A Certain Family for Higher-Order Derivatives of Multivalent Analytic Functions Associated with Dziok-Srivastava Operator

Abbas Kareem Wanas[1] and Faiz Chaseb Khudher[2]
[1,2]Department of Mathematics
College of Science
University of Al-Qadisiyah, Iraq
E-Mails: abbas.kareem.w@qu.edu.iq[1] , almdrsfayz@gmail.com

Article Info
Page Number: 1778 - 1788
Publication Issue:
Vol 72 No. 1 (2023)

Article History
Article Received: *15 October 2022*
Revised: *24 November 2022*
Accepted: *18 December 2022*

Abstract
By making use of the principle of differential subordination, we introduce and study a family for higher-order derivatives of multivalent analytic functions which are defined by means of a Dziok-Srivastava operator. We obtain some important results connected to inclusion relationship, argument estimate, integral representation and subordination property.

Keywords : Multivalent functions, Subordination, Integral representation, Higher-order derivatives, Dziok-Srivastava operator.

---

1. Introduction

Let $\mathcal{A}_p$ denote the family of functions f of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p}, \qquad p \in \mathbb{N} = \{1,2,\dots\}, \tag{1.1}$$

which are analytic and p-valent in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{A}_1 = \mathcal{A}$. Upon differentiating both sides of (1.1) j–times with respect to z, we obtain

$$f^{(j)}(z) = \varphi(p,j) z^{p-j} + \sum_{n=1}^{\infty} \varphi(n+p,j) a_{n+p}\, z^{n+p-j},$$

where

$$\varphi(p,j) = \frac{p!}{(p-j)!} = \begin{cases} 1, & j = 0 \\ p(p-1)\dots\quad(p-j+1), & j \neq 0 \end{cases},$$

and $p, j \in \mathbb{N}$, $p > j$.

Stimulated by "aforementioned works on multivalent functions with higher – order derivatives (see, for example [1,7,14,15,16]).

Given two functions f and g which are analytic in U, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z) (z \in U)$, if there exists a Schwarz function w which is analytic in U with

$w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z)), (z \in U)$. In particular, if the function g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

For functions f given by (1.1) and $g \in \mathcal{A}_p$ given by

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{n+p} z^{n+p},$$

the Hadamard product $f * g$ of f and g is defined by

$$(f * g)(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} b_{n+p} z^{n+p} = (g * f)(z) .$$

For complex parameters $\alpha_i \in \mathbb{C}, \beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$, where $\mathbb{Z}_0^- = \{0, -1, -2, \ldots\}$; $1 \le i \le l,\ 1 \le j \le k; l, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the generalized hypergeometric function ${}_lF_k(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_k; z)$ is defined by the following infinite series:

$${}_lF_k(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_k; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_k)_n} \frac{z^n}{n!} , (l \le k+1; l, k \in \mathbb{N}_0; z \in U),$$

where $(x)_n$ is the Pochhammer symbol defined by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1 & (n = 0), \\ x(x+1) \ldots (x+n-1) & (n \in \mathbb{N}). \end{cases}$$

Corresponding to a function $h_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_k; z)$ defined by

$$h_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_k; z) = z^p \ {}_lF_k(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_k; z). \tag{1.2}$$

Dziok and Srivastava [3] introduced a linear operator

$$H_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_k) : R(p,1) \longrightarrow R(p,1) ,$$

defined in terms of the Hadamard product as

$$H_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_k) f(z) = h_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_k; z) * f(z).$$

If $f \in R(p,1)$ is given by (1.1), then we have

$$H_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_k) f(z) = z^p + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_k)_n \, n!} a_{n+p} z^{n+p} . \tag{1.3}$$

In order to make the notation simple, we write $H_p^{l,k}(\alpha_1) = H_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_k)$.

We note from (1.3) that, we have

$$z\left(H_p^{l,k}(\alpha_1) f(z)\right)' = \alpha_1 H_p^{l,k}(\alpha_1 + 1) f(z) - (\alpha_1 - p) H_p^{l,k}(\alpha_1) f(z) . \tag{1.4}$$

Differentiating (1.4), $(j-1)$ times, we get

$$z\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(j)} = \alpha_1\left(H_p^{l,k}(\alpha_1+1)f(z)\right)^{(j-1)} - (\alpha_1 - p + j - 1)\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(j-1)}. \quad (1.5)$$

We note that special cases of the Dziok-Srivastava operator $H_p^{l,k}(\alpha_1)$ include the Hohlov linear operator [6], the Carlson-Shafer operator [2], the Ruscheweyh derivative operator [12], the Srivastava-Owa fractional operator [10] and many others.

