Another Generalization of ω-Closed Sets

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Abstract
In this paper, the further study is carried out using the generalized classes of τ_ω. Apart from this, some new generalizations of ω-open sets are introduced and investigated. Also, using the generalized subsets of τ_ω, a new decomposition of ω-continuity in topological spaces is obtained.

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1. Introduction

In 1961, Levine [8] obtained a decomposition of continuity. Tong [13] decomposed continuity into A-continuity and showed that his decomposition is independent of Levine's. In 1982, the notions of ω-open sets, ω-closed sets and ω-closed mappings were introduced and investigated by Hdeib [6]. In 2005, Al-Zoubi [2] introduced and studied the concepts of gω-closed sets and gω-open sets in topological spaces. In 2007, Al-Omari and Noorani [1] introduced and studied the concepts of rgω-closed sets and rgω-open sets in topological spaces. In 2009, Noiri et al. [11] introduced some weaker forms of ω-open sets and obtained some decompositions of continuity. Quite Recently, Umamaheswari et al. [12] studied gω-continuity and its topological

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properties. In this paper, the further study is carried out using the generalized classes of τ_ω. Apart from this, some new generalizations of ω-open sets are introduced and investigated. Also, using the generalized subsets of τ_ω, a new decomposition of ω-continuity in topological spaces is obtained.

2. Preliminaries

Throughout this paper, $\mathbb{R}$ (resp. $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}-\mathbb{Q}$) denotes the set of all real numbers (resp. the set of all natural numbers, the set of all rational numbers, the set of all irrational numbers). By a space (X, τ), we always mean a topological space (X, τ) with no separation axioms assumed. For a subset H of a space (X, τ), cl(H) and int(H) denote the closure of H and the interior of H, respectively.

Definition 2.1 A subset H of a space (X, τ) is called α-open [10] if $H \subseteq \text{int}(\text{cl}(\text{int}(H)))$.

The complement of an α-open set is called α-closed. The family of all α-open sets is denoted by τ_α or $\alpha O(X)$. $\alpha cl(H)$ is the smallest α-closed set containing H.

Definition 2.2 [6] Let H be a subset of a space (X, τ). A point p in X is called a condensation point of H if for each open set U containing p, $U \cap H$ is uncountable.

Definition 2.3 [6] A subset H of a space (X, τ) is called ω-closed if it contains all its condensation points.

The complement of an ω-closed set is called ω-open. The family of all ω-closed sets is denoted by $\omega C(X, \tau)$. The family of all ω-open sets is denoted by $\omega O(X)$. It is well known that a subset W of a space (X, τ) is ω-open [6] if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. The family of all ω-open sets, denoted by τ_ω, is a topology on X, which is finer than τ. The interior and closure operator in (X, τ_ω) are denoted by int_ω and cl_ω respectively.

Remark 2.4 [6] In a space (X, τ), every closed set is ω-closed but not conversely.

Lemma 2.5 [12] Let H be a subset of a space (X, τ). Then

1. H is ω-closed in X if and only if $H = \text{cl}_\omega(H)$.
2. $\text{cl}_\omega(X\setminus H) = X\setminus \text{int}_\omega(H)$.
3. $\text{cl}_\omega(H)$ is ω-closed in H.
4. $x \in \text{cl}_\omega(H)$ if and only if $H \cap G \neq \phi$ for each ω-open set G containing x.
5. $\text{cl}_\omega(H) \subseteq \text{cl}(H)$.
6. $\text{int}(H) \subseteq \text{int}_\omega(H)$.

Definition 2.6 [2] A subset H of a space (X, τ) is called generalized ω-closed (briefly, gω-closed) if $\text{cl}_\omega(H) \subseteq G$ whenever $H \subseteq G$ and $G \in \tau$.

The family of all gω-closed sets is denoted by $G\omega C(X, \tau)$.

Definition 2.7 [3] A space (X, τ) is called $T_{\frac{1}{2}}$-space if every singleton is open or closed.

Definition 2.8 [9] A subset H of a space (X, τ) is called generalized closed (briefly, g-closed) if $\text{cl}(H) \subseteq G$ whenever $H \subseteq G$ and $G \in \tau$.

The complement of a g-closed set is called g-open.

Theorem 2.9 [2] If (X, τ) is $T_{\frac{1}{2}}$-space, then every gω-closed set in (X, τ) is ω-closed in (X, τ).

Lemma 2.10 [4] If H is an open set, then $\text{cl}(H \cap G) = \text{cl}(H \cap \text{cl}(G))$ and hence $H \cap \text{cl}(G) \subseteq \text{cl}(H \cap G)$ for any subset G.

Definition 2.11 [1] Two nonempty sets K and L of X are said to be separated if

$cl(K) \cap L = \phi = K \cap cl(L)$.

