Total Domination Number of Kronecker Product of Wheel Graphs

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Abstract
The paper is related on the analytical study of total domination number of Kronecker product of Wheel Graph. Cockayne, Dawes and Hedetniemi introduced the total domination in graphs. In this paper, first we have defined the Kronecker product of wheel graph W_n for $n \geq 4$. Then, we will find the exact value of total domination number of Kronecker product of Wheel Graphs in generalize form. If G_1 and G_2 are two graphs with their vertex set $V_1 = \{u_1, u_2, u_3, \dots \dots u_n\}$ and $V_2 = \{v_1, v_2, v_3, \dots \dots v_n\}$ respectively then the Kronecker product of these two graphs is defined to be a graph $K(V_1 \times V_2)$ with its vertex set as $V_{K(V_1 \times V_2)}$ such that $V_{K(V_1 \times V_2)}$ is the Cartesian product of the sets V_1 and V_2. Where two vertices $(u_p, v_q), (u_r, v_s)$ in Kronecker product graph have an edge if and only if $u_p u_r$ and $v_q v_s$ are edges in graph V_1 and V_2 respectively.

Keyword: Graph, Kronecker Product, Total Global Domination Number, Wheel Graphs
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1. Introduction

Graph theory helps us to obtain the solution of many real life problems in engineering and technology, physical science, social science and biological science. In last century, theory of graphs has an unexpected growth in research field due to its extensive applications to inter-disciplinary domains. Many structures involving real-world situations can be conveniently represented on papers by means of a diagram consisting of a set of points together with lines or curves joining some or all pairs of these points. The general method of converting a real life problem into an abstract problem with a digram and then analyzing and evaluating its characteristics is known as graph theory [1]. It used in solution of many types of realistic problems arising in science, engineering and technology. A structural diagram graph $G =$

(V, E) which contains two sets one is set of vertices and another one is set of edges joining certain pairs of these vertices. Mathematically, we can write $G_1 = [V_1(G_1), E_1(G_1)]$, where, $V_1(G_1)$ and $E_1(G_1)$ are two finite sets defined as: $V_1(G_1)$ = set of vertices or objects of graph G_1 and $E_1(G_1)$ = set of edges of graph G_1 [1]. The notation of total dominating sets in graph theory is one of the important concepts which are very simple but it has lots of important applications. The mathematical study of theory of total domination process in graph theory was proposed by Cockayne, Dawes and Hedetniemi [2]. Then, the concept was revised by Brigham and Dutton [3] in generalized form as factor domination or global domination in graphs. The mathematical way of total domination in graph theory is a minimization process of operators or objects in any network which has more than one distinct spanning branch. The concept of Kronecker product of two graphs was introduced by Paul Weichsel [4]. S. Maheswari and S. Meenakshi defined the domination and split domination process for Kronecker product of some graphs [5]. In this paper, we generalized the concept of Kronecker product of wheel graph W_n for $n \geq 4$ and then derived the exact value of total domination number of Kronecker product of Wheel Graph in generalize form. P- Class problems mean a problem which can be solved on polynomial time that is this problem can be solved in time O (n^k), where k is constant. For an example all shorting and searching algorithms [6]. NP- class problems mean a problem which can be solved on polynomial time but it can be verified in polynomial time is known as non-deterministic polynomial or NP-class problem. Examples are Su-Du-Ku, graph colouring. NP- Hard problems mean a problem is known as NP-hard if each problem in NP can be converted polynomial or polynomial reducing to it [6]. Examples are travelling salesmen problem, optimization problem. There are lots of applications of total domination number of Kronecker product of two graphs. One of the important applications is that, if we want to find the minimum number of operators to run two system or two networks at a time such that the programs in each network are not run by its own operator then we will apply the application of total domination number of Kronecker product of two graphs. First we find the Kronecker product of both networks then total domination number of that Kronecker product gives the minimum number of operators to run two system or two network at a time such that the programs in each network is not run by its own operator.

