]$ such that $0\le\mu_A^3(x)+\sqrt[3]{\lambda_A}(x)\le 1$, and $\nu_A^N:X\to[-1,0]$ such that $-1\le(\nu_A^N)^3\le 0$.

Definition 3.7. Let $X\ne\Phi$ be Q- algebras , then the crossing - CR – fuzzy set $A=\{(x,\mu_A(x),\lambda_A(x),\nu_A^N(x))\,|\,x\in X\}$,over a set X is called crossing - CR –fuzzy Q – ideal if the following are holed

(CQ0) $\mu_A^3(0)\ge\mu_A^3(x),\sqrt[3]{\lambda_A}(0)\le\sqrt[3]{\lambda_A}(x)$and $(\nu_A^N)^3(0)\le(\nu_A^N)^3(x)$,

(CQ1) $\mu_A^3(x*z)\ge min\{\mu_A^3((x*y)*z),\mu_A^3(y)\}$, $\sqrt[3]{\lambda_A}(x*z)\le max\{\sqrt[3]{\lambda_A}((x*y)*z),\sqrt[3]{\lambda_A}(y)\}$ and

(CQ2) $(\nu_A^N)^3(x*z)\le max\{(\nu_A^N)^3((x*y)*z),(\nu_A^N)^3(y)\}$,where $\mu_A(x),\lambda_A(x):X\to[0,1]$ such that $0\le\mu_A^3(x)+\sqrt[3]{\lambda_A}(x)\le 1$, and $\nu_A^N:X\to[-1,0]$ such that $-1\le(\nu_A^N)^3\le 0$.

Example 3.8. Let $X = \{0,1,2,3\}$ be a set with a binary operation $*$ defined by the following Table (3) :

Table (3)

We can prove that $(X,*,0)$ is a Q-algebra .

Define $A=\{(x,\mu_A(x),\lambda_A(x),\nu_A^N)\,|\,x\in X\}$,as follows:

$A(0)=\{0.4,,0.1,-0.4\}$, $A(1)=\{0.3,0.2,-0.3\}$

$A(2)=\{0.2,0.1,-0.2\}, A(3)=\{0.1,0.1,-0.1\}$

It is easy to check that $A=\{(x,\mu_A(x),\lambda_A(x),\nu_A^N)\,|\,x\in X\}$, is crossing - CR – fuzzy Q-ideal on Q- algebra.

Lemma 3.9. If $A=\{(x,\mu_A(x),\lambda_A(x),\nu_A^N(x))\,|\,x\in X\}$, is crossing - CR – fuzzy Q-sub-algebra on Q- algebra X, then $\mu_A^3(0)\geq\mu_A^3(x)$, $\sqrt[3]{\lambda_A}(0)\leq\sqrt[3]{\lambda_A}(x)$ and $(\nu_A^N)^3(0)\leq(\nu_A^N)^3(x)$

Proof. In(Definition 3.5) put z = x ,

From ((S1)): $\mu_A^3(x*x)=\mu_A^3(0)\geq min\{\mu_A^3(x),\mu_A^3(x)\}=\mu_A^3(x)$, $\sqrt[3]{\lambda_A}(x*x)=\sqrt[3]{\lambda_A}(0)\leq max\{\sqrt[3]{\lambda_A}(x),\sqrt[3]{\lambda_A}(x)\}=\sqrt[3]{\lambda_A}(x)$ and

From (S2): $(\nu_A^N)^3(x*x)=(\nu_A^N)^3(0)\leq max\{(\nu_A^N)^3(x),(\nu_A^N)^3(x)\}=(\nu_A^N)^3(x)$.

Theorem 3.10 In BCK- algebra , every crossing - CR –fuzzy BCK-ideal of *X* is crossing - CR – fuzzy Q- ideal of X.

Proof. Let $A=\{(x,\mu_A(x),\lambda_A(x),\nu_A^N(x))\,|\,x\in X\}$,be crossing - CR – fuzzy BCK-ideal of *X* and $x,y,z\in X$. Then $\mu_A^3(x)\geq min\{\mu_A^3(x*y),\mu_A^3(y)\}$ ….(1) , put $x*z$ instate of x , we have

$$\mu_A^3(x*z)\geq min\{\mu_A^3((x*z)*y),\mu_A^3(y)\}=min\{\mu_A^3(\overbrace{(x*y)*z}^{Definition\,2.3(3)}),\mu_A^3(y)\}$$, i.e

$$\mu_A^3(x*z)\geq min\{\mu_A^3((x*y)*z),\mu_A^3(y)\}$$

Similar, we can prove that $\sqrt[3]{\lambda_A}(x*z)\leq max\{\sqrt[3]{\lambda_A}((x*y)*z),\sqrt[3]{\lambda_A}(y)\}$ and $(\nu_A^N)^3(x*z)\leq max\{(\nu_A^N)^3((x*y)*z),(\nu_A^N)^3(y)\}$, then

$A=\{(x,\mu_A(x),\lambda_A(x),\nu_A^N(x))\,|\,x\in X\}$,is crossing - CR – fuzzy Q- ideal of X.

Theorem 3.11 In Q -algebra X, every crossing - CR – fuzzy BCK-ideal of *X* is crossing - CR – fuzzy sub- algebra of X.