Let T be the class of functions h of the form:

$$h(z) = 1 + \sum_{n=1}^{\infty} h_n z^n,$$

which are analytic and convex univalent in U and satisfy the condition:

$$Re\{h(z)\} > 0, \quad (z \in U).$$

We will require the following lemmas in proving our main results.

Lemma 1.1 [5]. Let $u, v \in \mathbb{C}$ and suppose that ψ is convex and univalent in U with $\psi(0) = 1$ and $Re\{u\psi(z) + v\} > 0, (z \in U)$. If q is analytic in U with $q(0) = 1$, then the subordination

$$q(z) + \frac{zq'(z)}{uq(z) + v} \prec \psi(z)$$

implies that $q(z) \prec \psi(z)$.

Lemma 1.2 [8]. Let h be convex univalent in U and $\mathcal{T}$ be analytic in U with $Re\{\mathcal{T}(z)\} \geq 0, (z \in U)$. If q is analytic in U and $q(0) = h(0)$, then the subordination

$$q(z) + \mathcal{T}(z) z q'(z) \prec h(z)$$

implies that $q(z) \prec h(z)$.

Lemma 1.3 [4]. Let q be analytic in U with $q(0) = 1$ and $q(z) \neq 0$ for all $z \in U$. If there exists two points $z_1, z_2 \in U$ such that

$$-\frac{\pi}{2} b_1 = arg\big(q(z_1)\big) < arg\big(q(z)\big) < arg\big(q(z_2)\big) = \frac{\pi}{2} b_2,$$

for some b_1 and b_2 $(b_1 > 0, b_2 > 0)$ and for all $z(|z| < |z_1| = |z_2|)$, then

$$\frac{z_1 q'(z_1)}{q(z_1)} = -i\left(\frac{b_1 + b_2}{2}\right) m \qquad and \qquad \frac{z_2 q'(z_2)}{q(z_2)} = i\left(\frac{b_1 + b_2}{2}\right) m,$$

where

$$m \geq \frac{1 - |\varepsilon|}{1 + |\varepsilon|} \quad and \quad \varepsilon = i \tan\frac{\pi}{4}\left(\frac{b_2 - b_1}{b_1 + b_2}\right).$$

Lemma 1.4 [11]. The function

$$(1 - z)^{\eta} \equiv \exp(\log(1 - z)), \quad (\eta \neq 0)$$

is univalent if and only if η is either in the closed disk $|\eta - 1| \leq 1$ or in the closed disk $|\eta + 1| \leq 1$.

Lemma 1.5 [9]. Let q be univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\phi\big(q(z)\big)$ and $h(z) = \theta\big(q(z)\big) + Q(z)$. Suppose that

(1) $Q(z)$ is starlike univalent in U,

(2) $Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ for $z \in U$.

If F is analytic in U, with $F(0) = q(0), F(U) \subset D$ and

$$\theta\big(F(z)\big) + zF'(z)\phi\big(F(z)\big) \prec \theta\big(q(z)\big) + zq'(z)\phi\big(q(z)\big),$$

then $F \prec q$ and q is the best dominant.

2. Main Results

Definition 2.1. A function $f \in \mathcal{A}_p$ is said to be in the family $N(\alpha_1, l, k, p, j, \gamma; h)$ if it satisfies the following differential subordination condition:

$$\frac{1}{p-j+1-\gamma}\left(\frac{z\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(j)}}{\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(j-1)}} - \gamma\right) \prec h(z)\,, \tag{2.1}$$

where $p, j \in \mathbb{N}, p > j$, $0 \leq \gamma < p$ and $h \in T$.