Definition 2.12 [5] A subset H of a space (X, τ) is called locally closed if $H = M \cap N$ where M is open and N is closed.

3. αgω-closed sets

Definition 3.1 A subset H of a topological space (X, τ) is said to be αgω-closed if $cl_{\omega}(H) \subseteq G$ whenever $H \subseteq G$ and G is α-open.

The family of all αgω-closed sets is denoted by $\alpha G\omega C(X, \tau)$.

Example 3.2 If τ is any topology on a countable set X, then each subset is ω-open and (X, τ_{ω}) is a discrete space. Consequently $\tau_{\omega} = \mathcal{P}(X) = \omega C(X, \tau)$, where $\mathcal{P}(X)$ is the power set of X. It is clear that if (X, τ) is a countable space, then $\mathcal{P}(X) = \omega C(X, \tau) = \alpha G\omega C(X, \tau) = G\omega C(X, \tau)$.

Example 3.3 In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{R} - \mathbb{Q}\}$, the set $H = \mathbb{R} - \mathbb{Q}$ is neither αgω-closed nor gω-closed.

Solution: H is open and $H \subseteq H$. But $cl_{\omega}(H) = \mathbb{R} \not\subseteq H$. Hence H is not gω-closed. Also H is α-open, being open and $H \subseteq H$. But $cl_{\omega}(H) = \mathbb{R} \not\subseteq H$. Hence H is not αgω-closed also.

Proposition 3.4 Every ω-closed set is αgω-closed.

The proof follows immediately from the definitions. However the converse is not true as the following Example shows.

Example 3.5 In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \{1\}\}$, the set $H = \mathbb{R} - \mathbb{Q}$ is αgω-closed, since the only α-open set containing H is $\mathbb{R}$. But H is not ω-closed, for 3 is a condensation point of H and $3 \notin H$.

Solution: If G is any α-open set containing H then $H \subseteq G \Rightarrow cl(H) \subseteq cl(G) \subseteq cl(int(cl(int(G)))) = cl(int(G))$. It implies $\mathbb{R} - \{1\} = cl(H) \subseteq cl(int(G)) \subseteq cl(G)$. If $G \neq \phi$, then $cl(G) \neq \phi$. It means that $G = \mathbb{R}$ or $\{1\}$. If $G = \{1\}$, this is a contradiction. Hence $G = \mathbb{R}$.

Proposition 3.6 Every αgω-closed set is gω-closed.

The proof follows immediately from the definitions. However the converse is not true as the following Example shows.

Example 3.7 Let X be an uncountable set and let A, B, C and D be subsets of X such that each of them is uncountable and the family $\{A, B, C, D\}$ is a partition of X. We defined the topology $\tau = \{\phi, X, \{C\}, \{D\}, \{A, C\}, \{C, D\}, \{A, C, D\}\}$, where X is identified with $\{A, B, C, D\}$. Take $H = \{B, C, D\}$ and H is gω-closed since X is the only open set containing H. On the other hand, $H \subseteq H$ and $H \in \alpha O(X)$. But $cl_{\omega}(H) = X$. Hence H is not αgω-closed.

Theorem 3.8 If (X, τ) is $T_{\frac{1}{2}}$-space, then every αgω-closed set (X, τ) is ω-closed in (X, τ).

The proof follows immediately from Theorem 2.9 and Proposition 3.6.

Proposition 3.9 If $\mathcal{A} = \{A_\alpha : \alpha \in I\}$ is a locally finite collection of $\alpha g\omega$-closed sets of a space (X, τ), then $A = \cup_{\alpha \in I} A_\alpha$ is $\alpha g\omega$-closed (in particular, a finite union of $\alpha g\omega$-closed sets is $\alpha g\omega$-closed).

Proof. Let H be an α-open subset of (X, τ) such that $A \subseteq H$. Since $A_\alpha \in \alpha G\omega C(X, \tau)$ and $A_\alpha \subseteq H$ for each $\alpha \in I$, $cl_\omega(A_\alpha) \subseteq H$. As τ_ω is a topology on X finer than τ, $\mathcal{A}$ is locally finite in (X, τ_ω). Therefore $cl_\omega(A) = cl_\omega(\cup_{\alpha \in I} A_\alpha) = \cup_{\alpha \in I} cl_\omega(A_\alpha) \subseteq H$. Thus, A is $\alpha g\omega$-closed in (X, τ).

Remark 3.10 The following Example shows that a countable union of $\alpha g\omega$-closed sets need not be $\alpha g\omega$-closed.