Proof. Let $A=\{(x,\mu_A(x),\lambda_A(x),\nu_A^N(x))\,|\,x\in X\}$, be crossing - CR – fuzzy BCK-ideal of *X*. Then

$$\mu_A^3(\overbrace{(x*y)*x}^{0\ by\ Theorem\ 2.10})=\mu_A^3(0)\geq\mu_A^3(x)\ ,\ \sqrt[3]{\lambda_A}(\overbrace{(x*y)*x}^{0\ by\ Theorem\ 2.10})=\sqrt[3]{\lambda_A}(0)\leq\sqrt[3]{\lambda_A}(x)\ and$$

$$(\nu_A^N)^3(\overbrace{(x*y)*x}^{0\ by\ Theorem\ 2.10})=(\nu_A^N)^3(0)\leq(\nu_A^N)^3(x)$$

Thus,

$\mu_A^3(x*y) \geq min\{\mu_A^3((x*y)*x), \mu_A^3(x)\} = \mu_A^3(x) \geq min\{\mu_A^3(x), \mu_A^3(y)\}$,

$\sqrt[3]{\lambda_A}(x*y) \leq max\{\sqrt[3]{\lambda_A}((x*y)*x), \sqrt[3]{\lambda_A}(x)\} = \sqrt[3]{\lambda_A}(x) \leq max\{\sqrt[3]{\lambda_A}(x), \sqrt[3]{\lambda_A}(y)\}$,and

$(\nu_A^N)^3(x*y) \leq max\{(\nu_A^N)^3((x*y)*x), (\nu_A^N)^3(x)\} = (\nu_A^N)^3(x) \leq max\{(\nu_A^N)^3(x), (\nu_A^N)^3(y)\}$

fore all $x, y \in X$.❑

Theorem 3.12. If $A = \{(x, \mu_A(x), \lambda_A(x), \nu_A^N(x)) \mid x \in X\}$,is crossing - CR – fuzzy Q-ideal of X, then the set $X = \{(x, \mu_A^3(x) = \mu_O^3(0), \sqrt[3]{\lambda_A}(x) = \sqrt[3]{\lambda_A}(0), (\nu_A^N)^3(x) = (\nu_A^N)^3(0) \mid x \in X\}$,is a Q - ideal of *X*.

Proof. Let $(x*y)*z, y \in X$. Then

, $\mu_A^3((x*y)*z) = \mu_A^3(0) = \mu_A^3(y)$, $\sqrt[3]{\lambda_A}((x*y)*z) = \sqrt[3]{\lambda_A}(0) = \sqrt[3]{\lambda_A}(y)$,

and $(\nu_A^N)^3((x*y)*z) = (\nu_A^N)^3(0) = (\nu_A^N)^3(y)$ so.

$\mu_A^3(x*z) \geq min\{\mu_A^3((x*y)*z), \mu_A^3(y)\} = min\{\mu_A^3(0), \mu_A^3(0)\} = \mu_A^3(0)$,

$\sqrt[3]{\lambda_A}(x*z) \leq max\{\sqrt[3]{\lambda_A}((x*y)*z), \sqrt[3]{\lambda_A}(y)\} = max\{\sqrt[3]{\lambda_A}(0), \sqrt[3]{\lambda_A}(0)\} = \sqrt[3]{\lambda_A}(0)$ and

$(\nu_A^N)^3(x*z) \leq max\{(\nu_A^N)^3((x*y)*z), (\nu_A^N)^3(y)\} = max\{(\nu_A^N)^3(0), (\nu_A^N)^3(0)\} = (\nu_A^N)^3(0)$

Combining this with conditions of definition 3.7. $\mu_A^3(0) \geq \mu_A^3(x)$, $\sqrt[3]{\lambda_A}(0) \leq \sqrt[3]{\lambda_A}(x)$ and $(\nu_A^N)^3(0) \leq (\nu_A^N)^3(x)$ we get

$\mu_A^3(x*z) = \mu_A^3(0), \sqrt[3]{\lambda_A}(x*z) = \sqrt[3]{\lambda_A}(0)$ $and (\nu_A^N)^3(x*z) = (\nu_A^N)^3(0)$, that is $x*z \in X$. Hence X is a Q -ideal of *X*.

Lemma 3.13. Let $A = \{(x, \mu_A(x), \lambda_A(x), \nu_A^N(x)) \mid x \in X\}$,be crossing - CR – fuzzy Q-ideal of X. If $x \leq y$ in *X*, then, $\mu_A^3(x) \geq \mu_A^3(y)$, $\sqrt[3]{\lambda_A}(x) \leq \sqrt[3]{\lambda_A}(y)$, $(\nu_A^N)^3(x) \leq (\nu_A^N)^3(y)$,for all $x, y \in X$.

Proof. Let $x, y \in X$ be such that $x \leq y$, then $x*y = 0$. From (Definition 3.7) put z = 0

We have $\mu_A^3(x*0) = \mu_A^3(x) \geq min\{\mu_A^3((x*y)), \mu_A^3(y)\} = min\{\mu_A^3(0), \mu_A^3(y)\} = \mu_A^3(y)$,

Similar, we can prove that

$\sqrt[3]{\lambda_A}(x*0) = \sqrt[3]{\lambda_A}(x) \leq max\{\sqrt[3]{\lambda_A}(x*y), \sqrt[3]{\lambda_A}(y)\} = max\{\sqrt[3]{\lambda_A}(0), \sqrt[3]{\lambda_A}(y)\} = \sqrt[3]{\lambda_A}(y)$.

$(\nu_A^N)^3(x*0) = (\nu_A^N)^3(x) \leq max\{(\nu_A^N)^3(x*y), (\nu_A^N)^3(y)\} = max\{(\nu_A^N)^3(0), (\nu_A^N)^3(y)\} = (\nu_A^N)^3(y)$

Lemma 3.14 Let $A = \{(x, \mu_A(x), \lambda_A(x), \nu_A^N(x)) \mid x \in X\}$,be crossing - CR – fuzzy Q-ideal of X, if the inequality $x*y \leq z$ hold in *X*, then

$\mu_A^3(x) \geq min\{\mu_A^3(z), \mu_A^3(y)\}$, $\sqrt[3]{\lambda_A}(x) \leq max\{\sqrt[3]{\lambda_A}(z), \sqrt[3]{\lambda_A}(y)\}$ and

$(\nu_A^N)^3(x) \leq max\{(\nu_A^N)^3(z), (\nu_A^N)^3(y)\}$

Proof. Let $x, y, z \in X$ be such that $x*y \leq z$. Then by (Lemma 3. 13),we have $\mu_A^3(x*y) \geq \mu_A^3(z)$

Thus, put $z = 0$ in (definition *3.7.*) and using lemma 3.13,then we get

$$\mu_A^3(x*0)=\mu_A^3(x)\geq min\{\mu_A^3((x*y)*0),\mu_A^3(y)\}=$$

$$min\{\mu_A^3((x*y)),\mu_A^3(y)\}\geq min\{\mu_A^3(\overbrace{z}^{since\ (x*y)\leq z\Rightarrow\mu_A^3(x*y)\geq\mu_A^3(z)\ by\ Lemma\ 3.13}),\mu_A^3(y)\}$$

Hence, we have $\mu_A^3(x)\geq min\{\mu_A^3(z),\mu_A^3(y)\}$. Similar, we can prove that $\sqrt[3]{\lambda_A}(x)\leq max\{\sqrt[3]{\lambda_A}(z),\sqrt[3]{\lambda_A}(y)\}$, $(\nu_A^N)^3(x)\leq max\{(\nu_A^N)^3(z),(\nu_A^N)^3(y)\}$ .This completes the proof.

