Research on Solutions for Algebraic and Transcendental Diophantine Equations

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Abstract - When it comes to number theory, Diophantine approximation is the field of number theory that deals with questions such as whether or not an integer is rational, irrational, or transcendental. On the other hand, one can ask how well an irrational number approximates a rational number, or how well algebraic numbers approximate a transcendental number. Two expressions are placed at the same value, One way to describe an algebraic equation is a mathematical statement. Variables, coefficients, and constants are the main components of an algebraic equation in most cases.

Keywords: The diophantine equation, polynomial equation, algebra.

1 Transcendental Equations – an Overview

Theoretical physics is replete with transcendental equations. There are numerous quantum mechanical and nanophysical systems whose energy eigenvalues may be determined via transcendental equations. Transcendental equations include, but are not limited to,

$$\zeta e^{\zeta} = p, \quad \frac{\sin \zeta}{\zeta} = \pm p, \quad \frac{\cos \zeta}{\zeta} = \pm p$$

Solving such equations might involve the use of trigonometric functions (tan, cot), hyperbolic functions (sin, cosh, tanh), or more sophisticated algebraic formulations. You may do this by finding a function that has an explicit form $\zeta\,(p)$, which is equal to finding each function $p = p\,(\zeta\,)$, defined as follows:

Solving such equations can be done in at least three different ways. Lagrange-Burmann inversion is the initial step. There are a few instances where this approach is convenient, such as when Equations or related transcendental equations have a simple derivative. A similar situation may be seen in first equation, where $W\,(p)$, i.e. $\zeta\,(p) = W\,(p)$ represents the Lambert function's solution to the problem at hand *ζ (p)*. The inversion theorem may be used to derive its series expansion.

1.1 Algebraic Equations

Two expressions are placed at the same value, One way to describe an algebraic equation is a mathematical statement. Variables, coefficients, and constants are the main components of an algebraic equation in most cases.

Equations, or the equal sign, represent equality. To "equate one quantity with another" is what equations are for.

$$ax^2 + bx + c = 0$$

Example:

$$5x^2 + 7x - 9 = 4x^2 + x - 18$$

$$5x^2 + 7x - 9 - 4x^2 - x + 18 = 0$$

$$\boxed{x^2 + 6x + 9 = 0}$$

Equations are akin to a scale of equality. There must be an identical quantity of power positioned on whichever surface of a balancing level in order for it to be called "balanced". With only one side of the scale being heavier, the scale will tip and no longer be evenly distributed between both. In the same way, equations follow a logical progression. Anything on one side of an equal sign has to match up perfectly with whatever is opposite it or else it becomes an inequality.

2 Illustration

Consider

$72A + 36 = 6 * (12A + 6) = 9 * (8A + 4)$ which implies

$(12A + 12)^2 + (8A - 5)^2 = (12A)^2 + (8A + 13)^2 = 208A^2 + 208A + 169$ which is a R_2 number.

Remarkable Observation

Let (x_0, y_0, z_0) be any solution of (2.31), then the following triple of integers based on x_0, y_0 and z_0 also satisfy (2.31).

1. **Triple 1:** $(x_0, 2z_0 - 2x_0 + y_0 - 2, -z_0 + 2x_0 + 2)$
2. **Triple 2:** $(x_0 + 2z_0 - 2y_0 - 6, y_0, -z_0 + 2y_0 + 6)$ $(x_0, 2z_0 - 2x_0 + y_0 - 2, -z_0 + 2x_0 + 2)$
3. **Triple 3:** $(y_0 + 2, x_0 - 2, z_0)$

3 Deals with Ternary Cubic Diophantine Equations

They are discussed in subsections **III.A.1** and **III.A.2**.

Trivariate cubic diophantine equation of the Non-trival distinct integer solutions

$ax^2 + by^2 = (a + b)z^3$

are presented in **III.A.1**.