Example 3.11 Consider $X = \mathbb{R}$ with the usual topology τ_u. For each $n \in \mathbb{N}$ (the set of all natural numbers), put $A_n = [\frac{1}{n}, 1]$ and $A = \cup_{n \in \mathbb{N}}^{\infty} A_n = (0,1]$. Then A is a countable union of $\alpha g\omega$-closed sets but A is not $\alpha g\omega$-closed since $H = (0,2)$ is α-open, $A \subseteq H$ and $cl_\omega(A) = [0,1] \nsubseteq H$.

Remark 3.12 The following Example shows that a finite intersection of $\alpha g\omega$-closed sets need not be $\alpha g\omega$-closed.

Example 3.13 Let X be an uncountable and let A be a subset of X such that A and $X - A$ are uncountable. Let $\tau = \{\phi, A, X\}$. Choose $x_0, x_1 \notin A$ and $x_0 \neq x_1$. Then $A_0 = A \cup \{x_0\}$ and $A_1 = A \cup \{x_1\}$ are two $\alpha g\omega$-closed subsets of (X, τ). But $A_0 \cap A_1 = A$ is not $\alpha g\omega$-closed, for $A \subseteq A \in \tau_\alpha$ and $cl_\omega(A) \neq A$.

Proposition 3.14 The intersection of an α-open set and an open set is α-open.

Proof. It is obvious, since $\tau \subseteq \tau_\alpha$ and τ_α is a topology.

Proposition 3.15 The union of an α-open set and an open set is α-open.

Proof. It is obvious, since $\tau \subseteq \tau_\alpha$ and τ_α is a topology.

Proposition 3.16 If $H \in \alpha G\omega C(X, \tau)$ and G is closed in (X, τ), then $H \cap G \in \alpha G\omega C(X, \tau)$.

Proof. Let U be an α-open set in (X, τ) such that $H \cap G \subseteq U$. Put $W = X - G$. Then $H \subseteq U \cup W$ and by Proposition 3.15, $U \cup W$ is α-open. Since $H \in \alpha G\omega C(X, \tau)$, $cl_\omega(H) \subseteq U \cup W$. Now, $cl_\omega(H \cap G) \subseteq cl_\omega(H) \cap cl_\omega(G) \subseteq cl_\omega(H) \cap cl(G) = cl_\omega(H) \cap G \subseteq (U \cup W) \cap G = (U \cap G) \cup \phi = U \cap G \subseteq U$.

Theorem 3.17 Let (X, τ) be a topological space. Then every subset of X is $\alpha g\omega$-closed if and only if every α-open set is ω-closed.

Proof. Suppose every subset of X is $\alpha g\omega$-closed. If U is α-open, then U is $\alpha g\omega$-closed and $cl_\omega(U) \subseteq U$. Hence $cl_\omega(U) = U$ implies U is ω-closed. Conversely, suppose that every α-open set is ω-closed. If H is any subset of X and W is an α-open set such that $H \subseteq W$, then $cl_\omega(H) \subseteq cl_\omega(W) = W$ and so H is $\alpha g\omega$-closed.

Proposition 3.18 Let H be an $\alpha g\omega$-closed subset of a space (X, τ) and $G \subseteq X$. Then the following hold.

(a) $cl_\omega(H) - H$ contains no non-empty α-closed set.

(b) If $H \subseteq G \subseteq cl_\omega(H)$, then $G \in \alpha G\omega C(X, \tau)$.

Proof. (a) If $cl_\omega(H) - H$ contains an α-closed set C, then $H \subseteq X - C$ and $X - C$ is α-open in (X, τ). Then $cl_\omega(H) \subseteq X - C$ or equivalently, $C \subseteq X - cl_\omega(H)$. Therefore, $C \subseteq (X - cl_\omega(H)) \cap (cl_\omega(H) - H) = (X - cl_\omega(H)) \cap cl_\omega(H) \cap (X - H) = \phi$.

(b) Let U be α-open and $G \subseteq U$. Then $H \subseteq G \subseteq U$. Since $H \in \alpha G\omega C(X, \tau)$, $cl_\omega(H) \subseteq U$. But $cl_\omega(G) \subseteq cl_\omega(cl_\omega(H)) = cl_\omega(H) \subseteq U$ and the result follows.

Lemma 3.19 If H is an α-open and $\alpha g\omega$-closed subset of a space (X, τ), then H is ω-closed in X.

The proof is obvious.

Theorem 3.20 Let H be an $\alpha g\omega$-closed subset of (X, τ). Then $H = cl_\omega(int_\omega(H))$ if and only if $cl_\omega(int_\omega(H)) - H$ is α-closed.