Proposition 4.17. Let $(M_i)_{i\in\Lambda}=\{(x,\mu_{M_i}(x),\lambda_{M_i}(x),\nu^N_{M_i}(x)\mid x\in X\}$ be a family of crossing – fuzzy CR –Q-ideal of X, then $\bigcap_{i\in\Lambda} M_i$ is crossing - CR – fuzzy Q-ideals of a Q-algebra X.

Proof. Let $(M_i)_{i\in\Lambda}$ be a family of crossing - CR- fuzzy Q -ideals of a Q-algebras, then for any $x,y,z\in X$, we get

$$(\bigcap\mu^3_{M_i})(0)=inf(\mu^3_{M_i}(0))\geq inf(\mu^3_{M_i}(x))=(\bigcap\mu^3_{M_i})(x)$$

$$(\bigcup(\sqrt[3]{\lambda_A})_{M_i})(0)=sup((\sqrt[3]{\lambda_A})_{M_i}(0))\leq sup((\sqrt[3]{\lambda_A})_{M_i}(x))=(\bigcup(\sqrt[3]{\lambda_A})_{M_i})(x),$$

$$(\bigcup((\nu_A^N)^3)_{M_i})(0)=sup(((\nu_A^N)^3)_{M_i}(0))\leq sup(((\nu_A^N)^3)_{M_i}(x))=(\bigcup((\nu_A^N)^3)_{M_i})(x),$$

Now

$$(\bigcap\mu^3_{M_i})(x*z)=inf(\mu^3_{M_i}(x*z))\geq inf\{min\{\mu^3_{M_i}((x*y)*z),\mu^3_{M_i}(y)\}\}=$$

$$min\{inf\{\mu^3_{M_i}((x*y)*z)\},inf\{\mu^3_{M_i}(y)\}\}\}=min\{(\bigcap\mu^3_{M_i})((x*y)*z),(\bigcap\mu^3_{M_i})(y)\}\}$$

and

$$(\bigcup(\sqrt[3]{\lambda_A})_{M_i})(x*z)=sup((\sqrt[3]{\lambda_A})_{M_i}(x*z))\leq sup\{max\{\{(\sqrt[3]{\lambda_A})_{M_i}((x*y)*z),(\sqrt[3]{\lambda_A})_{M_i}(y)\}\}=$$

$$max\{sup\{(\sqrt[3]{\lambda_A})_{M_i}((x*y)*z),sup\{(\sqrt[3]{\lambda_A})_{M_i}(y)\}\}\}=max\{(\bigcup(\sqrt[3]{\lambda_A})_{M_i})((x*y)*z),(\bigcup(\sqrt[3]{\lambda_A})_{M_i})(y)\}\}\},$$

$$(\bigcup((\nu_A^N)^3)_{M_i})(x*z)=sup(((\nu_A^N)^3)_{M_i}(x*z))\leq sup\{max\{\{((\nu_A^N)^3)_{M_i}((x*y)*z),((\nu_A^N)^3)_{M_i}(y)\}\}=$$

$$max\{sup\{((\nu_A^N)^3)_{M_i}((x*y)*z),sup\{((\nu_A^N)^3)_{M_i}(y)\}\}\}=$$

$$max\{(\bigcup((\nu_A^N)^3)_{M_i})((x*y)*z),(\bigcup((\nu_A^N)^3)_{M_i})(y)\}\}\}$$

Hence $\bigcap_{i\in\Lambda} M_i$ is crossing - CR – fuzzy Q -ideals of a Q -algebra X .This completes the proof.

4.*Image (Pre-image)* crossing -*CR – fuzzy Q- ideal under homomorphism of Q -algebras*

Definition 4.1 Let $(X,*,0)$ and $(Y,*',0')$ be Q-algebras. A mapping $f:X\rightarrow Y$ is said to be a homomorphism if $f(x*y)=f(x)*'f(y)$ for all $x,y\in X$.

Definition 4.2. Let X and Y be two Q-algebras, μ a fuzzy subset of X, β a fuzzy subset of Y and $f:X\rightarrow Y$ a Q-homomorphism.

The image of μ under f denoted by $f(\mu)$ is a fuzzy set of Y defined by

$$f(\mu)(y)=\begin{cases}\sup\limits_{x\in f^{-1}(y)}\mu(x) & if\quad f^{-1}(y)\neq\phi\\ 0 & otherwise\end{cases}$$

The pre-image of β under f denoted $f^{-1}(\beta)$ is a fuzzy set of X defined by: For all

$x \in X$, $f^{-1}(\beta)(x) = \beta(f(x))$.

Let $f : X \to Y$ be a homomorphism of Q-algebras for any crossing - CR – fuzzy Q-ideals $A = \{(x, \mu_A(x), \lambda_A(x), \nu_A^N(x)) \mid x \in X\}$, in Y, we define new crossing -CR – fuzzy Q- ideals $A^f = \{(x, (\mu_A(x))^f, (\lambda_A(x))^f, (\nu_A^N(x))^f \mid x \in X\}$, in X defined by

$(\mu_A^3(x))^f = \mu_A^3(f(x)), (\sqrt[3]{\lambda_A}(x))^f = \sqrt[3]{\lambda_A}(f(x)), ((\nu_A^N)^3(x))^f = (\nu_A^N)^3(f(x))$ *for all* $x \in X$.