The ternary cubic diophantine to the non-trival integral solutions of the problem in III.A.2

$$5(x^2 + y^2) - 9xy + x + y + 1 = 35z^3$$

are obtained.

In **III.B** cubic diophantine equations with four unknowns are considered. They are explained in subsections **III.B.1** and **III.B.2**.

III.B.1 involves the study of the cubic diophantine problem with four variables

$x^3 + y^3 = (z + w)^2 (z - w)$

Because of the non-zero integral answers it provides.

In **III.B.2**, The cubic diophantine problem has non-zero integral solutions

$$x^3 + y^3 = 14zw^2$$

When it is solved for its integrals

III.C deals with cubic diophantine equations with five unknowns. They are solved in three parts **III.C.1** to **III.C.3**.

In **III.C.1**, The cubic diophantine problem with five unknowns has non-rival different integral solutions.

$x^3 + y^3 = z^3 + w^3 + t^2 (x + y)$

have been explored in detail.

III.C.2 involves the study of the cubic diophantine problem with five variables

$x^3 + y^3 + u^3 + v^3 = 3t^3$

for its non-zero integer solutions.

The cubic diophantine equation has different integral solutions that are Non-trival.

$x^3 + y^3 + u^3 + v^3 = kt^3$

are obtained **III.C.3**.

In each of the sections above, a few notable correlations between the solutions, showed polygonal numbers, and pyramidal numbers.

4 With Four Unknowns Deals with Bi-Quadratic Diophantine Equations

They are discussed in subsections from **IV.A.1** to **IV.A.3**.

4.1 The Bi-Quadratic Diophantine Equation with four Unknowns the Non-Trival Integral Solutions

$x^4 - y^4 = (k^2 + 1)(z^2 - w^2)$

are attained.

The bi-quadratic diophantine equation has different integer solutions that are Non-trival

$$8(x^3 + y^3) = (1 + 3k^2)^n z^3 w$$

are presented in **IV.A.2**.

In **IV.A.3** the bi-quadratic equation

$(x + y)(x^3 + y^3) = 52z^2w^2$

Solved for its distinct integral solutions that do not equal 0.

In section **IV.B** bi-quadratic diophantine equations with five unknowns are considered. They are explained in two subsections **IV.B.1** and **IV.B.2**.

5 Five Unknowns in A Non-Homogenous Quintic Equation

$$(x - y)(x^3 + y^3) = 2(z^2 - w^2)T^3$$

The diophantine equation that represents the quintic equation using five unknowns is

$$(x - y)(x^3 + y^3) = 2(z^2 - w^2)T^3$$

Introduction of the linear transformations

$$x = u + v, y = u - v, z = 2u + v, w = 2u - v$$

in guides to

$$u^2 + 3v^2 = 4T^3$$

As a result, five separate solutions to the following equation may be obtained by solving it in five different ways.

Pattern 1

Let

$$T = a^2 + 3b^2$$

Write 4 as

$$4 = (1 + i\sqrt{3})(1 - i\sqrt{3})$$

If (5.3) is substituted for (5.4) and (5.5) and factorization is used, conclude

$$(u + i\sqrt{3}v) = (1 + i\sqrt{3})(a + i\sqrt{3}b)^3$$

Real and imaginary components are equal in (5.6), which gives us

$$u = a^3 - 9a^2 - 9ab^2 + 9b^3$$

$$v = a^3 + 3a^2b - 9ab^2 - 3b^3$$

Given and x, y, z and w corresponds to these values for each of the four variables.

$$\left.\begin{array}{c} x(a,b) = 2a^3 - 6a^2b - 18ab^2 + 6b^3 \\ y(a,b) = -12a^2b + 12b^3 \\ z(a,b) = 3a^3 - 15a^2b - 27ab^2 + 15b^3 \\ w(a,b) = a^3 - 21a^2b - 9ab^2 + 21b^3 \end{array}\right\}$$