Proof. If $H = cl_\omega(int_\omega(H))$, then $cl_\omega(int_\omega(H)) - H = \phi$ and hence $cl_\omega(int_\omega(H)) - H$ is α-closed. Conversely, let $cl_\omega(int_\omega(H)) - H$ be α-closed. Since $cl_\omega(H) - H$ contains the α-closed set $cl_\omega(int_\omega(H)) - H$. By Proposition 3.18 (a), $cl_\omega(int_\omega(H)) - H = \phi$ and hence $H = cl_\omega(int_\omega(H))$.

Theorem 3.21 Every $\alpha g\omega$-closed set and g-closed set are gω-closed.

The proof is obvious.

Corollary 3.22 Let (X, τ) be a space and H be an $\alpha g\omega$-closed set. The following are equivalent.

(a) H is an ω-closed set.

(b) $cl_\omega(H) - H$ is an α-closed set.

Proof. (a) ⇒ (b) If H is ω-closed, then $cl_\omega(H) - H = \phi$ and so $cl_\omega(H) - H$ is α-closed.

(b) ⇒ (a) If $cl_\omega(H) - H$ is α-closed, then since H is $\alpha g\omega$-closed by assumption, using Proposition 3.18 (a) we have $cl_\omega(H) - H = \phi$ and so H is ω-closed.

Theorem 3.23 Let (X, τ) be a space and $H \subseteq X$. If H is $\alpha g\omega$-closed, then $H = G - N$ where G is ω-closed and N contains no nonempty α-closed set.

Proof. If H is $\alpha g\omega$-closed, then by Proposition 3.18 (a), $cl_\omega(H) - H = N$ (say) contains no nonempty α-closed set. Let $G = cl_\omega(H)$, then G is ω-closed such that $G - N = cl_\omega(H) - (cl_\omega(H) - H) = H$.

Theorem 3.24 If K and L are $\alpha g\omega$-closed sets, then $K \cup L$ is $\alpha g\omega$-closed.

Proof. Suppose that $K \cup L \subseteq U$, where U is α-open. Then $K \subseteq U$ and $L \subseteq U$. Since K and L are $\alpha g\omega$-closed sets, $cl_\omega(K) \subseteq U$ and $cl_\omega(L) \subseteq U$. Now $cl_\omega(K \cup L) = cl_\omega(K) \cup cl_\omega(L) \subseteq U$. It shows that $K \cup L$ is $\alpha g\omega$-closed.

Theorem 3.25 Let K and L be subsets of a space (X, τ) such that $K \subseteq L \subseteq cl_\omega(K)$ and K is an $\alpha g\omega$-closed set. Then L is also $\alpha g\omega$-closed.

Proof. This is the same consequence with proposition of 3.18 (b).

Proposition 3.26 The intersection of an α-closed set and a closed set is α-closed.

Proof. This is obvious similarly with Proposition 3.14.

Theorem 3.27 If (X, τ) is any space and $H \subseteq X$ such that $cl_\omega(H) = cl(H)$, then the following are equivalent.

(a) H is $\alpha g\omega$-closed.

(b) $cl_\omega(H) \subseteq G$ whenever $H \subseteq G$ and G is α-open in X.

(c) For all $x \in cl_\omega(H)$, $\alpha cl(\{x\}) \cap H \neq \phi$.

(d) $cl_\omega(H) - H$ contains no nonempty α-closed set.

Proof. (a) ⇒ (b): If H is $\alpha g\omega$-closed, then $cl_\omega(H) \subseteq G$ whenever $H \subseteq G$ and G is α-open in X. This proves (b).

(b) ⇒ (c): Suppose $x \in cl_\omega(H)$. If $\alpha cl(\{x\}) \cap H = \phi$, then $H \subseteq X - \alpha cl(\{x\})$. By (b), $cl_\omega(H) \subseteq X - \alpha cl(\{x\})$. This is contrary to that $x \in cl_\omega(H)$.

(c) ⇒ (d): Suppose $F \subseteq cl_\omega(H) - H$, F is α-closed and $x \in F$. Since $F \subseteq X - H$ and F is α-closed, $\alpha cl(\{x\}) \cap H = \phi$. Since $x \in cl_\omega(H)$ by (c), $\alpha cl(\{x\}) \cap H \neq \phi$. This is a contradiction.

(d) ⇒ (a): Let $H \subseteq U$, where U is α-open. Then $X - U \subseteq X - H$ and $cl_\omega(H) \cap (X - U) \subseteq cl_\omega(H) \cap (X - H) = cl_\omega(H) - H$. This shows $cl(H) \cap (X - U) \subseteq cl_\omega(H) - H$. Since $cl(H)$ is closed and $X - U$ is α-closed, by Proposition 3.26, $cl(H) \cap (X - U)$ is an α-closed set contained in $cl_\omega(H) - H$. Therefore, $cl(H) \cap (X - U) = \phi$ by (d). Thus $cl(H) \subseteq U$ and $cl_\omega(H) \subseteq cl(H) \subseteq U$. Therefore H is $\alpha g\omega$-closed.

