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# Investigation of Representation Theory for Algebraic and Geometric Applications

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**Abstract**
Understanding the algebraic and geometric structures that form in diverse mathematical areas depends heavily on the study of representation theory. The significance and uses of representation theory in both algebra and geometry are briefly discussed in this abstract.The primary goal of representation theory is to understand how linear transformations on vector spaces can represent abstract algebraic objects like groups, rings, and algebras. Representation theory offers a strong framework to analyse and interact with these structures using the methods and tools of linear algebra by linking algebraic structures with linear transformations.The representation theory has significant effects on algebra. Through the examination of the representations that go along with a group, it allows us to examine its composition and behaviour. One can learn more about the internal structures and underlying symmetries of groups by breaking representations down into irreducible parts. This has implications for number theory, combinatorics, and quantum physics, among other fields.Understanding symmetries and transformations of geometric objects in geometry depends critically on representation theory. The investigation of shape and space symmetry is made possible by representation theory, which links geometric objects with linear transformations. This has uses in a variety of disciplines, including physics' study of symmetry groups, differential geometry, and crystallography.A crucial area of research, representation theory has extensive uses in both algebra and geometry. It is indispensable in many branches of mathematics and opens up new directions for research and discovery because it can reveal the underlying structures and symmetries of abstract algebraic objects.

**Keywords**: Linear Algebra, geometric algebra, algebraic structure

## I. Introduction

Mathematicians study how algebraic structures, such as groups, rings, and algebras, can be represented by linear transformations on vector spaces in representation theory, a dynamic and crucial field of study. It offers a strong framework for comprehending and working with these structures by applying the methods and procedures of linear algebra. With significant ramifications in both algebra and geometry, representation theory has applications in many areas of mathematics. The representation theory sheds light on group behaviour and structure in algebra. The study of the symmetries and inner workings of these abstract algebraic objects is made possible by the association of groups with linear transformations in representation theory [2].

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For examining [3][4] the symmetries and transformations of geometric objects, representation theory is a key tool in geometry. The examination of the symmetries and transformations of shapes and spaces is made possible by representation theory, which connects geometric structures to linear transformations. This has numerous uses in differential geometry, the study of symmetry groups in different physical processes, and crystallography.By serving as a link between algebra and geometry and representation theory, these two fields can share concepts and methods more easily[6]. Particularly in geometric representation theory, the geometric and algebraic aspects are combined, giving geometric insights into the algebraic structures and vice versa. This multidisciplinary approach has advanced our grasp of both algebra and geometry and enabled new insights into complex mathematical phenomena.In this study, we explore representation theory's depths to reveal its nuanced relationships to algebra and geometry. We want to reveal the underlying structures and symmetries of abstract algebraic objects and geometric entities by probing the theory's ideas, methods, and applications[8].

## II. Review of Literature

Strassen discovered a startling finding while attempting to demonstrate the superiority of the conventional row-column approach for multiplying matrices. Instead of getting the expected result, he discovered a ground-breaking approach that could multiply n x n matrices over any field with a lot less arithmetic than the conventional algorithm [7]. In contrast to the usual approach's O(n3) difficulty, Strassen's technique had a time complexity of O(n2.81). This startling discovery raised a crucial issue: How well can matrices be multiplied?

The discipline [1] of computational linear algebra was significantly affected by Strassen's discovery, which prompted studies into the efficiency bounds of matrix multiplication. It led to the hypothesis that matrix multiplication could become almost as simple as matrix addition as matrix size increases [9].

The difficulty of matrix multiplication as a geometry problem. The amazing conjecture cited above can be more specifically stated as follows:

$\omega := \inf\tau$ {n × n matrices multiplied with O(nτ) operations related to Arithmetic }

Matrix multiplication, written as Mn: Cn2 x Cn2 x Cn2, can be thought of as a bilinear map that takes two n n matrices and produces their product, or (X, Y) →XY.

In general, a trilinear form or tensor, represented as β: A* × B* → Ccan be thought of as a bilinear map, Tβ ∈ A⊗B⊗C. The trilinear form of matrix multiplication is provided by (X, Y, Z) →trace(XYZ), where trace stands for the diagonal element sum of the XYZ matrix product.

A tensor T ∈ A⊗B⊗C complexity can be calculated using its rank, or R(T). The rank, defined as T = ∑(j=1 to r) ej⊗fj⊗g, is the minimum value of r for which T can be represented as the sum of r rank-one tensors.This hypothesis has stunning potential consequences. Several mathematical and scientific fields that extensively rely on matrix computations would be revolutionised if it were found to be true that multiplication enormous matrices could be done with astonishing efficiency. Since then, scientists have worked to pinpoint the limits of matrix multiplication efficiency in an effort to comprehend the underlying complexity and find algorithms that can multiply data even more quickly[11][12].

Strassen's algorithm offers proof of the potency of unforeseen mathematical breakthroughs. While his first objective was to prove that the conventional approach was best, his ground-breaking technique unlocked new doors and inspired continued research into the effectiveness of matrix multiplication. The development of computing linear algebra is still driven by the search for effective matrix multiplication techniques [13].