Theorem 4.3 Let $f : X \to Y$ be a homomorphism of Q-algebras. If

$A = \{(x, \mu_A(x), \lambda_A(x), \nu_A^N(x)) \mid x \in X\}$, is crossing -CR – fuzzy Q- ideal of Y, then $A^f = \{(x, (\mu_A(x))^f, (\lambda_A(x))^f, (\nu_A^N(x))^f \mid x \in X\}$, is crossing -CR – fuzzy Q- ideal of X.

Proof. Notes that:

(1) $(\mu_A^3(x))^f := \mu_A^3(f(x)) \le \mu_A^3(f(0)) = (\mu_A^3(0))^f$

$(\mu_A^3(x*z))^f := \mu_A^3(f(x*z)) \ge \min\{\mu_A^3((f(x)*f(y))*f(z)), \mu_A^3(f(y))\}$

$= \min\{\mu_A^3(f(x*y)*f(z))\mu_A^3(f(y))\} = \min\{\mu_A^3(f((x*y)*z), \mu_A^3(f(y))\}$

$= \min\{(\mu_A^3((x*y)*z))^f, (\mu_A^3(y))^f\}$;

(2) $(\sqrt[3]{\lambda_A}(x))^f := \sqrt[3]{\lambda_A}(f(x)) \ge \sqrt[3]{\lambda_A}(f(0)) = (\sqrt[3]{\lambda_A}(0))^f$

$(\sqrt[3]{\lambda_A}(x*z))^f := \sqrt[3]{\lambda_A}(f(x*z)) \le \max\{\sqrt[3]{\lambda_A}((f(x)*f(y))*f(z)), \sqrt[3]{\lambda_A}(f(y))\}$

$= \max\{\sqrt[3]{\lambda_A}(f(x*y)*f(z)), \sqrt[3]{\lambda_A}(f(y))\} = \max\{\sqrt[3]{\lambda_A}(f((x*y)*z)), \sqrt[3]{\lambda_A}(f(y))\}$

$= \max\{(\sqrt[3]{\lambda_A}((x*y)*z))^f, (\sqrt[3]{\lambda_A}(y))^f\}$.

(3) $((\nu_A^N)^3(x))^f := (\nu_A^N)^3(f(x)) \ge (\nu_A^N)^3(f(0)) = ((\nu_A^N)^3(0))^f$

$((\nu_A^N)^3(x*z))^f := (\nu_A^N)^3(f(x*z)) \le \max\{(\nu_A^N)^3((f(x)*f(y))*f(z)), (\nu_A^N)^3(f(y))\}$

$= \max\{(\nu_A^N)^3(f(x*y)*f(z)), (\nu_A^N)^3(f(y))\} = \max\{(\nu_A^N)^3(f((x*y)*z)), (\nu_A^N)^3(f(y))\}$

$= \max\{((\nu_A^N)^3((x*y)*z))^f, ((\nu_A^N)^3(y))^f\}$.

Hence $A^f = \{(x, (\mu_A(x))^f, (\lambda_A(x))^f, (\nu_A^N(x))^f \mid x \in X\}$, is crossing - CR- fuzzy Q- ideal of X.

Theorem4.4 Let $f : X \to Y$ be an epimorphism of Q-algebras .

If $A^f = \{(x, (\mu_A(x))^f, (\lambda_A(x))^f, (\nu_A^N(x))^f \mid x \in X\}$, is crossing - CR – fuzzy ideal of X , then $A = \{(x, \mu_A(x), \lambda_A(x), \nu_A^N(x)) \mid x \in X\}$, is crossing CR – fuzzy Q- ideal of Y.

Proof.

(1) For any $a \in Y$, there exists $x \in X$ such that $f(x) = a$. Then

$\mu_A^3(a) = \mu_A^3(f(x)) = (\mu_A^3(x))^f \le (\mu_A^3(0))^f = \mu_A^3(f(0)) = (\mu_A^3(0))^f$,

Let $a, b, c \in Y$. Then $f(x) = a, f(y) = b, f(z) = c$, for some $x, y, z \in X$. It follows that

$\mu_A^3(a*c) = \mu_A^3(f(x)*f(z)) = (\mu_A^3(x*z))^f \ge \min\{(\mu_A^3((x*y)*z))^f, (\mu_A^3(y))^f\}$

$= \min\{\mu_A^3(f((x*y)*z), \mu_A^3(f(y)\} = \min\{\mu_A^3(f(x*y)*f(z)), \mu_A^3(f(y))\}$

$= \min\{\mu_A^3((f(x)*f(y))*f(z)), \mu_A^3(f(y))\} = \min\{\mu_A^3((a*b)*c)), \mu_A^3(b)\}$,

(2) For any $a \in Y$, there exists $x \in X$ such that $f(x) = a$. Then

$$\sqrt[3]{\lambda_A}(a) = \sqrt[3]{\lambda_A}(f(x)) = (\sqrt[3]{\lambda_A}(x))^f \geq (\sqrt[3]{\lambda_A}(0))^f = \sqrt[3]{\lambda_A}(f(0)) = (\sqrt[3]{\lambda_A}(0))^f ,$$

Let $a, b, c \in Y$. Then $f(x) = a, f(y) = b, f(z) = c$, for some $x, y, z \in X$. It follows that

$$\sqrt[3]{\lambda_A}(a*c) = \sqrt[3]{\lambda_A}(f(x)*f(z)) = (\sqrt[3]{\lambda_A}(x*z))^f \leq max\{(\sqrt[3]{\lambda_A}((x*y)*z))^f ,(\sqrt[3]{\lambda_A}(y))^f \} = =$$
$$max\{\sqrt[3]{\lambda_A}(f((x*y)*z), \sqrt[3]{\lambda_A}(f(y)\} = max\{\sqrt[3]{\lambda_A}(f(x*y)*f(z)), \sqrt[3]{\lambda_A}(f(y))\}$$
$$= max\{\sqrt[3]{\lambda_A}((f(x)*f(y))*f(z)), \sqrt[3]{\lambda_A}(f(y))\} = max\{\sqrt[3]{\lambda_A}((a*b)*c)), \sqrt[3]{\lambda_A}(b)\}$$