2. Related Definitions and Results

2.1. Kronecker product of two graphs: If G_1 and G_2 are two graphs with their vertex set $V_1 = \{u_1, u_2, u_3, \dots \dots u_n\}$ and $V_2 = \{v_1, v_2, v_3, \dots \dots v_n\}$ respectively then the Kronecker product of these two graphs is defined to be a graph $K(V_1 \times V_2)$ with its vertex set as $V_{V_1 \times V_2}$ such that $V_{V_1 \times V_2}$ is the Cartesian product of the sets V_1 and V_2. Where two vertices $(u_p, v_q), (u_r, v_s)$ in Kronecker product graph have an edge if and only if $u_p u_r$ and $v_q v_s$ are edges in graph V_1 and V_2 respectively [1].

2.2. Dominating Set and Domination Number: A set of vertices$D \in V$ is called a dominating set of graph$G$ if each vertex in the complement of D in V is in the neighborhood of some vertex in set D and the dominating number $\gamma(G)$of any G is the least number of members in a dominating set of graph G [5].

2.3. Total Dominating Set and total Dominating Number: The total dominating set $D_0(G)$ of a graph G is a dominating set such that all the vertices of dominating set are not dominated by itself and the minimum cardinality of such set is called total domination number of graph G [5].

3. Wheel Graph W_4:

The graph W_4 contains 4 vertices with one central vertex as shown in figure 1. First we will find the Kronecker product of graph W_4 corresponding to its two factors F_1 and F_2-

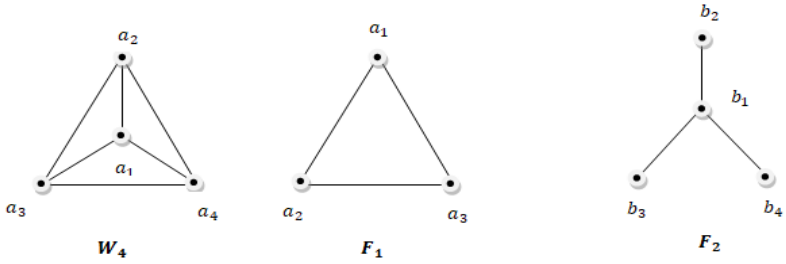

Figure 1: Wheel Graph W_4 and two factors F_1 and F_2

Here factor F_1contains three vertices and factor F_2 contains four vertices. Hence, the Kronecker product $K(W_4)$ of graph W_4 contains total 12 vertices such that the vertex set of Kronecker product is

$$V_{K(W_4)} = \begin{Bmatrix} (a_1,b_1),(a_1,b_2),(a_1,b_3),(a_1,b_4),(a_2,b_1),(a_2,b_2),(a_2,b_3),(a_2,b_4),(a_3,b_1), \\ (a_3,b_2),(a_3,b_3),(a_3,b_4) \end{Bmatrix}$$

and the two vertices in Kronecker product $K(W_4)$ have an edge if 1st tuple in both vertices have an edge in factor F_1 and 2nd tuple in both vertices have an edge in factor F_2. Therefore, the graph of Kronecker product $K(W_4)$ is as follows-

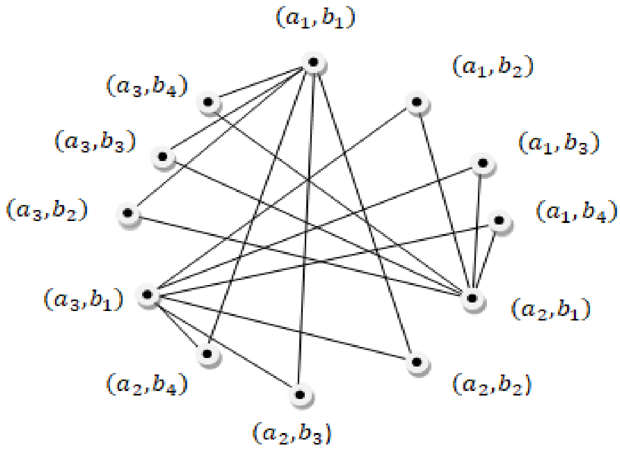

Figure 2: Kronecker Product of Wheel Graph W_4

Theorem 3.1 The total domination number of Kronecker of wheel graph W_4 is 5 and its total dominating set is $D'(W_4) = \{ (a_1,b_1),(a_1,b_2),(a_2,b_1),(a_2,b_2),(a_3,b_1) \}$.