Theorem 2.1. Let $Re\{(p-j+1-\gamma)h(z) + \gamma + (\alpha_1 - p + j - 1)\} > 0$. Then

$$N(\alpha_1 + 1, l, k, p, j, \gamma; h) \subset N(\alpha_1, l, k, p, j, \gamma; h).$$

Proof. Let $f \in N(\alpha_1 + 1, l, k, p, j, \gamma; h)$ and put

$$q(z) = \frac{1}{(p-j+1-\gamma)}\left(\frac{z\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(j)}}{\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(j-1)}} - \gamma\right). \tag{2.2}$$

Then q is analytic in U with $q(0) = 1$. According to (2.2) and using the relation (1.5), we obtain

$$\frac{\alpha_1\left(H_p^{l,k}(\alpha_1+1)f(z)\right)^{(j-1)}}{\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(j-1)}} = (p-j+1-\gamma)q(z) + \gamma + (\alpha_1 - p + j - 1). \tag{2.3}$$

By logarithmically differentiating both sides of (2.3) with respect to z and multiplying by z, we get

$$q(z) + \frac{zq'(z)}{(p-j+1-\gamma)q(z) + \gamma + (\alpha_1 - p + j - 1)}$$
$$= \frac{1}{(p-j+1-\gamma)}\left(\frac{z\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(j)}}{\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(j-1)}} - \gamma\right) \prec h(z). \tag{2.4}$$

Since $Re\{(p-j+1-\gamma)h(z)+\gamma+(\alpha_1-p+j-1)\}>0$, then applying Lemma 1.1 to the subordination (2.4), yields $q(z)\prec h(z)$, which implies $f\in N(\alpha_1,l,k,p,j,\gamma;h)$.

Theorem 2.2. Let $\in \mathcal{A}_p$, $0<a_1,a_2\le 1$ and $0\le\gamma<p$. If

$$-\frac{\pi}{2}a_1<arg\left(\frac{z\left(H_p^{l,k}(\alpha_1+1)f(z)\right)^{(j)}}{\left(H_p^{l,k}(\alpha_1+1)g(z)\right)^{(j-1)}}-\gamma\right)<\frac{\pi}{2}a_2,$$

for some $g\in N\left(\alpha_1+1,l,k,p,j,\gamma;h;\frac{1+AZ}{1+Bz}\right)$, $(-1\le B<A\le 1)$, then

$$-\frac{\pi}{2}b_1<arg\left(\frac{z\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(j)}}{\left(H_p^{l,k}(\alpha_1)g(z)\right)^{(j-1)}}-\gamma\right)<\frac{\pi}{2}b_2,$$

where b_1 and b_2 $(0<b_1,b_2\le 1)$ are the solutions of the equations :

$$a_1=\begin{cases} b_1+\frac{2}{\pi}\tan^{-1}\left(\frac{(1-|\varepsilon|)(b_1+b_2)\cos\frac{\pi}{2}t}{2(1+|\varepsilon|)\left(\frac{(1+A)(p-j+1-\gamma)}{1+B}+\gamma+(\alpha_1-p+j-1)\right)+(1-|\varepsilon|)(b_1+b_2)\sin\frac{\pi}{2}t}\right), & B\neq -1\\ b_1, & B=-1\end{cases}$$

and

$$a_2=\begin{cases} b_2+\frac{2}{\pi}\tan^{-1}\left(\frac{(1-|\varepsilon|)(b_1+b_2)\cos\frac{\pi}{2}t}{2(1+|\varepsilon|)\left(\frac{(1+A)(p-j+1-\gamma)}{1+B}+\gamma+\alpha_1-p+j-1\right)+(1-|\varepsilon|)(b_1+b_2)\sin\frac{\pi}{2}t}\right), & B\neq -1\\ b_2, & B=-1\end{cases}$$

with

$$\varepsilon=i\tan\frac{\pi}{2}\left(\frac{b_2-b_1}{b_1+b_2}\right)\quad\text{and } t=\frac{2}{\pi}\sin^{-1}\left(\frac{(A-B)(p-j+1-\gamma)}{(\gamma+\alpha_1-p+j-1)(1-B^2)+(p-j+1-\gamma)(1-AB)}\right).\quad (2.7)$$

Proof. Define the function G by

$$G(z)=\frac{1}{(p-j+1-\tau)}\left(\frac{z\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(j)}}{\left(H_p^{l,k}(\alpha_1)g(z)\right)^{(j-1)}}-\tau\right),\qquad (2.8)$$

where $g\in N\left(\alpha_1+1,l,k,p,j,\gamma;h;\frac{1+AZ}{1+Bz}\right)$, $(-1\le B<A\le 1)$ and $0\le\tau<p$.