(3) For any $a \in Y$, there exists $x \in X$ such that $f(x) = a$. Then

$$(\nu_A^N)^3(a) = (\nu_A^N)^3(f(x)) = ((\nu_A^N)^3(x))^f \geq ((\nu_A^N)^3(0))^f = (\nu_A^N)^3(f(0)) = ((\nu_A^N)^3(0))^f ,$$

Let $a, b, c \in Y$. Then $f(x) = a, f(y) = b, f(z) = c$, for some $x, y, z \in X$. It follows that

$$(\nu_A^N)^3(a*c) = (\nu_A^N)^3(f(x)*f(z)) = ((\nu_A^N)^3(x*z))^f \leq max\{((\nu_A^N)^3((x*y)*z))^f ,((\nu_A^N)^3(y))^f \} = =$$
$$max\{(\nu_A^N)^3(f((x*y)*z), (\nu_A^N)^3(f(y)\} = max\{(\nu_A^N)^3(f(x*y)*f(z)), (\nu_A^N)^3(f(y))\}$$
$$= max\{(\nu_A^N)^3((f(x)*f(y))*f(z)), (\nu_A^N)^3(f(y))\} = max\{(\nu_A^N)^3((a*b)*c)), (\nu_A^N)^3(b)\}$$

hence $A = \{(x, \mu_A(x), \lambda_A(x), \nu_A^N(x)) \mid x \in X\}$, is crossing -CR – fuzzy Q- ideal of Y. This completes the proof.

5. *Product of crossing - CR – fuzzy Q- ideals*

Definition 5.1 Let $A = \{(x, \mu_A(x), \lambda_A(x), \nu_A^N(x)) \mid x \in X\}$, and $B = \{(x, \mu_B(x), \lambda_B(x), \nu_B^N(x)) \mid x \in X\}$, be two crossing - CR – fuzzy sets of *X*, the Cartesian product $A \times B = \{X \times X, \mu_A^3 \times \mu_B^3, \sqrt[3]{\lambda_A} \times \sqrt[3]{\lambda_B}, (\nu_A^N)^3 \times (\nu_B^N)^3\}$ is defined by

$$,(\mu_A^3 \times \mu_B^3)(x, y) = min\{\mu_A^3(x), \mu_B^3(y)\}$$
$$(\sqrt[3]{\lambda_A} \times \sqrt[3]{\lambda_B})(x, y) = max\{\sqrt[3]{\lambda_A}(x), \sqrt[3]{\lambda_B}(y)\},$$
$$((\nu_A^N)^3 \times (\nu_B^N)^3)(x, y) = max\{(\nu_A^N)^3(x), (\nu_B^N)^3(y)\}$$ for all $x, y \in X$.

Remark 5.2 Let *X* and *Y* be Q-algebras, we define* on $X \times Y$ by: For every $(x, y), (u, v) \in X \times Y$, $(x, y)*(u, v) = (x*u, y*v)$. Clearly $(X \times Y; *, (0,0))$ Q-algebra.

Proposition 5.3. Let $A = \{(x, \mu_A(x), \lambda_A(x), \nu_A^N(x)) \mid x \in X\}$, and $B = \{(x, \mu_B(x), \lambda_B(x), \nu_B^N(x)) \mid x \in X\}$, be two crossing - CR –fuzzy Q- ideals of *X*, then $A \times B$ is crossing - CR – fuzzy Q- ideal of $X \times X$.

Proof.

$$(\mu_A^3 \times \mu_B^3)(0,0) = min\{\mu_A^3(0), \mu_B^3(0)\} \geq min\{\mu_A^3(x), \mu_B^3(y)\}\} = (\mu_A^3 \times \mu_B^3)(x, y)$$
$$(\sqrt[3]{\lambda_A} \times \sqrt[3]{\lambda_B})(0,0) = max\{\sqrt[3]{\lambda_A}(0), \sqrt[3]{\lambda_B}(0)\} \leq max\{\sqrt[3]{\lambda_A}(x), \sqrt[3]{\lambda_B}(y)\}\} =$$
$$(\sqrt[3]{\lambda_A} \times \sqrt[3]{\lambda_B})(x, y)$$

$$((\nu_A^N)^3 \times (\nu_B^N)^3)(0,0) = max\{(\nu_A^N)^3(0),(\nu_B^N)^3(0)\} \le max\{(\nu_A^N)^3(x),(\nu_B^N)^3(y)\}\} =$$
$$= ((\nu_A^N)^3 \times (\nu_B^N)^3)(x,y)$$

for all $x, y \in X$. Now let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$, then

$$min\{(\mu_A^3 \times \mu_B^3)(((x_1,x_2)*(y_1,y_2))*(z_1,z_2)),(\mu_A^3 \times \mu_B^3)((y_1,y_2))\}$$
$$= min\{(\mu_A^3 \times \mu_B^3)((x_1*y_1,x_2*y_2)*(z_1,z_2)),(\mu_A^3 \times \mu_B^3)(y_1,y_2)\}$$
$$= min\{(\mu_A^3 \times \mu_B^3)((x_1*y_1)*z_1,(x_2*y_2)*z_2),(\mu_A^3 \times \mu_B^3)(y_1,y_2)\}$$
$$= min\{min\{\mu_A^3((x_1*y_1)*z_1),\mu_B^3((x_2*y_2)*z_2)\},min\{\mu_A^3(y_1),\mu_B^3(y_2)\}\}$$
$$= min\{min\{\mu_A^3((x_1*y_1)*z_1),\mu_A^3(y_1)\},min\{\mu_B^3((x_2*y_2)*z_2),\mu_B^3(y_2)\}$$
$$= min\{min\{\mu_A^3((x_1*y_1)*z_1),\mu_A^3(y_1)\},min\{\mu_B^3((x_2*y_2)*z_2),\mu_B^3(y_2)\}$$
$$\le min\{\mu_A^3(x_1*z_1),\mu_B^3(x_2*z_2) = (\mu_A^3 \times \mu_B^3)(x_1*z_1,x_2*z_2).$$