Proof: We know that total dominating set $D_0(G)$ of a graph G is a dominating set such that all the vertices of dominating set are not dominated by itself and the minimum cardinality of such set is called total domination number of graph G. So, first we find the dominating set of Kronecker product graph of W_4. Then, we will find the dominating set such that all the vertices of dominating set are not dominates by itself. The minimum dominating set of Kronecker product graph of W_4 is $D(W_4) = \{(a_1, b_1), (a_2, b_1), (a_3, b_1)\}$. Now, if we add two more vertices (a_1, b_2) and (a_2, b_2) in the minimum dominating set then all the vertices of dominating set satisfy the condition of total dominating set. Hence, the set $D'(W_4) = \{ (a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2), (a_3, b_1) \}$ is the total dominating set of Kronecker product graph of W_4 and this is the minimal possible criteria for total dominating set of Kronecker product graph of W_4. Therefore, total domination number of Kronecker of wheel graph W_4 is 5 and its total dominating set is $D'(W_4) = \{ (a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2), (a_3, b_1) \}$.

4. Wheel Graph W_5:

The graph W_5 contains 5 vertices with one central vertex as shown in figure 3. First we will find the Kronecker product of graph W_5 corresponding to its two factors F_1 and F_2-

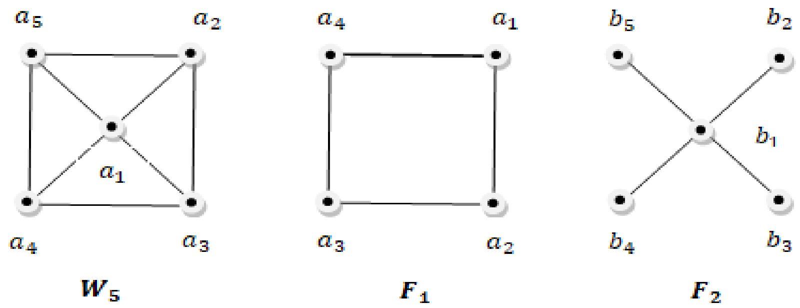

Figure 3: Wheel Graph W_5 and two factors F_1 and F_2

Here factor F_1 contains 4 vertices and factor F_2 contains 5 vertices. Hence, the Kronecker product $K(W_5)$ of graph W_5 contains total 20 vertices such that the vertex set of Kronecker product is $V_{K(W_5)} = \{(a_1, b_1), (a_1, b_2), (a_1, b_3), (a_1, b_4), (a_1, b_5)\ (a_2, b_1), (a_2, b_2), (a_2, b_3), (a_2, b_4), (a_2, b_5)\ (a_3, b_1), (a_3, b_2), (a_3, b_3), (a_3, b_4), (a_3, b_5), (a_4, b_1), (a_4, b_2), (a_4, b_3), (a_4, b_4), (a_4, b_5)\}$ and the two vertices in Kronecker product $K(W_5)$ of graph W_5 have an edge if 1st tuple in both vertices have an edge in factor F_1 and 2nd tuple in both vertices have an edge in factor F_2. Therefore, the graph of Kronecker product $K(W_5)$ is as follows-