Definition 3.28 A space (X, τ) is called $\alpha g\omega$-$T_{\frac{1}{2}}$ if every $\alpha g\omega$-closed set in (X, τ) is ω-closed in (X, τ).

Example 3.29 Every countable space is $\alpha g\omega$-$T_{\frac{1}{2}}$.

Example 3.30 In Example 3.5, the set $H = \mathbb{R} - \mathbb{Q}$ is an $\alpha g\omega$-closed set which is not ω-closed. Therefore $(\mathbb{R}, \tau)$ is not $\alpha g\omega$-$T_{\frac{1}{2}}$.

Theorem 3.31 For a space (X, τ), the following are equivalent. [(a)]

(a)X is an $\alpha g\omega$-$T_{\frac{1}{2}}$ space.

(b) Every singleton is either α-closed or ω-open.

Proof. (a) ⇒ (b): Suppose $\{x\}$ is not an α-closed subset for $x \in X$. Then $X - \{x\}$ is not α-open and hence X is the only α-open set containing $X - \{x\}$. Therefore $X - \{x\}$ is $\alpha g\omega$-closed. Since (X, τ) is an $\alpha g\omega$-$T_{\frac{1}{2}}$ space, $X - \{x\}$ is ω-closed and thus $\{x\}$ is ω-open.

(b) ⇒ (a): Let H be an $\alpha g\omega$-closed subset of (X, τ) and $x \in cl_\omega(H)$. We show that $x \in H$.

Case (i) If $\{x\}$ is α-closed and $x \notin H$, then $x \in (cl_\omega(H) - H)$. Thus $cl_\omega(H) - H$ contains a nonempty α-closed set $\{x\}$. This is contrary to Proposition 3.18 (a). So $x \in H$.

Case (ii) If $\{x\}$ is ω-open, then since $x \in cl_\omega(H)$, for every ω-open set U containing x, we have $U \cap H \neq \phi$. But $\{x\}$ is ω-open, then $\{x\} \cap H \neq \phi$. Hence $x \in H$.

So in both cases, we have $x \in H$. Therefore H is ω-closed.

4. αgω-open sets

Definition 4.1 A subset H of a space (X, τ) is called αgω-open if its complement $X - H$ is αgω-closed in (X, τ).

Theorem 4.2 Let (X, τ) be a topological space and $H \subseteq X$. Then H is αgω-open if and only if $G \subseteq int_{\omega}(H)$ whenever G is α-closed and $G \subseteq H$.

Proof. Suppose H is αgω-open. If G is α-closed and $G \subseteq H$, then $X - H \subseteq X - G$ and so $cl_{\omega}(X - H) \subseteq X - G$. Therefore $G \subseteq int_{\omega}(H)$. Conversely, Suppose the condition holds. Let U be an α-open set such that $X - H \subseteq U$. Then $X - U \subseteq H$ and so by assumption $X - U \subseteq int_{\omega}(H)$ which implies that $cl_{\omega}(X - H) \subseteq U$. Therefore $X - H$ is αgω-closed and so H is αgω-open.

Theorem 4.3 Let K and L be subsets of a space (X, τ) such that $int_{\omega}(K) \subseteq L \subseteq K$. If K is αgω-open, then L is also αgω-open.

The proof is similar to the proof of Theorem 3.25.

Theorem 4.4 If H is an αgω-closed subset of (X, τ), then $cl_{\omega}(H) - H$ is αgω-open.

Proof. Let H be an αgω-closed subset of (X, τ) and let U be an α-closed subset such that $U \subseteq cl_{\omega}(H) - H$. By Proposition 3.18 (a), $U = \phi$ and thus $U \subseteq int_{\omega}(cl_{\omega}(H) - H)$. By Theorem 4.2 $cl_{\omega}(H) - H$ is αgω-open.

Lemma 4.5 If every α-open set is closed, then all subsets of (X, τ) are αgω-closed (and hence all subsets are αgω-open).

Proof. Let A be any subset of X such that $A \subseteq U$ and U is α-open, then $cl_{\omega}(A) \subseteq cl_{\omega}(U) \subseteq cl(U) = U$. Therefore A is αgω-closed.

Theorem 4.6 If K and L are αgω-open and separated sets, then $K \cup L$ is αgω-open.

Proof. Let F be an α-closed subset of $K \cup L$. Then $F \subseteq K \cup L \subseteq cl(K \cup L)$. Now $F \cap cl(K) \subseteq (K \cup L) \cap cl(K) = [K \cap cl(K)] \cup [L \cap cl(K)] = K \cup \phi = K$. Since F is α-closed and $cl(K)$ is closed, by Proposition 3.26, $F \cap cl(K)$ is α-closed. By Theorem 4.2, we have $F \cap cl(K) \subseteq int_{\omega}(K)$. Similarly, $F \cap cl(L) \subseteq int_{\omega}(L)$.