Then G is analytic in U with $G(0)=1$. Therefore by making use of (1.5) and (2.8), we obtain

$$\left((p-j+1-\tau)G(z)+\tau\right)\left(H_p^{l,k}(\alpha_1)g(z)\right)^{(j-1)}$$
$$=\alpha_1\left(H_p^{l,k}(\alpha_1+1)f(z)\right)^{(j-1)}-(\alpha_1-p+j-1)\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(j-1)}.$$

Differentiating above relation with respect to z and multiplying by z, we get

$$\left((p-j+1-\tau)G(z)+\tau\right)z\left(H_p^{l,k}(\alpha_1)g(z)\right)^{(j)}+(p-j+1-\tau)zG'(z)\left(H_p^{l,k}(\alpha_1)g(z)\right)^{(j-1)}$$
$$=\alpha_1 z\left(H_p^{l,k}(\alpha_1+1)f(z)\right)^{(j)}-(\alpha_1-p+j-1)z\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(j)}. \tag{2.9}$$

Suppose that

$$L(z)=\frac{1}{(p-j+1-\gamma)}\left(\frac{z\left(H_p^{l,k}(\alpha_1)g(z)\right)^{(j)}}{\left(H_p^{l,k}(\alpha_1)g(z)\right)^{(j-1)}}-\gamma\right).$$

Using (1.5) again, we have

$$\frac{\alpha_1\left(H_p^{l,k}(\alpha_1+1)g(z)\right)^{(j-1)}}{\left(H_p^{l,k}(\alpha_1)g(z)\right)^{(j-1)}}=(p-j+1-\gamma)L(z)+\gamma+(\alpha_1-p+j-1). \tag{2.10}$$

From (2.8) and (2.10), we easily get

$$G(z)+\frac{zG'(z)}{(p-j+1-\gamma)L(z)+\gamma+(\alpha_1-p+j-1)}$$
$$=\frac{1}{(p-j+1-\tau)}\left(\frac{z\left(H_p^{l,k}(\alpha_1+1)f(z)\right)^{(j)}}{\left(H_p^{l,k}(\alpha_1+1)g(z)\right)^{(j-1)}}-\tau\right). \tag{2.11}$$

Notice that from Theorem 2.1, $g\in N\left(\alpha_1+1,l,k,p,j,\gamma;\frac{1+AZ}{1+Bz}\right)$ implies $g\in N\left(\alpha_1,l,k,p,j,\gamma;\frac{1+AZ}{1+Bz}\right)$. Thus,

$$L(z)\prec\frac{1+AZ}{1+Bz}\quad(-1\le B<A\le 1).$$

By using the result of Silverman and Silvia [13], we have

$$\left|L(z)-\frac{1-AB}{1-B^2}\right|<\frac{A-B}{1-B^2}\quad(B\ne -1,\ \ z\in U) \tag{2.12}$$

and

$$Re\{L(z)\}>\frac{1-A}{2}\quad(B=-1,\ \ z\in U). \tag{2.13}$$

It follows from (2.12) and (2.13) that

$$\left|(p-j+1-\gamma)L(z)+\gamma+\alpha_1-p+j-1 - \frac{(\gamma+\alpha_1-p+j-1)(1-B^2)+(p-j+1-\gamma)(1-AB)}{1-B^2}\right| < \frac{(A-B)(p-j+1-\gamma)}{1-B^2}, \quad (B\neq -1,\ z\in U)$$

and

$$Re\{(p-j+1-\gamma)L(z)+\gamma+\alpha_1-p+j-1\} > \frac{(1-A)(p-j+1-\gamma)}{2}+\gamma+\alpha_1-p+j-1,\ \ (B=-1,\ z\in U).$$