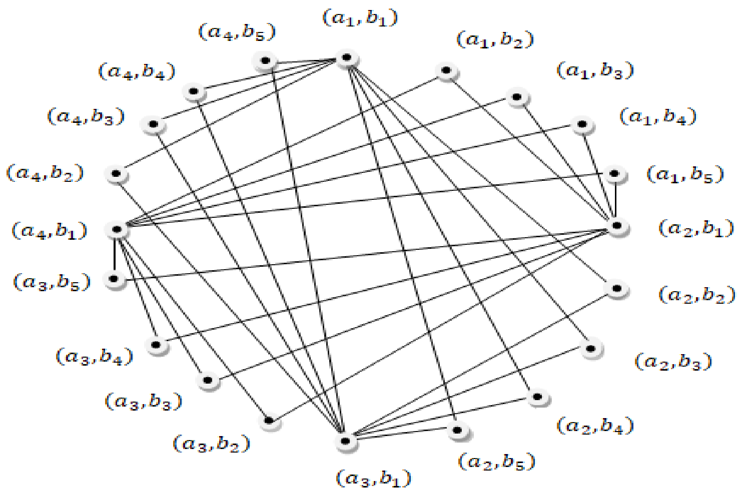

Figure 4: Kronecker Product of Wheel Graph W_5

Theorem 4.1 The total domination number of Kronecker of wheel graph W_5 is 6 and its total dominating set is $D'(W_5) = \{ (a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2), (a_3, b_1), (a_4, b_1)\}$.

Proof: We know that total dominating set $D_0(G)$ of a graph G is a dominating set such that all the vertices of dominating set are not dominated by itself and the minimum cardinality of such set is called total domination number of graph G. So, first we find the dominating set of Kronecker product graph of W_5. Then, we will find the dominating set such that all the vertices of dominating set are not dominates by itself. The minimum dominating set of Kronecker product graph of W_5 is $D(W_5) = \{(a_1, b_1), (a_2, b_1), (a_3, b_1), (a_4, b_1)\}$.

Now, if we add two more vertices (a_1, b_2) and (a_2, b_2) in the minimum dominating set then all the vertices of dominating set satisfy the condition of total dominating set. Hence, the set $D'(W_5) = \{ (a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2), (a_3, b_1), (a_4, b_1) \}$ is the total dominating set of Kronecker product graph of W_5 and this is the minimal possible criteria for total dominating set of Kronecker product graph of W_5. Therefore, total domination number of Kronecker of wheel graph W_5 is 6 and its total dominating set is

$D'(W_5) = \{ (a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2), (a_3, b_1), (a_4, b_1)\}$.

5. Wheel Graph W_6:

The graph W_6 contains 6 vertices with one central vertex as shown in figure 5. First we will find the Kronecker product of graph W_6 corresponding to its two factors F_1 and F_2-

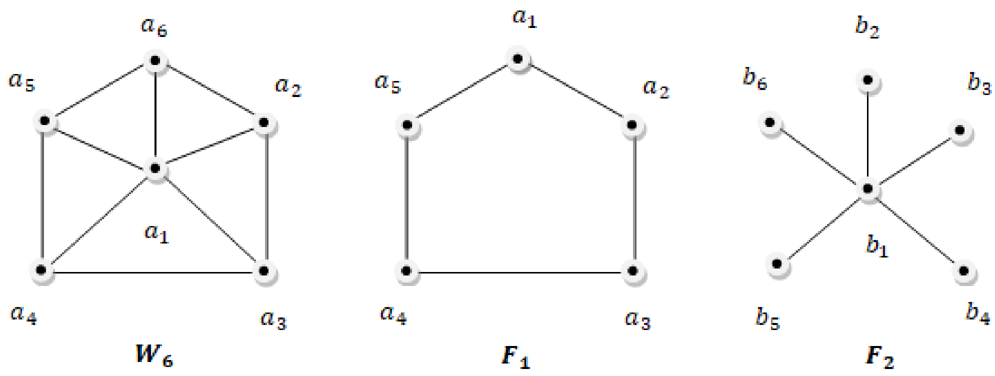

Figure 5: Wheel Graph W_6 and two factors F_1 and F_2

Here factor F_1 contains 5 vertices and factor F_2 contains 6 vertices. Hence, the Kronecker product $K(W_6)$ of graph W_6 contains total 30 vertices such that the vertex set of Kronecker product is