6 With Four Unknowns Observations on The Non-Homogeneous Sextic Equation

$$x^4 - y^4 = 2^{2k+1}zT^5$$

There is an equation that has to be solved:

$$x^4 - y^4 = 2^{2k+1}zT^5$$

Introduction of the transformations

$$x = u + v, y = u - v, z = 4uv \ (u \neq v)$$

in leads to

$$u^2 + v^2 = 2^{2k}\,T^5$$

Take

$$T = a^2 + b^2$$

Write

$$2 = (1 + i)(1 - i)$$

To define use the factorization technique together

$$u + iv = (1 + i)^{2k}(a + ib)^5$$

$$= \sqrt{2}^{2k}\left(\cos\frac{k\pi}{2} + i\sin\frac{k\pi}{2}\right)(a + ib)^5$$

6.1 With Four Unknowns on the Observations Non-Homogenous Sextic Equation

$$(x-y)(x^2+y^2)=z(x^2-xy+y^2+7w^5)$$

There is an equation that has to be solved:

$$(x-y)(x^2+y^2)=z(x^2-xy+y^2+7w^5)$$

Introduction of the linear transformations

$$x=u+v, y=u-v, z=v$$

in leads to

$$v^2+3u^2=7w^5$$

7 Observations on the Transcendental Equation

$$\mathbf{5\sqrt{y^2+2x^2}-\sqrt[3]{X^2+Y^2}=(k^2+1)z^2}$$

With five unknowns to solve, the transcendental surd equation can yield nonzero integral solutions

$5\sqrt{y^2+2x^2}-\sqrt[3]{X^2+Y^2}=(k^2+1)z^2$

Begin by reversing some of the transformations

$x=2pq, y=2p^2-q^2, X=p(p^2+q^2)\,, Y=q(p^2+q^2)$

In leads to

$9p^2+4q^2=(k^2+1)z^2$

Equation may be solved in five distinct ways, resulting in five alternative sets of answers to the question:

Pattern 1

Take

$z=a^2+b^2$

Use (factorization to obtain

$$(3p+iq)=(k+i)(a+ib)^2$$

Real and imaginary portions of this equation may be equated to give us

$$3q=k(a^2-b^2)-2ab$$

$$2q=(a^2-b^2)+2kab$$

Understating a = 6A, and b = 6B in the equations above, one gets

$$p = 12(k(A^2 - B^2) - 2AB)$$

$$q = 18(A^2 - B^2) + 2kAB$$

This yields the non-zero distinct integral solutions of when p,q are substituted.

$$x(k,A,B) = 2f_1^2(k,A,B)f_2^2(k,A,B)$$

$$y(k,A,B) = 2f_1^2(k,A,B) - f_2^2(k,A,B)$$

$$X(k,A,B) = f_1(k,A,B)f_1^2(k,A,B) + f_2^2(k,A,B))$$

$$Y(k,A,B) = f_2(k,A,B)f_1^2(k,A,B) + f_2^2(k,A,B))$$

$$z(A,B) = 36(A^2 + B^2)$$

Where

$$f_1(k,A,B) = 12(k(A^2 - B^2) - 2AB)$$

$$f_2(k,A,B) = 18((A^2 - B^2) + 2kAB)$$

For k = 1, a few characteristics of the solutions are discussed.

1. $t_{3,A+1}\left(x(A,1) - 432\left(6F_{4,A,6} - 2CP_A^9 - 8Pr_A + 1\right)\right) = 9072P_A^3$
2. $-Y(A,A) + 101088OH_A^*\, CP_A^6 = 33696CP_A^6$
3. $9X(A,1)(4t_{3,A} - t_{4,A} - 1) = 2Y(A,1)(S_A - 3t_{4,A} - 4)$
4. $21X(2A,A) + 2Y(2A,A) = 0$
5. $x(A,A) + 1728\left(24F_{4,A,3} - 36P_A^3 + t_{4,A} + 6Pr_A\right) = 0$

Each of the numbers below is a nasty.