## III. The basic principle of Linear Algebra

3.1 (Linear algebra's fundamental theorem). Fix the bases of A and B (ai and bj), then for r min (a, b), set Ir = ∑r k=1 ak⊗bk. The sum of the following numbers equals the rank of T ∈ A⊗B: The study of vector spaces and linear transformations is central to the fundamental ideas of linear algebra. At its foundation, linear algebra uses equations and mathematical representations to attempt to comprehend the characteristics and behaviour of linear equations and systems. Vectors and matrices are the basic building elements of linear algebra. A column or row of numbers is often used to indicate a vector, a mathematical object that represents a quantity or a point in space [14]. Vectors have attributes like length and direction and can be combined together and scaled by a scalar.

- The smallest r necessary for T to be a sum of r components with rank one. Specifically, the least r such that
- T ∈ End(A)× is a limit of a sum of r rank one elements, i.e., such that T ∈ GL(A) × GL(B) · Ir.
- (mlA) dim A − dim ker(TA : A ∗ → B)
- (mlB) dim B − dim ker(TB : B ∗ → A)
- (Q) The biggest r such that Ir∈ GL(A) × GL(B) · T., Ir∈ End(A) × End(B) · T., and T are satisfied.
- most significant r such that Ir End(A) End(B) T.

3.2 For tensors, the fundamental theorem utterly fails.

In fact, [15] tensors are not directly covered by the basic theorem of linear algebra, which ensures the existence and uniqueness of solutions to systems of linear equations. Tensors bring complexity and difficulties that go beyond what the basic theorem can handle.Vectors and matrices are generalised by tensors, which are multidimensional arrays. They can describe higher-order relationships between vectors, matrices, or other tensors and have components that span several indices. Tensors have uses in physics, engineering, machine learning, and image processing, among other disciplines [16].

$$Q(T) \leq Q(T) \leq \min\{mlA(T), mlB(T), mlC(T)\}$$
$$\leq \max\{mlA(T), mlB(T), mlC(T)\} \leq R(T) \leq R(T)$$

and even when a = b = c, all inequalities could still be strict.

It has been proved that if T is a generic tensor, then its rank R(T) is roughly equal to m(2/3) and thus is the highest possible rank in the scenario when a = b = c = m. However, it wasn't until Lickteig's work in 1985 that the precise values for generic tensors were established. Terracini began studying the symmetric case in 1916 and made great strides, but he never fully solved the problem

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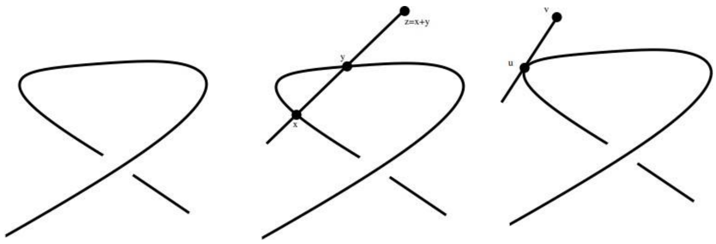
for polynomials of any degree. The symmetric case's definitive resolution, which took into account polynomials of any degree [17].

3.3 Insight of Bini's geometry.

Consider the following pictures



(a) (b) (c)

**Figure 1: (a) A curve represents the 3m − 2 dimensional set, (b) Represent points on a secant line to the setof tensors (c) tangent line on tensor**

It has been found that the majority of points residing on a secant line are exclusive to that particular secant line when studying a curve or variety with a large codimension. In contrast to the case of a plane curve, when all points lie on a family of secant lines, points on a secant line typically do not lie on a tangent line [18].

Because secant lines can converge to tangent lines, Bini's revelation made clear that tensor rank is not semi-continuous. He developed the idea of border rank, which takes these limitations into consideration, to account for these limiting instances. It is noteworthy that this absence of semi-continuity was observed by classical Italian algebraic geometers a century prior.Bini went on to show that border rank is a reliable indicator of the degree of matrix multiplication difficulty. In other words, it offers important information on the computational challenge of multiplying matrices [12].

The rediscovery of Bini clarified the non-semi-continuity of tensor rank and introduced the idea of border rank, which takes secant line boundaries into account. The comprehension of border rank, which provides a more thorough measurement than conventional tensor rank, has proven to be essential in assessing the complexity of matrix multiplication [10].

It is represent as:

$R(M\langle n\rangle) = O(n\ \omega\ )$[15]........................................................................ (1)

Secant varieties formula written as:

$\sigma r(X) := \{z \in PV \mid \exists x1, \cdots, xr \in X \mid z \in \langle x1, \cdots, xr\rangle\}$............................(2)

By noticing that the Segre variety Seg(PA × PB) ⊂ P(A⨂B)of rank one matrices exhibits flawed secant varieties, the idea that the fundamental lemma of linear algebra is a pathological instance can be reiterated. To put it another way, the secant varieties connected to this Segre variety exhibit unexpected behaviour.It is crucial to note that the secant varieties of tensors of order three or higher often do not exhibit this flaw. This is corroborated by Lickteig's study, which showed that there is only one exception to the (m, m, m) scenario, which happens when m = 3 and r = 4. Refer to [1] for a thorough summary of the current level of understanding on the general tensor issue.