$V_{K(W_6)} = \{(a_1,b_1),(a_1,b_2),(a_1,b_3),(a_1,b_4),(a_1,b_5),(a_1,b_6),(a_2,b_1),(a_2,b_2),(a_2,b_3),$
$(a_2,b_4),(a_2,b_5),(a_2,b_6),(a_3,b_1),(a_3,b_2),(a_3,b_3),(a_3,b_4),(a_3,b_5),(a_3,b_6),(a_4,b_1),(a_4,b_2),(a_4,b_3),$
$(a_4,b_4),(a_4,b_5),(a_4,b_6),(a_5,b_1),(a_5,b_2),(a_5,b_3),(a_5,b_4),(a_5,b_5),(a_5,b_6)\}$

and the two vertices in Kronecker product $K(W_6)$ of graph W_6 have an edge if 1^{st} tuple in both vertices have an edge in factor F_1 and 2^{nd} tuple in both vertices have an edge in factor F_2. Therefore, the graph of Kronecker product $K(W_6)$ is as follows-

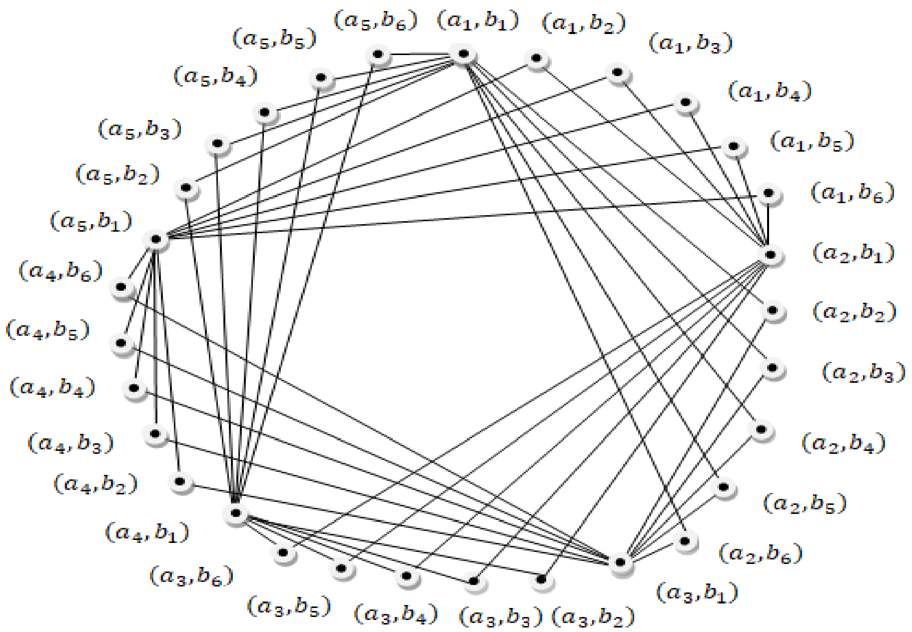

Figure 6: Kronecker Product of Wheel Graph W_6

Theorem 5.1 The total domination number of Kronecker of wheel graph W_6 is 8 and its total dominating set is

$$D'(W_6) = \{(a_1,b_1),(a_1,b_2),(a_2,b_1),(a_2,b_2),(a_3,b_1),(a_3,b_2),(a_4,b_1),(a_5,b_1)\}.$$