Putting

$$(p-j+1-\gamma)L(z)+\gamma+\alpha_1-p+j-1=\rho e^{i\frac{\pi}{2}\phi},$$

where

$$-\frac{(A-B)(p-j+1-\gamma)}{(\gamma+\alpha_1-p+j-1)(1-B^2)+(p-j+1-\gamma)(1-AB)} < \phi < \frac{(A-B)(p-j+1-\gamma)}{(\gamma+\alpha_1-p+j-1)(1-B^2)+(p-j+1-\gamma)(1-AB)}, (B\neq -1)$$

and $-1<\phi<1,\ (B=-1)$,

then

$$\frac{(1-A)(p-j+1-\gamma)}{1-B}+\gamma+\alpha_1-p+j-1<\rho < \frac{(1+A)(p-j+1-\gamma)}{1+B}+\gamma+\alpha_1-p+j-1,\ \ (B\neq -1)$$

and

$$\frac{(1-A)(p-j+1-\gamma)}{1-B}+\gamma+\alpha_1-p+j-1<\rho<\infty,\ \ (B=-1).$$

An application of Lemma 1.2 with $(z)=\frac{1}{(p-j+1-\gamma)L(z)+\gamma+\alpha_1-p+j-1}$, yields $G(z)\prec h(z)$.

If there exist two points $z_1, z_2 \in U$ such that

$$-\frac{\pi}{2}b_1 = arg\big(G(z_1)\big) < arg\big(G(z)\big) < arg\big(G(z_2)\big)=\frac{\pi}{2}b_2\,,$$

then by Lemma 1.3, we get

$$\frac{z_1G'(z_1)}{G(z_1)} = -\frac{mi}{2}(b_1+b_2) \qquad and \qquad \frac{z_2G'(z_2)}{G(z_2)}=\frac{mi}{2}(b_1+b_2)\,,$$

where

$$m \geq \frac{1-|\varepsilon|}{1+|\varepsilon|} \quad and \quad \varepsilon = i\tan\frac{\pi}{4}\left(\frac{b_2-b_1}{b_1+b_2}\right).$$

Now, for the case $B \neq -1$, we obtain

$$arg\left(\frac{1}{(p-j+1-\tau)}\left(\frac{z_1\left(H_p^{l,k}(\alpha_1+1)f(z_1)\right)^{(j)}}{\left(H_p^{l,k}(\alpha_1+1)g(z_1)\right)^{(j-1)}}-\tau\right)\right)$$
$$= arg\left(G(z_1)+\frac{z_1G'(z_1)}{(p-j+1-\gamma)L(z_1)+\gamma+\alpha_1-p+j-1}\right)$$
$$= arg\big(G(z_1)\big)+arg\left(1+\frac{z_1G'(z_1)}{[(p-j+1-\gamma)L(z_1)+\gamma+\alpha_1-p+j-1]G(z_1)}\right)$$
$$= -\frac{\pi}{2}b_1+arg\left(1-\frac{mi}{2\rho}(b_1+b_2)e^{-i\frac{\pi}{2}\phi}\right)$$
$$= -\frac{\pi}{2}b_1+arg\left(1-\frac{m}{2\rho}(b_1+b_2)\cos\frac{\pi}{2}(1-\phi)+\frac{mi}{2\rho}(b_1+b_2)\sin\frac{\pi}{2}(1-\phi)\right)$$
$$\leq -\frac{\pi}{2}b_1-\tan^{-1}\left(\frac{m(b_1+b_2)\sin\frac{\pi}{2}(1-\phi)}{2\rho+m(b_1+b_2)\cos\frac{\pi}{2}(1-\phi)}\right)$$
$$\leq -\frac{\pi}{2}b_1$$
$$-\tan^{-1}\left(\frac{(1-|\varepsilon|)(b_1+b_2)\cos\frac{\pi}{2}t}{2(1+|\varepsilon|)\left(\frac{(1+A)(p-j+1-\gamma)}{1+B}+\gamma+\alpha_1-p+j-1\right)+(1-|\varepsilon|)(b_1+b_2)\sin\frac{\pi}{2}t}\right)$$
$$= -\frac{\pi}{2}a_1,$$

where a_1 and t are given by (2.5) and (2.7), respectively.