Since $F \subseteq cl(K \cup L)$, $F = F \cap cl(K \cup L) = F \cap [cl(K) \cup cl(L)] = [F \cap cl(K)] \cup [F \cap cl(L)] \subseteq int_{\omega}(K) \cup int_{\omega}(L) \subseteq int_{\omega}(K \cup L)$. Hence $K \cup L$ is αgω-open.

5. Generalized locally closed sets

Definition 5.1 A subset H of a space (X, τ) is called αgω-locally closed if $H = M \cap N$ where M is αgω-open and N is ω-closed.

Proposition 5.2 Let (X, τ) be a space and $H \subseteq X$. The following hold. [(a)]

(a) If H is αgω-open, then H is αgω-locally closed.

(b) If H is ω-closed, then H is αgω-locally closed.

Remark 5.3 None of the converses of the statements in the above proposition is true as shown in the following Examples.

Example 5.4 In the space $(\mathbb{R}, \tau_u)$, H=(0,1) is ω-open and hence αgω-open. Thus H is αgω-locally closed. Also H is not ω-closed since 0 is a condensation point of H and

$0 \notin H$.

Example 5.5 In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{R} - \mathbb{Q}\}$, the set $H = \mathbb{Q}$ is closed and hence ω-closed. Thus H is αgω-locally closed. Also H is not αgω-open since $R \backslash H$ is not αgω-closed which is proved in the Example 3.3.

Definition 5.6 A subset H of a space (X, τ) is called locally ω-closed if $H = M \cap N$ where M is ω-open and N is closed

Theorem 5.7 Every locally ω-closed set is αgω-locally closed.

Proof. It follows from the fact that (a) Every ω-open set is αgω-open and (b) Every closed set is ω-closed.

The converse of the above theorem is not true as seen from the following example.

Example 5.8 In $\mathbb{R}$ with the usual topology τ_u, $H = \mathbb{Q}$ is ω-closed since $\mathbb{Q}$ is countable. Hence H is an αgω-locally closed set. But H is not locally ω-closed. Suppose H is locally ω-closed. Let $H = M \cap N$, where $M \in \tau_\omega$ and N is closed. Then we have $H \subseteq N$ and $cl(H) \subseteq cl(N) = N$ by assumption. Thus $cl(H) = \mathbb{R} \subseteq N$ and so $\mathbb{R} = N$. But $H = M \cap \mathbb{R} = M \in \tau_\omega$ which is a contradiction since $H = \mathbb{Q}$ is not ω-open. This proves that $H = \mathbb{Q}$ is not locally ω-closed.

Theorem 5.9 Every locally closed set is αgω-locally closed.

It follows from the fact that (a) Every open set is ω-open

(b) Every ω-open set is αgω-open and

(c) Every closed set is ω-closed.

The converse of the above theorem is not true as seen from the following Example.

Example 5.10 In $\mathbb{R}$ with the usual topology τ_u, $H = \mathbb{R} - \mathbb{Q}$ is ω-open and hence αgω-open. Therefore H is αgω-locally closed. But $H = \mathbb{R} - \mathbb{Q}$ is not locally closed. Suppose $H = \mathbb{R} - \mathbb{Q}$ is locally closed. Let $H = M \cap N$, where M is open and N is closed. Then $H \subseteq N \Rightarrow \mathbb{R} = cl(H) \subseteq cl(N) = N$. Hence $N = \mathbb{R}$ and $H = \mathbb{R} \cap M = M$ which is open. This is a contradiction since H is not open. This proves that H is not locally closed.

Definition 5.11 A subset H of a space (X, τ) is called locally ω-α-closed if $H = M \cap N$ where M is α-open and N is ω-closed.

Proposition 5.12 Let (X, τ) be a space and $H \subseteq X$. The following hold. [(a)]

(a) If H is ω-open, then H is locally ω-closed.

(b) If H is closed, then H is locally ω-closed.

(c) If H is α-open, then H is locally ω-α-closed.

(d) If H is ω-closed, then H is locally ω-α-closed.

Remark 5.13 None of the converses of the statements in the above proposition is true as shown in the following examples.

Example 5.14 In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}\}$, the set $H = \mathbb{Q}$ is ω-open and hence locally ω-closed. But H is not closed.

Example 5.15 In $\mathbb{R}$ with the usual topology τ_u, $H = [0,1]$ is closed and hence locally ω-closed. But H is not ω-open, since $\mathrm{int}_\omega(H) \neq H$.