Proof: We know that total dominating set $D_0(G)$ of a graph G is a dominating set such that all the vertices of dominating set are not dominated by itself and the minimum cardinality of such set is called total domination number of graph G. So, first we find the dominating set of Kronecker product graph of W_6. Then, we will find the dominating set such that all the vertices of dominating set are not dominates by itself. The minimum dominating set of Kronecker product graph of W_5 is $D(W_6) = \{(a_1, b_1), (a_2, b_1), (a_3, b_1), (a_4, b_1), (a_5, b_1)\}$.
Now, if we add three more vertices $(a_1, b_2), (a_2, b_2)$ *and* (a_3, b_2) in the minimum dominating set then all the vertices of dominating set satisfy the condition of total dominating set. Hence, the set
$D'(W_6) = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2), (a_3, b_1), (a_3, b_2), (a_4, b_1), (a_5, b_1)\}$ is the total dominating set of Kronecker product graph of W_6 and this is the minimal possible criteria for total dominating set of Kronecker product graph of W_6. Therefore, total domination number of Kronecker of wheel graph W_6 is 8 and its total dominating set is $D'(W_6) = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2), (a_3, b_1), (a_3, b_2), (a_4, b_1), (a_5, b_1)\}$.

6. Wheel Graph W_n:

Let W_n be a wheel graph having n vertices with two factors F_1 and F_2 such that factor F_1 contains outer cycle of W_n graph with $(n-1)$ vertices and factor F_2 contains inner part of W_n graph with n vertices-

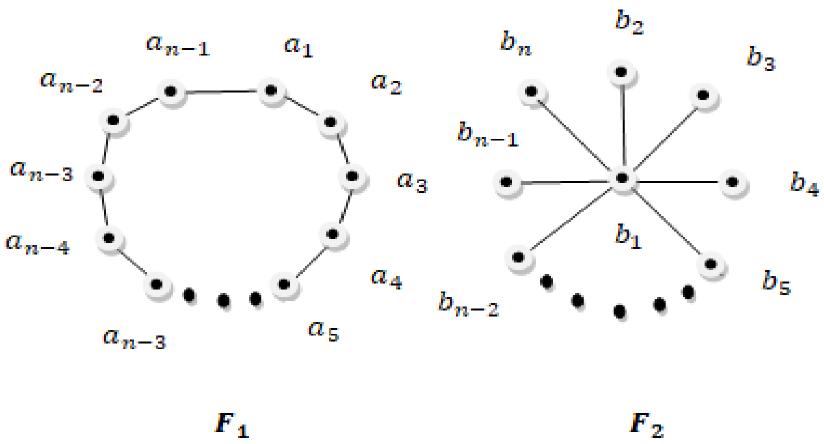

Figure 7: Factors F_1 and F_2 of Wheel Graph W_n

The Kronecker product of graph W_n with factor F_1 and F_2 is a graph $K(W_n)$ of order $n(n-1)$ and its vertex set is

$$V_{K(W_n)} = \{(a_1, b_1), (a_1, b_2), \dots (a_1, b_{n-1}), (a_1, b_n), (a_2, b_1), \dots (a_2, b_n) \dots \dots, (a_{n-1}, b_1), (a_{n-1}, b_2), \dots (a_{n-1}, b_{n-1}), (a_{n-1}, b_n)\}$$

and two vertices in Kronecker product $K(W_n)$ have an edge if 1st tuple in both vertices have an edge in factor F_1 and 2nd tuple in both vertices have an edge in factor F_2. Therefore, the graph of Kronecker product $K(W_n)$ is as follows-