2) $max\{(\sqrt[3]{\lambda_A} \times \sqrt[3]{\lambda_B})(((x_1,x_2)*(y_1,y_2))*(z_1,z_2)),(\sqrt[3]{\lambda_A} \times \sqrt[3]{\lambda_B})((y_1,y_2))\}$

$$= max\{(\sqrt[3]{\lambda_A} \times \sqrt[3]{\lambda_B})((x_1*y_1,x_2*y_2)*(z_1,z_2)),(\sqrt[3]{\lambda_A} \times \sqrt[3]{\lambda_B})(y_1,y_2)\}$$
$$= max\{(\sqrt[3]{\lambda_A} \times \sqrt[3]{\lambda_B})((x_1*y_1)*z_1,(x_2*y_2)*z_2),(\sqrt[3]{\lambda_A} \times \sqrt[3]{\lambda_B})(y_1,y_2)\}$$
$$= max\{max\{\sqrt[3]{\lambda_A}((x_1*y_1)*z_1),\sqrt[3]{\lambda_B}((x_2*y_2)*z_2)\},max\{\sqrt[3]{\lambda_A}(y_1),\sqrt[3]{\lambda_B}(y_2)\}\}$$
$$= max\{max\{\sqrt[3]{\lambda_A}((x_1*y_1)*z_1),\sqrt[3]{\lambda_A}(y_1)\},max\{\sqrt[3]{\lambda_B}((x_2*y_2)*z_2),\sqrt[3]{\lambda_B}(y_2)\}$$
$$\ge max\{\sqrt[3]{\lambda_A}(x_1*z_1),\sqrt[3]{\lambda_B}(x_2*z_2) = ((\sqrt[3]{\lambda_A} \times \sqrt[3]{\lambda_B})(x_1*z_1,x_2*z_2)$$

3) $max\{((\nu_A^N)^3 \times (\nu_B^N)^3)(((x_1,x_2)*(y_1,y_2))*(z_1,z_2)),((\nu_A^N)^3 \times (\nu_B^N)^3)((y_1,y_2))\}$

$$= max\{((\nu_A^N)^3 \times (\nu_B^N)^3)((x_1*y_1,x_2*y_2)*(z_1,z_2)),((\nu_A^N)^3 \times (\nu_B^N)^3)(y_1,y_2)\}$$
$$= max\{((\nu_A^N)^3 \times (\nu_B^N)^3)((x_1*y_1)*z_1,(x_2*y_2)*z_2),((\nu_A^N)^3 \times (\nu_B^N)^3)(y_1,y_2)\}$$
$$= max\{max\{(\nu_A^N)^3((x_1*y_1)*z_1),(\nu_B^N)^3((x_2*y_2)*z_2)\},max\{(\nu_B^N)^3(y_1),(\nu_B^N)^3(y_2)\}\}$$
$$= max\{max\{(\nu_A^N)^3((x_1*y_1)*z_1),(\nu_A^N)^3(y_1)\},max\{((\nu_B^N)^3((x_2*y_2)*z_2),(\nu_B^N)^3(y_2)\}$$

$\ge max\{(\nu_A^N)^3(x_1*z_1),(\nu_B^N)^3(x_2*z_2) = ((\nu_A^N)^3 \times (\nu_B^N)^3)(x_1*z_1,x_2*z_2)$. This completes the proof.

6.Correlation coefficient for crossing CR – fuzzy sets

Correlation plays an important role in statistics , engineering sciences and so on see [5, 6 , 26].In this section, we propose some correlation coefficients for any two crossing - CR – fuzzy sets.

Let $A = \{(x,\mu_A(x),\lambda_A(x),\nu_A^N(x)) \mid x \in X\}$,and $B = \{(x,\mu_B(x),\lambda_B(x),\nu_B^N(x)) \mid x \in X\}$, be two crossing - CR – fuzzy sets of *X* sets in the universe of discourse $X = \{x_1, x_2, ...x_n\}$.

We define the following expression:

◊Informational crossing - CR – fuzzy energies of A in the universe of the discourse $X = \{x_1, x_2, ...x_n\}$ is defined by : for every $x_i \in X$

$$E_{crossing-CR-fuzzy}(A) = \sum_1^n((\mu_A^3(x_i))^2 + (\sqrt[3]{\lambda_A}(x_i))^2 + ((\nu_A^N)^3)^2$$

◊Informational crossing - CR – fuzzy energies of B in the universe of the discourse

$X=\{x_1,x_2,......x_n\}$ is defined by : for every $x_i \in X$

$$E_{crossing-CR-fuzzy}(B)=\sum_1^n((\mu_B^3(x_i))^2+(\sqrt[3]{\lambda_B}(x_i))^2+((\nu_B^N)^3)^2$$

◊The correlation between two crossing - CR – fuzzy sets A and B of X is defined by : for every $x_i \in X$

$$C_{crossing-CR-fuzzy\ sets}(A,B)=\sum_1^n(\mu_A^3(x_i)\times\mu_B^3(x_i))+(\sqrt[3]{\lambda_A}(x_i)\times\sqrt[3]{\lambda_B}(x_i))+(\nu_A^N)^3\times(\nu_B^N)^3$$

Definition 8 Let $A=\{(x,\mu_A(x),\lambda_A(x),\nu_A^N(x))\,|\,x\in X\}$, and $B=\{(x,\mu_B(x),\lambda_B(x),\nu_B^N(x))\,|\,x\in X\}$, be two crossing - CR – fuzzy sets on *X*.