Example 5.16 In ℝ with the usual topology τ_u, H = [0,1] is ω-closed and hence locally ω-α-closed. But H is not α-open, since int(cl(int(H))) = (0,1) ⊉ H.

Example 5.17 In ℝ with the usual topology τ_u, H = (0,1) is α-open and hence H is locally ω-α-closed. But H is not ω-closed, since 0 is a condensation point of H and 0 ∉ H.

Theorem 5.18 Every locally closed set is locally ω-α-closed.

Proof. It follows from the fact that (a) Every open set is α-open and

(b) Every closed set is ω-closed.

The converse of the above theorem is not true as seen from the following Example.

Example 5.19 In ℝ with the usual topology τ_u, H = ℚ is ω-closed since ℚ is countable. Hence H is locally ω-α-closed. But H = ℚ is not locally closed. Suppose H = ℚ is locally closed. Let H = M ∩ N, where M is open and N is closed. Thus H ⊆ M. But ℝ is the only open set containing H, we have M = ℝ and H = ℝ ∩ N = N is closed. This is a contradiction since H is not closed. This proves that H is not locally closed.

Theorem 5.20 For a subset H of a space (X, τ), the following are equivalent. [(a)]

(a) H is ω-closed.

(b) H is αgω-closed and locally ω-α-closed.

Proof. (a) ⇒ (b): It follows from the fact that

(1) Every ω-closed set is αgω-closed and

(2) Every ω-closed set is locally ω-α-closed.

(b) ⇒ (a): Given H is locally ω-α-closed. So H = M ∩ N, where M is α-open and N is ω-closed. Since H ⊆ N, $\mathrm{cl}_\omega(H) \subseteq \mathrm{cl}_\omega(N)$ = N. Given H is αgω-closed. Since H ⊆ M, $\mathrm{cl}_\omega(H) \subseteq M$. We have $\mathrm{cl}_\omega(H) \subseteq M \cap N$ = H and hence H is ω-closed.

The following Examples show that the concepts of αgω-closedness and locally ω-α-closedness are independent.

Example 5.21 In ℝ with the usual topology τ_u, H = (0,1) is α-open and $\mathrm{cl}_\omega(H)$ = [0,1] ⊈ H. Thus H is not αgω-closed. Since H is α-open, H is locally ω-α-closed.

Example 5.22 In ℝ with the topology τ = {ϕ, ℝ, {1}}, the set H = ℝ − ℚ is αgω-closed but not locally ω-α-closed.

Solution: It is proved in Example 3.5 that H is αgω-closed but not ω-closed. Suppose H is locally ω-α-closed. Then H = M ∩ N, where M is α-open and N is ω-closed. Since H ⊆ M, M = ℝ. We have H = ℝ ∩ N = N which is ω-closed, a contradiction since H is not ω-closed. This proves that H is not locally ω-α-closed.

Theorem 5.23 Every locally closed set is locally ω-closed.

Proof. It follows from the fact that every open set is ω-open.

The converse of Theorem 5.23 is not true as seen from the following Example.

Example 5.24 In ℝ with the usual topology τ_u, the set H = ℝ − ℚ is locally ω-closed but not locally closed.

Solution: H = $\mathbb{R} - \mathbb{Q}$ is ω-open and hence locally ω-closed. But H is not locally closed. Suppose H is locally closed. Then H = M ∩ N, where M is open and N is closed. Since H ⊆ M and $\mathbb{R}$ is the only open set containing H, M = $\mathbb{R}$. Hence H = $\mathbb{R}$ ∩ N = N which is closed. This is a contradiction since H is not closed. This proves that H is not locally closed.

Remark 5.25 We have the following implications for a subset H of (X, τ).

Locally closed set $\underset{\not\leftarrow}{\rightarrow}$ ***Locally ω-α-closed set***

$\not\uparrow\downarrow$

Locally ω-closed set $\underset{\not\leftarrow}{\rightarrow}$ ***αgω-locally closed set***

6. αgω-normal spaces

Definition 6.1 A space (X, τ) is called an αgω-normal space if for every pair of disjoint closed sets P and Q, there exist disjoint αgω-open sets K and L such that P ⊆ K and Q ⊆ L.

Remark 6.2 Every normal space is αgω-normal.

Proof. Since every open set is ω-open and every ω-open set is αgω-open, every normal space is αgω-normal.

The following Example shows that an αgω-normal space is not necessarily a normal space.

Example 6.3 Let X = {a, b, c} and τ = {ϕ, X, {b}, {a, b}, {b, c}}. Then αO(X) = {ϕ, X, {b}, {a, b}, {b, c}} and ωO(X) = $\mathcal{P}$(X). Hence every α-open set is ω-closed and by Theorem 3.17, every subset of X is αgω-closed and hence every subset of X is αgω-open. This implies (X, τ) is αgω-normal. Now {a} and {c} are disjoint closed subsets of X which are not separated by disjoint open sets and so (X, τ) is not normal.