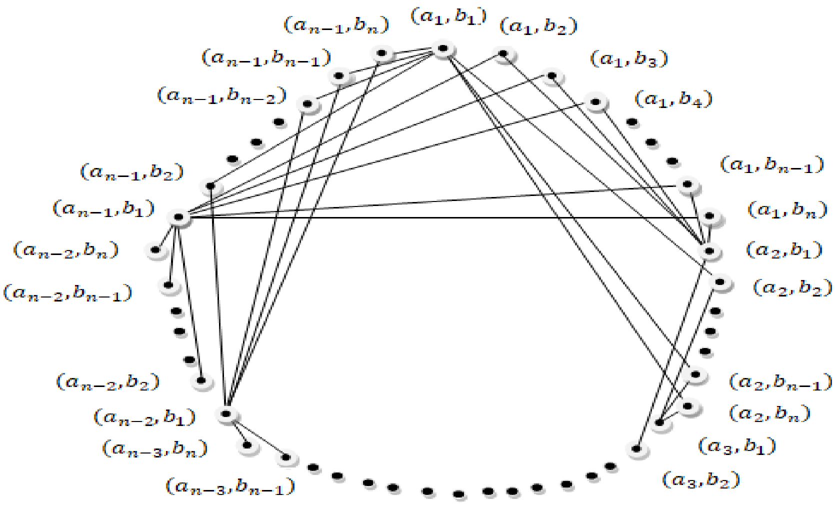

Figure 8: Kronecker Product $K(W_n)$ of Wheel Graph W_n

Theorem 6.1 The total domination number of Kronecker of wheel graph W_{n+1} is $n + \left\lceil \frac{n}{2} \right\rceil$ and its total dominating set is

$$D'(W_{n+1}) = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2), (a_3, b_1), (a_3, b_1), (a_4, b_1), (a_5, b_1), \dots \dots (a_{n-1}, b_1)\ (a_n, b_1)\}.$$

Proof: We know that total dominating set $D_0(G)$ of a graph G is a dominating set such that all the vertices of dominating set are not dominated by itself and the minimum cardinality of such set is called total domination number of graph G. So, first we find the dominating set of Kronecker product graph of W_{n+1}. Then, we will find the dominating set such that all the vertices of dominating set are not dominates by itself. The minimum dominating set of Kronecker product graph of W_n is

$$D(W_{n+1}) = \{(a_1, b_1), (a_2, b_1), (a_3, b_1), (a_4, b_1), \dots \dots (a_{n-1}, b_1), (a_n, b_1)\}.$$

Now, the members of set $D(W_{n+1})$ dominates all other vertices of Kronecker product graph of W_{n+1}. In total domination process all the vertices of dominating set are not dominated by itself and these n vertices in set $D(W_{n+1})$ are not adjacent to each other. Hence, for dominating these n vertices we required extra $\left\lceil \frac{n}{2} \right\rceil$ vertices. Therefore, if we add $\left\lceil \frac{n}{2} \right\rceil$ more vertices $(a_1, b_2), (a_2, b_2), (a_3, b_2) \dots \dots \left(a_{\left\lceil \frac{n}{2} \right\rceil}, b_2\right)$ in the minimum dominating set then all the vertices of dominating set satisfy the condition of total dominating set. Hence, the set

$$D'(W_{n+1}) = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2), (a_3, b_1), (a_3, b_2), \dots\ (a_{n-1}, b_1), (a_n, b_1)\}$$

is the total dominating set of Kronecker product graph of W_{n+1} and this is the minimal possible criteria for total dominating set of Kronecker product graph of W_{n+1}. Therefore, total domination number of Kronecker of wheel graph W_{n+1} is $n + \left\lceil \frac{n}{2} \right\rceil$ and its minimum total dominating set is

$$D'(W_{n+1}) = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2), (a_3, b_1), (a_3, b_1), (a_4, b_1), (a_5, b_1), \dots \dots (a_{n-1}, b_1)\ (a_n, b_1)\}.$$

7. Conclusion:

In this paper we have found the total domination number of Kronecker product of wheel graphs in generalized form. There are lots of applications of total domination number of Kronecker product of two graphs. One of the important applications is that, if we want to find the minimum number of operators to run two system or two networks at a time such that the programs in each network are not run by its own operator then we will apply the application of total domination number of Kronecker product of two graphs. First we find the Kronecker product of both networks then total domination number of that Kronecker product gives the minimum number of operators to run two system or two network at a time such that the programs in each network is not run by its own operator.

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