Also,

$$arg\left(\frac{1}{(p-j+1-\tau)}\left(\frac{z_2\left(H_p^{l,k}(\alpha_1+1)f(z_2)\right)^{(j)}}{\left(H_p^{l,k}(\alpha_1+1)g(z_2)\right)^{(j-1)}}-\tau\right)\right)$$
$$\geq \frac{\pi}{2}b_2$$
$$+\tan^{-1}\left(\frac{(1-|\varepsilon|)(b_1+b_2)\cos\frac{\pi}{2}t}{2(1+|\varepsilon|)\left(\frac{(1+A)(p-j+1-\gamma)}{1+B}+\gamma+\alpha_1-p+j-1\right)+(1-|\varepsilon|)(b_1+b_2)\sin\frac{\pi}{2}t}\right)$$
$$= \frac{\pi}{2}a_2,$$

where a_2 and t are given by (2.6) and (2.7), respectively.

Similarly, for the case $B = -1$, we have

$$arg\left(\frac{1}{(p-j+1-\tau)}\left(\frac{z_1\left(H_p^{l,k}(\alpha_1+1)f(z_1)\right)^{(j)}}{\left(H_p^{l,k}(\alpha_1+1)g(z_1)\right)^{(j-1)}}-\tau\right)\right)\leq -\frac{\pi}{2}b_1$$

and

$$arg\left(\frac{1}{(p-j+1-\tau)}\left(\frac{z_1\left(H_p^{l,k}(\alpha_1+1)f(z_2)\right)^{(j)}}{\left(H_p^{l,k}(\alpha_1+1)g(z_2)\right)^{(j-1)}}-\tau\right)\right)\geq \frac{\pi}{2}b_2.$$

The above two cases contradict the assumptions. Consequently, the proof of the theorem is complete.

In the following theorem, we find integral representation of the family $N(\alpha_1,l,k,p,j,\gamma;h)$.

Theorem 2.3. Let $f\in N(\alpha_1,l,k,p,j,\gamma;h)$. Then

$$\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(j-1)}=z^{p-j+1}.\exp\left[(p-j+1-\gamma)\int_0^z\frac{h(w(s))-1}{s}ds\right],$$

where w is analytic in U with $w(0)=0$ and $|w(z)|<1$, $(z\in U)$.

Proof. Assume that $f\in N(\alpha_1,l,k,p,j,\gamma;h)$. It is easy to see that subordination condition (2.1) can be written as follows

$$\frac{z\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(j)}}{\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(j-1)}}=(p-j+1-\gamma)h(w(z))+\gamma\,, \tag{2.14}$$

where w is analytic in U with $w(0)=0$ and $|w(z)|<1$, $(z\in U)$.

From (2.14), we find that

$$\frac{\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(j)}}{\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(j-1)}}-\frac{p-j+1}{z}=(p-j+1-\gamma)\frac{h(w(z))-1}{z}\,, \tag{2.15}$$

After integrating both sides of (2.15), we have

$$\log\left(\frac{\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(j-1)}}{z^{p-j+1}}\right)=(p-j+1-\gamma)\int_0^z\frac{h(w(s))-1}{s}ds. \tag{2.16}$$

Therefore, from (2.16), we obtain the required result.

Theorem 2.4. Let $1<\beta<2$ and $\eta\in\mathbb{R}\setminus\{0\}$ such that either $|2\eta(\beta-1)\alpha_1+1|\leq 1$ or $|2\eta(\beta-1)\alpha_1-1|\leq 1$. If $f\in\mathcal{A}_p$ satisfies the condition

$$Re\left\{1+\frac{\left(H_p^{l,k}(\alpha_1+1)f(z)\right)^{(j-1)}}{\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(j-1)}}\right\}>2-\beta+\frac{1-p}{\alpha_1}, \tag{2.17}$$

then,

$$\left(z\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(j-1)}\right)^{\eta}\prec(1-z)^{-2\eta(\beta-1)\alpha_1}$$

and $(1-z)^{-2\eta(\beta-1)\alpha_1}$ is the best dominant."