Theorem 6.4 Let (X, τ) be a space. Then the following are equivalent. [(a)]

(a) X is αgω-normal.

(b) For every pair of disjoint closed sets P and Q, there exist disjoint αgω-open sets K and L such that P ⊆ K and Q ⊆ L. \

(c) For every closed set P and an open set M containing P, there exists an αgω-open set N such that $P \subseteq N \subseteq cl_\omega(N) \subseteq M$.

Proof. (a) ⇒ (b): The proof follows from the definition of an αgω-normal space. (b) ⇒ (c): Let P be a closed set and M be an open set containing P. Since P and X − M are disjoint closed sets, there exist disjoint αgω-open sets N and W such that P ⊆ N and X − M ⊆ W. Again N ∩ W = ϕ implies that $N \cap int_\omega(W) = \phi$ and so $cl_\omega(N) \subseteq X - int_\omega(W)$. Since X − M is α-closed and W is αgω-open, X − M ⊆ W implies that $X - M \subseteq int_\omega(W)$ and so $X - int_\omega(W) \subseteq M$. Thus we have

$P \subseteq N \subseteq cl_\omega(N) \subseteq X - int_\omega(W) \subseteq M$ which proves (c).

(c) ⇒ (a): Let P and Q be two disjoint closed subsets of X. By hypothesis, there exists an $\alpha g\omega$-open set K such that $P \subseteq K \subseteq cl_\omega(K) \subseteq X - Q$. If $L = X - cl_\omega(K)$, then K and L are the required disjoint $\alpha g\omega$-open sets containing P and Q, respectively. So (X, τ) is $\alpha g\omega$-normal.

Theorem 6.5 Let (X, τ) be an $\alpha g\omega$-normal space. If F is closed and A is a g-closed set such that $A \cap F = \phi$, then there exist disjoint $\alpha g\omega$-open sets K and L such that $A \subseteq K$ and $F \subseteq L$.

Since $A \cap F = \phi$, $A \subseteq X - F$, where $X - F$ is open. Therefore, by hypothesis, $cl(A) \subseteq X - F$. Since $cl(A) \cap F = \phi$ and X is $\alpha g\omega$-normal, there exist disjoint $\alpha g\omega$-open sets K and L such that $A \subseteq cl(A) \subseteq K$ and $F \subseteq L$.

Theorem 6.6 Let (X, τ) be an $\alpha g\omega$-normal space. Then the following hold.

(a) For every closed set P and every g-open set Q containing P, there exists an $\alpha g\omega$-open set U such that $P \subseteq int_\omega(U) \subseteq U \subseteq Q$.

(b) For every g-closed set P and every open set Q containing P, there exists $\alpha g\omega$-closed set U such that $P \subseteq U \subseteq cl_\omega(U) \subseteq Q$.

Proof. (a) Let P be a closed set and Q be a g-open set containing P. Then $P \cap (X - Q) = \phi$, where P is closed and $X - Q$ is g-closed. By Theorem 6.5, there exist disjoint $\alpha g\omega$-open sets U and V such that $P \subseteq U$ and $X - Q \subseteq V$. Since $U \cap V = \phi$, we have $U \subseteq X - V$. By Theorem 4.2, $P \subseteq int_\omega(U)$. Therefore $P \subseteq int_\omega(U) \subseteq U \subseteq X - V \subseteq Q$.

This proves (a).

(b) Let P be a g-closed set and Q be an open set containing P. Then $X - Q$ is a closed set contained in the g-open set $X - P$. By (a), there exists an $\alpha g\omega$-open set V such that $X - Q \subseteq int_\omega(V) \subseteq V \subseteq X - P$. Therefore $P \subseteq X - V \subseteq cl_\omega(X - V) \subseteq Q$. If $U = X - V$, then $P \subseteq U \subseteq cl_\omega(U) \subseteq Q$ and so U is the required $\alpha g\omega$-closed set.

7. Decomposition of ω-continuity

Definition 7.1 A function $f: X \to Y$ is said to be ω-continuous [7] (resp. $\alpha g\omega$-continuous, locally ω-α-continuous) if $f^{-1}(G)$ is ω-closed (resp. $\alpha g\omega$-closed, locally ω-α-closed) in X for each closed subset G of Y.

Theorem 7.2 For a function $f: X \to Y$, the following are equivalent. [(a)]

(a) f is ω-continuous.

(b) f is $\alpha g\omega$-continuous and locally ω-α-continuous.

Proof. This is an immediate consequence of Theorem 5.20.

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