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## IV. Problems related to complexity

### 1. P v. NP and variants.

The famous P $\neq$ NP conjecture, which was first put up by Cook, Karp, and Levin, has a number of antecedents. Its foundations can be found in John Nash's 1950s study, Soviet Union researchers, and Kurt Gödel's speculations on the viability of quantifying intuition. Polynomial complexity is incorporated into one of L. Valiant's algebraic interpretations of the issue, which is as follows:
Using affine linear forms in the variables xi, Valiant showed that any polynomial p(x1,..., xN) can be written as the determinant of a n n matrix, where the size of the matrix depends on the polynomial p. Valiant demonstrated that a valid indicator of a polynomial's complexity is the size of the matrix.The "permanent vs. determinant" version of the P $\neq$ NP conjecture asserts that the size of the matrix needed to compute the permanent permm$\in$ S(mC)m² of an m × m matrix grows more rapidly than any polynomial in m.

In the field of algebraic geometry, homogeneous polynomials are frequently preferred. As a result, to homogenise the problem in line with K. Mulmuley and M. Sohoni's methodology, a new variable is added. This makes it possible to write the permanent as a determinant of homogeneous linear functions and formulate the problem in that way.
Mulmuley and Sohoni proposed a more solid supposition to bring a geometric viewpoint into the issue. Their suggestion, which can be summed up as follows, tries to strengthen the relationship with algebraic geometry.

Mulmuley and Sohoni proposed a larger conjecture that makes use of the idea of complexity classes and geometric complexity theory in addition to the size of the matrix needed to compute the permanent and determinant.According to their conjecture, some geometric complexity classes, like VP (Varieties of Polynomials), have issues that are challenging to resolve with algebraic circuits. To put it another way, they believe that the class VP adequately depicts the underlying complexity of computing polynomials on algebraic varieties.

### 2. Algorithms in invariant theory

Invariant theory algorithms are essential for comprehending and researching the symmetries and invariants of mathematical objects under group actions. Finding polynomial functions that don't change or are invariant in the face of specific transformations is the core goal of invariant theory. Usually, groups such as the symmetric group or the generic linear group provide these transformations.Computing these invariant polynomials and comprehending their characteristics are the major goals of algorithms in invariant theory. Numerous disciplines, such as algebraic geometry, representation theory, and combinatorics, can benefit from these techniques.
The formula for producing a generating set of invariants is another significant algorithm. This algorithm identifies the smallest group of polynomials that can produce all of a group action's invariants. The generating set offers a clear representation of the invariants and aids in understanding the structure of the invariant ring.
Additionally, Grobner bases of invariant ideals are computed using invariant theory techniques. Grobner bases are useful tools for solving polynomial equations in computational algebra. Systems of equations involving the invariants can be solved by computing Grobner bases of the invariant

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ideals.Algorithms are also created for calculating particular kinds of invariants, such as full invariants or primary invariants. Primary invariants record crucial details about the group action, whereas complete invariants give a comprehensive description of the invariant ring.

Invariant theory uses algorithms to calculate and analyse invariants under group actions. These algorithms offer useful insights into the symmetry and characteristics of mathematical objects and have applications in many branches of mathematics. They promote linkages between algebra, geometry, and combinatorics and help us grasp algebraic structures and their transformations better.

$$\mathrm{cap}(v) := \mathrm{min} w \in G \cdot v \mid\mid w \mid\mid.$$

Null cone membership is a remarkable and particular example that belongs to the domain of invariant theory. This includes figuring out whether a specific vector v has zero capacity. To solve this issue and determine the vector's membership in the null cone, invariant theory algorithms are used.Understanding null cone membership and how it relates to invariant theory helps to clarify key ideas in linear programming and gives important new information on the geometrical characteristics of vector spaces. Researchers can efficiently handle issues surrounding null cone membership and explore its ramifications in many mathematical and real-world contexts by utilising invariant theory techniques.

## V. Conclusion

In both algebraic and geometric applications, the study of representation theory has shown to be a potent and useful tool. In order to gain a better knowledge of the underlying algebraic and geometric structures of mathematical objects, this field of research explores the structural characteristics and symmetries of those objects.Numerous fields of mathematics, including algebraic geometry, group theory, combinatorics, and quantum physics, have benefited greatly from research into representation theory. Deep understandings, innovative methods, and ground-breaking findings have resulted from the rich interaction between representation theory and these fields.Researchers have found hidden symmetries, categorised things, and found solutions to difficult problems by investigating the representations of groups and algebras. A systematic framework for comprehending the behaviour of functions, transformations, and other mathematical operation. In addition to mathematics, representation theory has applications in physics, computer science, and other fields of study. Insights into representations of symmetries and structures have applications in coding theory, quantum information theory, and quantum mechanics, among other areas.The potential for new links between algebra, geometry, and other branches of mathematics exists as representation theory research develops. It provides a strong framework for comprehending the symmetries and qualities of mathematical objects and serves as a rich source of inspiration for addressing basic problems.

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