The correlation coefficient of A and B is defined as

$$\kappa(A,B)=\frac{C_{crossing-CR-fuzzy\ sets}(A,B)}{\sqrt{E_{crossing-CR-fuzzy}(A).E_{crossing-CR-fuzzy}(B)}}=$$

$$\frac{\sum_1^n(\mu_A^3(x_i)\times\mu_B^3(x_i))+(\sqrt[3]{\lambda_A}(x_i)\times\sqrt[3]{\lambda_B}(x_i))+(\nu_A^N)^3\times(\nu_B^N)^3}{\sqrt{\sum_1^n((\mu_A^3(x_i))^2+(\sqrt[3]{\lambda_A}(x_i))^2+((\nu_A^N)^3)^2}.\sqrt{\sum_1^n((\mu_B^3(x_i))^2+(\sqrt[3]{\lambda_B}(x_i))^2+((\nu_B^N)^3)^2}}$$

On the other hand, by using idea of Xu et al[26]. we can suggeste an alternative form of the correlation coefficient between two crossing - CR – fuzzy sets A and B as follows

$$\kappa(A,B)=\frac{\sum_1^n(\mu_A^3(x_i)\times\mu_B^3(x_i))+(\sqrt[3]{\lambda_A}(x_i)\times\sqrt[3]{\lambda_B}(x_i))+(\nu_A^N)^3\times(\nu_B^N)^3}{max\{\sqrt{\sum_1^n((\mu_A^3(x_i))^2+(\sqrt[3]{\lambda_A}(x_i))^2+((\nu_A^N)^3)^2},\sqrt{\sum_1^n((\mu_B^3(x_i))^2+(\sqrt[3]{\lambda_B}(x_i))^2+((\nu_B^N)^3)^2}\}}$$

It is obvious that the correlation of crossing - CR– fuzzy sets of X satisfies the following properties:

$$C_{crossing-CR-fuzzy\ sets}(A,A)=C_{crossing-CR-fuzzy\ sets}(A)$$

$$C_{crossing-CR-fuzzy\ sets}(A,B)=C_{crossing-CR-fuzzy\ sets}(B,A)$$

Theorem 3.1. For any two crossing - CR –fuzzy sets $A=\{(x,\mu_A(x),\lambda_A(x),\nu_A^N(x))\,|\,x\in X\}$, and $B=\{(x,\mu_B(x),\lambda_B(x),\nu_B^N(x))\,|\,x\in X\}$, in the universe of discourse $X=\{x_1,x_2,......x_n\}$ the correlation coefficient of A and B satisfies the following conditions:

1- $\kappa(A,B)=\kappa(B,A)$

2- $0\le\kappa(A,B)\le 1$

3- $A=B\Rightarrow\kappa(A,B)=1$

Proof . 1)- It is straightforward.

The inequality $\kappa(B,A)\ge 0$ is evident. We will prove $\kappa(A,B)\le 1$

$$C_{crossing\text{-}CR\text{-}\ fuzzy\ sets}(A,B)=\sum_{1}^{n}(\mu_A^3(x_i)\times\mu_B^3(x_i))+(\sqrt[3]{\lambda_A}(x_i)\times\sqrt[3]{\lambda_B}(x_i))+(\nu_A^N)^3\times(\nu_B^N)^3=$$
$$\mu_A^3(x_1)\times\mu_B^3(x_1)+\sqrt[3]{\lambda_A}(x_1)\times\sqrt[3]{\lambda_B}(x_1)+(\nu_A^N)^3(x_1)\times(\nu_B^N)^3(x_1)+\ldots\ldots\ldots\ldots\ldots\ldots\ldots+$$
$$\mu_A^3(x_n)\times\mu_B^3(x_n)+\sqrt[3]{\lambda_A}(x_n)\times\sqrt[3]{\lambda_B}(x_n)+(\nu_A^N)^3(x_n)\times(\nu_B^N)^3(x_n)$$

By Cauchy–Schwarz inequality

$$(x_1y_1+x_2y_2+\ldots..+x_ny_n)^2\le(x_1^2+x_2^2+\ldots..+x_n^2)\bullet(y_1^2+y_2^2+\ldots..+y_n^2)$$

$x_1+x_2+\ldots..+x_n\in\Re^n$ and $y_1+y_2+\ldots..+y_n\in\Re^n$,we get

$$(C_{crossing\text{-}\ CR\text{-}\ fuzzy\,ideals}(A,B))^2\le\{\mu_A^6(x_1)\times\mu_B^6(x_1)+(\sqrt[3]{\lambda_A}(x_1))^2\times(\sqrt[3]{\lambda_B}(x_1))^2+$$
$$(\nu_A^N)^6(x_1)\times(\nu_B^N)^6(x_1)+\ldots\ldots..+\mu_A^6(x_n)\times\mu_B^6(x_n)+$$
$$(\sqrt[3]{\lambda_A}(x_n))^2\times(\sqrt[3]{\lambda_B}(x_n))^2+(\nu_A^N)^6(x_n)\times(\nu_B^N)^6(x_n)=$$
$$(\sum_{1}^{n}\mu_A^6(x_i)+(\sqrt[3]{\lambda_A}(x_i))^2+(\nu_A^N)^6(x_i)\times\sum_{1}^{n}\mu_B^6(x_i)+(\sqrt[3]{\lambda_B}(x_i))^2+(\nu_B^N)^6(x_i)=$$
$$E_{crossing\text{-}CR\text{-}\ fuzzy}(A)\times E_{crossing\text{-}CR\text{-}\ fuzzy}(B)$$

Therefore $(C_{crossing\text{-}CR\text{-}\ fuzzy\,ideals}(A,B))^2\le E_{crossing\text{-}CR\text{-}\ fuzzy}(A)\times E_{crossing\text{-}CR\text{-}\ fuzzy}(B)$

Hence, we obtain the property $0\le\kappa(A,B)\le 1$.

As $A=B$ this implies that $\mu_A(x)=\mu_B(x),\lambda_A(x)=\lambda_B(x)\,and\ \nu_A^N(x)=\nu_B^N(x),\forall x\in X$.

Thus $\kappa(A,B)=1$.

Example 4.6. Let $X=\{0,1,2,3\}$ be a set with a binary operation $*$ defined by the table (1) of example 3.8 ,then $(X,*,0)$ is a Q-algebra .

