Characterization of Hyperbolic Semigroups

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Article Info
Page Number: 12588-12594
Publication Issue:
Vol. 71 No. 4 (2022)

Article History
Article Received: 25 October 2022
Revised: 28 November 2022
Accepted: 15 December 2022

Abstract: In this article, the Banach space is decomposed into the direct sum of two closed subspaces such that the semigroup becomes forward exponentially stable on one subspace and backward exponentially stable on other subspace. Hyperbolic semigroup is characterized in terms of the spectrum of its cogenerator. Further, we study the rescaled hyperbolic semigroup to analyze its spectrum.

1. Introduction

The theory of hyperbolicity is one of the basic source in the study of partial differential equations. Hyperbolic semigroups are studied in the models of beams and waves as well as the transport equation and networks of non homogeneous transmission lines. If $(T(\zeta))_{\zeta\geq 0}$ is a hyperbolic semigroup, then every operator T (ζ) in the semigroup must itself be hyperbolic. [1].

The literature on hyperbolic semigroups is very rich. The classical results are presented in the books [5, 8, 15]. Over the years, the notion of hyperbolicity was broadened (non-uniform hyperbolicity) [3] and relaxed (partial hyperbolicity) [16, 17] to encompass a much larger class of systems. In [10] Herbert and Daniel obtain a complete characterization of Fredholm spectrum of the semigroup generated by sharp energy estimates.

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2010 Mathematics Subject Classification: 47A05, 47D06.

Key words and phrases: C_0-semigroups, rescaled semigroups, restricted semigroups,

cogenerator of semigroups, discrete spectrum and essential spectrum.

The existence and uniqueness of conformal measures are briefly explained by Hikori in [11]. In [6], Dimitrios proved that the rate of convergence of the semigroup to the point one is exponential. Quasi hyperbolic semigroups [2] and hyperbolic dynamical systems [9, 14] gives a path way to study the variations of hyperbolic semigroups.

The aim of the present study is to bridge the gap explored from the above literature. In contrast to the known result on this topic, this paper is motivated to study the decomposition of Banach spaces which enable us to efficiently analyze the hyperbolic semigroup in terms of its spectrum of its co-generator.

2.Preliminaries

Definition 2.1. A strongly continuous semigroup $T(\zeta)$ on a Banach space X is called bounded if $\sup_{\zeta\geq 0}\|T(\zeta)\| < \infty$

We remark that if A be the generator of a bounded semigroup of $T(\zeta)$ then

$\sigma(A) \subset \{z : \mathrm{Re}z \leq 0\}$ but not conversely.

Theorem 2.2. For a strongly continuous semigroup $T(\zeta)$ on a Banach space X

the following assertions are equivalent [7].

(i) $T(\zeta)$ is uniformly exponentially stable.

(ii) $\lim_{\zeta\geq 0}\|T(\zeta)\| = 0$

(iii) $\|T(\zeta_0)\| < 1$ for some $\zeta_0 > 0$

(iv) $r(T_0) < 1$ for some $\zeta_0 > 0$

(v) $r(T(\zeta) < 1$ for all $\zeta > 0$

Definition 2.3. A C_0-semigroup $T(\zeta)$ on a Banach space X with generator A satisfies

the spectral mapping theorem if $\sigma(T(\zeta))/\{0\} = e^{\zeta\sigma(A)}$ for every $\zeta \geq 0$ [12, 13].

3. Hyperbolic Decomposition

Definition 3.1. A C_0-semigroup $T(\zeta)$ of bounded operators on a Banach space X is called hyperbolic if X can be written as a direct sum $X = X_s \oplus X_u$ of two $T(\zeta)_{\zeta\geq 0}$ - invariant closed subspaces X_s, X_u called stable and unstable subspaces and such that for some $\varepsilon > 0$ and $M > 0$ and one has

$$\|T(\zeta)x\| \leq Me^{-\varepsilon\zeta}\|x\| \quad \text{for } x \in X_s,\ \zeta > 0.$$

$\|T(\zeta)x\| \geq \frac{1}{M} e^{\varepsilon\zeta} \|x\|$ for x ∈ X_u, ζ > 0.

It is known that T $(\zeta)_{\zeta\geq 0}$ is hyperbolic iff σ(T (ζ)) ∩ Γ = φ for all or equivalently some ζ > 0 where Γ is the unit circle [4].

Theorem 3.2. For a C_0-semigroup (T $(\zeta))_{\zeta\geq 0}$ the following assertions are equivalent.

(a) $(T(\zeta))_{\zeta\geq 0}$ is hyperbolic.

(b) $\sigma(T(\zeta)) \cap \{e^{-\varepsilon\zeta} \leq |z| \leq e^{\varepsilon\zeta}\} = \phi$ for one /all ζ > 0.