Proof. Define the function k by

$$F(z)=\left(z\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(j-1)}\right)^{\eta}. \tag{2.18}$$

Differentiating (2.18) with respect to z logarithmically and using (1.5), we obtain

$$\frac{zF'(z)}{F(z)}=\frac{\eta\,\alpha_1\left(H_p^{l,k}(\alpha_1+1)f(z)\right)^{(j-1)}}{\left(H_p^{l,k}(\alpha_1)f(z)\right)^{(j-1)}}-\eta(\alpha_1-p+j-1).$$

Now, in view of the condition (2.16), we have the following subordination

$$1+\frac{zF'(z)}{\eta\alpha_1F(z)}\prec\frac{1+(2\beta-3)z}{1-z}.$$

Assume that

$$\theta(w)=1\,,\quad \phi(w)=\frac{1}{\eta\alpha_1 w}$$

and

$$q(z)=(1-z)^{-2\eta(\beta-1)\alpha_1},$$

then by making use of Lemma 1.4, we know that q is univalent in U. It now follows that

$$Q(z)=zq'(z)\phi\big(q(z)\big)=\frac{2(\beta-1)z}{1-z}$$

and

$$h(z)=\theta\big(q(z)\big)+Q(z)=\frac{1+(2\beta-3)z}{1-z}.$$

If we define the domain D by

$$q(U)=\left\{w\colon\left|w^{\frac{1}{\sigma}}-1\right|<\left|w^{\frac{1}{\sigma}}\right|,\sigma=2\eta(\beta-1)\alpha_1\right\}\subset D,$$

then, it is easy to check that the conditions of Lemma 1.5 hold true. Therefore, we get the desired result.

References

[1] O. Altintas, Neighborhoods of certain p-valentiy functions with negative coefficients, Appl. Math comput., 187(1)(2007), 47-53.
[2] B. C. Carlson and D. B. Shaffer, Starlike and pre-starlike hypergeometric functions, SIAM J. Math Anal., 15(1984), 737-745.
[3] J. Dziok and H. M. Srivastava, Classes of analytic functions associated with the generalized hypergeometric function, Appl. Math. Comput. , 103(1)(1999), 1–13.
[4] A. Ebadian, S. Shams, Z. G. Wang and Y. Sun, A class of multivalent analytic functions involving the generalized Jung-Kim-Srivastava operator, Acta Univ. Apulensis, 18(2009), 265-277.
[5] P. J. Eenigenburg, S. S. Miller, P. T. Mocanu and M. O. Reade, On a Briot-Bouquet differential subordination, General Inequalities, Birkhauser, Basel, 3(1983), 339–348.
[6] Ju. E. Hohlov, Operators and operations on the class of univalent functions, Izvestiya Vysshikh Uchebnykh Zavedeniĭ. Matematika, 10(197)(1978), 83–89.
[7] H. Irmak and N. E. Cho, A differential operator and its applications to certain multivalently analytic function , Hacet. J. Math. Stat., 36(1) (2007), 1-6.
[8] S. S. Miller and P. T. Mocanu , Differential subordinations and univalent functions, Michigan Math. J. , 28(1981) , 157-171.
[9] S. S. Miller and P. T. Mocanu, On some classes of first-order differential subordinations, Michigan Math. J., 32(1985), 185-195.
[10] S. Owa and H. M. Srivastava, Univalent and starlike generalized hypergeometric function, Canad. J. Math., 39(5)(1987), 1057-1077.
[11] M. S. Robertson, Certain classes of starlike functions, Michigan Math. J., 32(1985), 135-140.
[12] S. Ruscheweyh, New criteria for univalent functions, Proc. Amer. Math. Soc., 49(1975), 109-115.
[13] H. Silverman and E. M. Silvia, Subclasses of starlike functions subordinate to convex functions, Canad. J. Math., 37(1985), 48-61.
[14] A. K. Wanas, On sandwich theorems for higher – order derivatives of multivalent analytic functions associated with the generalized Noor integral operator, Asian–Eur. J. Math., 8(2)(2015), 77-82.
[15] A. K . Wanas and A .H. Majeed, Differential subordination for higher –order derivatives of multivalent analytic functions associated with Dziok–Srivastava operator, An. Univ. Oradea Fasc. Mat., XXX(1)(2018), 33-42.
[16] A. K. Wanas and G. Murugusundaramoorthy, Higher-order derivatives of multivalent analytic functions defined on complex Hilbert space, J. Adv. Math. Stud., 14(2)(2021), 197-205.