Define $A=\{(x,\mu_A(x),\lambda_A(x),\nu_A^N(x))\,|\,x\in X\}$,and $B=\{(x,\mu_B(x),\lambda_B(x),\nu_B^N(x))\,|\,x\in X\}$,as follows:

$\mu_A(0)=0.4,\mu_A(1)=0.3,\mu_A(2)=0.2,\mu_A(3)=0.1,$

$\lambda_A(0)=0.1,\lambda_A(1)=0.2,\lambda_A(2)=0.3,\lambda_A(3)=0.4$

$\nu_A^N(0)=-0.15,\nu_A^N(1)=-0.14,\nu_A^N(2)=-0.13,\nu_A^N(3)=-0.12,$

$\mu_B(0)=0.5,\mu_B(1)=0.4,\mu_B(2)=0.3,\mu_B(3)=0.2,$

$\lambda_B(0)=0.2,\lambda_B(1)=0.3,\lambda_B(2)=0.4,\lambda_B(3)=0.5$

$\nu_B^N(0)=-0.14,\nu_B^N(1)=-0.12,\nu_B^N(2)=-0.1,\nu_B^N(3)=-0.1$

It is easy to check that A and B are crossing - CR – fuzzy sets on Q- algebras

The informational energy of A is written as

$$E_{crossing\text{-}CR\text{-}\ fuzzy}(A)=\sum_{1}^{n}((\mu_A^3(x_i))^2+(\sqrt[3]{\lambda_A}(x_i))^2+((\nu_A^N)^3)^2=$$
$$(0.4)^6+(0.3)^6+(0.2)^6+(0.1)^6+(\sqrt[3]{0.1})^2+(\sqrt[3]{0.2})^2+(\sqrt[3]{0.3})^2+(\sqrt[3]{0.4})^2+$$
$$(-0.15)^6+(-0.14)^6+(-0.13)^6+(-0.12)^6=0.00489+1.548462+0.000027=1.553379$$

$$\sqrt{E_{crossing-CR-fuzzy}(A)}=\sqrt{1.553379}=1.2463462600738207$$

The informational energy of B is written as

$$E_{crossing-CR-fuzzy}(B)=\sum_1^n((\mu_B^3(x_i))^2+(\sqrt[3]{\lambda_B}(x_i))^2+((\nu_B^N)^3)^2=$$
$$(0.5)^6+(0.4)^6+(0.3)^6+(0.2)^6+(\sqrt[3]{0.2})^2+(\sqrt[3]{0.3})^2+(\sqrt[3]{0.4})^2+(\sqrt[3]{0.5})^2+$$
$$(-0.14)^6+(-0.12)^6+(-0.1)^6+(-0.1)^6=0.020514+1.9629797122574738+0.00001251552=$$
$$1.98350551552$$
$$\sqrt{E_{crossing-CR-fuzzy}(B)}=\sqrt{1.98350551552}=1.40836784967 56449$$

$$\sqrt{E_{crossing-CR-fuzzy}(A)}\times\sqrt{E_{crossing-CR-fuzzy}(B)}=\sqrt{1.553379}\times\sqrt{1.98350551552}=$$
$$1.2463462600738207\times1.4083698077990738=1.7553164427512613071645810872 6766$$

The correlation between crossing - CR – fuzzy sets A and B is written as

$$C_{crossing-CR-fuzzy\ sets}(A,B)=\sum_1^n(\mu_A^3(x_i)\times\mu_B^3(x_i))+(\sqrt[3]{\lambda_A}(x_i)\times\sqrt[3]{\lambda_B}(x_i))+(\nu_A^N)^3\times(\nu_B^N)^3=$$
$$(0.4)^3\times(0.5)^3+(0.3)^3\times(0.4)^3+(0.2)^3\times(0.3)^3+(0.1)^3\times(0.2)^3+$$
$$(\sqrt[3]{0.1})^2\times(\sqrt[3]{0.2})^2+(\sqrt[3]{0.2})^2\times(\sqrt[3]{0.3})^2+(\sqrt[3]{0.3})^2\times(\sqrt[3]{0.4})^2+(\sqrt[3]{0.4})^2\times\sqrt[3]{(0.5)}^2+$$
$$(-0.15)^3\times(-0.14)^3+(-0.14)^3\times(-0.12)^3+(-0.13)^3\times(-0.1)^3+,(-0.12)^3\times(-0.1)^3=0.009952+$$
$$0.0736806299728077+0.1532618864787106+0.2432880799822936+0.3419951893353394+$$
$$0.00001797632=0.0736985576048077+0.1532618864787106+0.2432880799822936+$$
$$0.3419951893353394=0.8128364316481783$$

Hence, the correlation coefficient between crossing - CR– fuzzy sets A and B is given by

$$\kappa(A,B)=\frac{C_{crossing-CR-fuzzy\ sets}(A,B)}{\sqrt{E_{crossing-CR-fuzzy}(A).E_{crossing-CR-fuzzy}(B)}}=$$
$$\frac{\sum_1^n(\mu_A^3(x_i)\times\mu_B^3(x_i))+(\sqrt[3]{\lambda_A}(x_i)\times\sqrt[3]{\lambda_B}(x_i))+(\nu_A^N)^3\times(\nu_B^N)^3}{\sqrt{\sum_1^n((\mu_A^3(x_i))^2+(\sqrt[3]{\lambda_A}(x_i))^2+((\nu_A^N)^3)^2}.\sqrt{\sum_1^n((\mu_B^3(x_i))^2+(\sqrt[3]{\lambda_B}(x_i))^2+((\nu_B^N)^3)^2}}=$$
$$\frac{0.8128364316481783}{1.7553164427512613071645810872 6766}\approx0.4630711665869204............(*)$$

Or the alternative form of the correlation coefficient of crossing - CR – fuzzy sets A and B as follows

$$\kappa(A,B)=\frac{\sum_1^n(\mu_A^3(x_i)\times\mu_B^3(x_i))+(\sqrt[3]{\lambda_A}(x_i)\times\sqrt[3]{\lambda_B}(x_i))+(\nu_A^N)^3\times(\nu_B^N)^3}{max\{\sqrt{\sum_1^n((\mu_A^3(x_i))^2+(\sqrt[3]{\lambda_A}(x_i))^2+((\nu_A^N)^3)^2},\sqrt{\sum_1^n((\mu_B^3(x_i))^2+(\sqrt[3]{\lambda_B}(x_i))^2+((\nu_B^N)^3)^2}\}}=$$
$$\frac{0.8128364316481783}{1.4083698077990738}\approx0.5771470189150605......(**)$$

From (*) and (**) , we see that the CR – fuzzy sets A and B are smi- correlated.

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