Proof: (a) ⇒ (b)

Let the restricted semigroups of (T $(\zeta))_{\zeta\geq 0}$ on X_s and X_u be $(T_s(\zeta))_{\zeta\geq 0}$ and $(T_u(\zeta))_{\zeta\geq 0}$ respectively. Then σ(T (ζ)) = $\sigma(T_s(\zeta)) \cup \sigma(T_u(\zeta))$.

By assumption $\|T_s(\zeta)x\| \leq Me^{-\varepsilon\zeta} \|x\|$ for ζ > 0 then $r(T_s(\zeta)) \leq e^{-\varepsilon\zeta}$ for ζ > 0

and so

$\sigma(T_s(\zeta))$ ∩ aΓ = φ for $a \geq e^{-\varepsilon\zeta}$ ——————(1)

By the same argument, from

$$\|T_u(\zeta)x\| \geq \frac{1}{M} e^{\varepsilon\zeta} \|x\|$$

$$\Rightarrow r(T_s(\zeta))^{-1} \leq e^{-\varepsilon\zeta}$$

Since $\sigma(T_u(\zeta)) = \{\lambda^{-1} : \lambda \in \sigma(T_u(\zeta)^{-1})\}$

$$\Rightarrow |\lambda| \geq e^{\varepsilon\zeta} \quad \text{for each } \lambda \in \sigma(T_u(\zeta)).$$

So $\sigma(T_u(\zeta))$ ∩ aΓ = φ for a $\leq e^{\varepsilon\zeta}$ —————(2)

Combining (1) and (2)

σ(T (ζ)) ∩ aΓ = φ for a $\in [e^{-\varepsilon\zeta}, e^{\varepsilon\zeta}]$ and so

σ(T (ζ)) ∩ $\{e^{-\varepsilon\zeta} \leq |z| \leq e^{\varepsilon\zeta}\} = \phi$

Conversely, assume s > 0 such that

σ(T (s)) ∩ $\{e^{-\varepsilon\zeta} \leq |z| \leq e^{\varepsilon\zeta}\} = \phi$.

The operators $P_s = \frac{1}{2\pi i} \int_{|z|\leq e^{-\varepsilon s}} \frac{dz}{z - T(s)}$ and $P_u = I - P_s$ define projection maps onto the corresponding subspaces X_s, X_u and

$\sigma(T(s)/X_s) = \sigma(T(s)) \cap \{|z| \le e^{-\varepsilon\zeta}\}$

$\sigma(T(s)/X_u) = \sigma(T(s)) \cap \{|z| \ge e^{\varepsilon\zeta}\}$

Then the spectral radius of $(T(s)/X_s)$ is given by $r(T_s(s)) \le e^{-\varepsilon s}$

Then by Theorem 2.2, for $x \in X_s$, $\zeta \ge 0$, $\exists$ M > 0 such that $\|T(\zeta)x\| \le Me^{-\varepsilon\zeta}\|x\|$

Now, $T_u(s)$ of T (s) in X_u has the spectrum

$\sigma(T_u(s)) = \{\lambda \in \sigma(T(s))/ |\lambda| \ge e^{\varepsilon s}\}$ and so is invertible on X_u.

This shows that $(T_u(\zeta))$ is invertible for $0 \le \zeta < s$ and for $\zeta > s$.

Let $n \in N$ so that $ns > \zeta$.

Then $(T_u(s))^n = T_u(ns) = T_u(\zeta)T_u(ns - \zeta)$ and so $T_u(\zeta)$ is invertible for $\zeta > s$.

Again by Theorem 2.2,

$$\|T_u(\zeta)x\| \ge \frac{1}{M}e^{\varepsilon\zeta}\|x\| \text{ for } x \in X_u,\ \zeta > 0.$$

Thus $(T(\zeta))_{\zeta\ge0}$ is hyperbolic.

Theorem 3.3. Let A be the generator and V be the co-generator of a C_0-semigroup $(T(\zeta))_{\zeta\ge0}$. Assume further $\sigma(T(\zeta)) \subset \Gamma e^{\zeta\sigma(A)} = \{ze^{\zeta\lambda} : \lambda \in \sigma(A), |z| = 1\}$. Then $(T(\zeta))_{\zeta\ge0}$ is hyperbolic if and only if $\sigma(V) \cap \Gamma = \varphi$.

Proof: By Engel, Nagel [7], [Theorem V.1.17] $(T(\zeta))_{\zeta\ge0}$ is hyperbolic if and only if $\sigma(A) \cap iR = \varphi$ whenever $\sigma(T(\zeta)) \subset \Gamma e^{\zeta\sigma(A)}$ holds. Using the identity

$V = (A - I + 2I)(A - I)^{-1} = I - 2R(1, A)$ we get

$$\sigma(V)/\{1\} = \left\{\frac{\lambda+1}{\lambda-1} : \lambda \in \sigma(A)\right\}$$

Now $ir \notin \sigma(A)$ is equivalent to $\frac{ir+1}{ir-1} \notin \sigma(V)/\{1\}$ for every real r.

This means $\sigma(V)/\{1\} \cap \Gamma = \phi$ as desired.

4. Rescaled Hyperbolic Semigroup

If $(T(\zeta))_{\zeta\ge0}$ is a hyperbolic semigroup, it can be observed that the rescaled semigroup operator $e^{\alpha\zeta}T(\zeta)$ has no spectrum in the annulus $\left(e^{(\alpha-\epsilon)\zeta} \le |z| \le e^{(\alpha+\epsilon)\zeta}\right)$ for every $\alpha \in$ R.

Definition 4.1. For $0<\lambda<\mu$, we call $(T(\zeta))_{\zeta\geq 0}$ is (λ,μ) hyperbolic whenever the rescaled operator $a^{\zeta}T(\zeta)$ is hyperbolic for every $a \in [\lambda, \mu]$. If $0<\lambda<1$ and $\mu > 1$ then (λ,μ) hyperbolic semigroup becomes a hyperbolic semigroup.

Theorem 4.2. A C_0-semigroup $(T(\zeta))_{\zeta\geq 0}$ is (λ,μ) hyperbolic iff

$\sigma(T(\zeta)) \cap \{\lambda \le |z| \le \mu\} = \varphi \ \forall\ \zeta > 0$. .

Proof: $(T(\zeta))_{\zeta\geq 0}$ is (λ,μ) hyperbolic iff $a^{\zeta}T(\zeta)$ is hyperbolic for every $a \in (\lambda,\mu)$ iff

$\sigma(T(\zeta)) \cap \left(a^{\zeta}e^{-\epsilon\zeta} \le |z| \le a^{\zeta}e^{\epsilon\zeta}\right) = \varphi$ for every $a \in [\lambda, \mu]$.

Since $(T(\zeta))_{\zeta\geq 0}$ is a hyperbolic semigroup if and only if T (1) is hyperbolic,

$\Rightarrow \sigma(T(\zeta)) \cap a\Gamma = \varphi$ for every $a \in (\lambda,\mu)$.

So, $\sigma(T(\zeta))$ cannot have any points in common with $\{\lambda \le |z| \le \mu\}$ as desired.

Theorem 4.3. If $\sigma(T(\zeta)) \subset a\Gamma e^{\zeta\sigma(A)} = \left\{aze^{\zeta\lambda} / \lambda \in \sigma(A), |z| = 1, a \in (\lambda,\mu)\right\}$

Then the following assertions are equivalent.

(a) $(T(\zeta))_{\zeta\geq 0}$ is (λ,μ) hyperbolic.

(b) $\sigma(T(\zeta)) \cap (\lambda < |z| < \mu) = \varphi \ \forall\ \zeta > 0$

(c) $\sigma(A) \cap (\log a + iR) = \varphi$

Proof: (a) $\Leftrightarrow$ (b) follows from Theorem 4.2

(b) $\Rightarrow$ (c):

By assumption, $\sigma(T(\zeta)) \cap a\Gamma = \varphi$ for every $a \in [\lambda, \mu]$ and $\forall\ \zeta > 0$.

Using spectral mapping theorem,

$e^{\zeta\sigma(A)} \cap a\Gamma = \varphi$ for every $a \in [\lambda,\mu]\ \forall \zeta > 0$.

Let $\lambda \in \sigma(A)$ so that $e^{\zeta\lambda} \notin a\Gamma\ \ \forall\ a \in [\lambda,\mu], \zeta > 0$

$\Rightarrow \lambda\zeta \neq \log a + i\theta$ for all $\zeta > 0$

and hence $\lambda \notin (\log a + iR)$ as desired.

(c) $\Rightarrow$ (b):

Observe that, if $\lambda \in \sigma(A)$ then $\lambda \notin (\log a + iR)$ for $\lambda \le a \le \mu$.

Then $\sigma(T(\zeta)) \subset a\Gamma e^{\zeta\sigma(A)} = \{aze^{\zeta\lambda}/\lambda \notin (\log a + iR), |z| = 1\}$

for all $\zeta > 0$ and $\lambda \leq a \leq \mu$ and which shows $\sigma(T(\zeta)) \cap a\Gamma = \varphi$ for $\lambda \leq a \leq \mu$.

5.Conclusion

In this work, the hyperbolic semigroup is characterized in terms of the spectrum of its co-generator. Also rescaled hyperbolic semigroup is developed to analyze its spectrum. By analyzing the spectrum, one can study the equality of spectral and growth bound for strongly continuous semigroup which has extensive applications in quantum theory and stochastic processes.

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