1. $\frac{3x(A,B)Y(A,B)}{X(A,B)}, \frac{6y(A,B)X^2(A,B)}{2X^2(A,B)-Y^2(A,B)}, \frac{6y(A,B)Y^2(A,B)}{2X^2(A,B)-Y^2(A,B)}$
2. $66(y(A,A) - x(A,A))$
3. $3(432z^2(A,B) - 3x(A,B))$

Pattern 2

Equation can be written as

$$9p^2 + 4q^2 = (k^2 + 1)z^2.1$$

Write 1 as

$$1 = \frac{\left((m^2-n^2)+2mni\right)((m^2-n^2)-2mni)}{(m^2+n^2)^2}$$

substituted in and factorization is used to define

$$(3p+i2q)=\frac{1}{m^2+n^2}((k+i)(m^2-n^2+2mni)(a+ib)^2)$$

Real and imaginary portions of this equation may be equated to give us

$$3p=\frac{1}{(m^2+n^2)}(k(m^2-n^2(a^2-b^2)-4mnab$$

$$-2mn(a^2-b^2)-2ab(m^2-n^2))$$

$$2q=\frac{1}{(m^2+n^2)}(k(m^2-n^2)(a^2-b^2)-4mnab$$

$$-2k(mn(a^2-b^2)-2ab(m^2-n^2)))$$

Taking a=6$(m^2+n^2)A, b=6(m^2+n^2)B$in (7.7), (7.8) and (7.4), the values of p, q, z are known by

$$p=12(m^2+n^2)(kF_1(A,B,m,n)-2F_2(A,B,m,n))$$

$$q=18(m^2+n^2)(F_1(A,B,m,n)+2kF_2(A,B,m,n))$$

$$z=36(m^2+n^2)^2(A^2+B^2)$$

Where

$$F_1(A,B,m,n)=(m^2-n^2)(A^{2-}B^2)-4mnAB$$

$$F_1(A,B,m,n)=(m^2-n^2)(A^{2-}B^2)-4mnAB$$

$$F_2(A,B,m,n=mn(A^2-B^2)+AB(m^2-n^2)$$

The non-zero distinct integral solutions of may be found by substituting p, q

$$x(A,B,m,n)=432(m^2+n^2)^2(kF_1(A,B,m,n)-2F_2(A,B,m,n))$$

$$F_1(A,B,m,n)+2kF_2(A,B,m,n))$$

$$y(A,B,m,n)=(m^2+n^2)^2[288(kF_1(A,B,m,n)-2F_2(A,B,m,n))^2$$

$$-324(F_1(A,B,m,n)+2kF_2(A,B,m,n)]^2$$

$$X(A,B,m,n)=12\big(m^2+n^2)^3kF_1(A,B,m,n)-2F_2(A,B,m,n)\big)[144(kF_1(A,B,m,n)$$

$$-2F_2(A,B,m,n))^2+324(F_1(A,B,m,n)+2kF_2(A,B,m,n))^2$$

$$Y(A,B,m,n=18(m^2+n^2)^3(F_1(A,B,m,n)+2kF_2(A,B,m,n))[144(kF_1(A,B,m,n)$$

$$-2F_2(A,B,m,n))^2+324(F_1(A,B,m,n)+2kF_2(A,B,m,n))^2$$

$$z(A,B,m,n)=36(m^2+n^2)(A^2+B^2)$$

Properties

1. $\frac{x(A,1,3,2)}{73008} + 1428F_{4,A,4} - 672p_A^5 - 570t_{3,4} \equiv 289(\text{mod}429)$
2. $3y(2,B,3,1) + 112233600F_{4,B,6} + 326707200 = 43200\left(4502P_B^5 + 4399t_{4,B}\right) + 27417600S_A$
3. $X(A,1,2,1) + y(A,1,2,1) + 228285000 = 135000(1217520F_{6,A,3} - 3470760F_{5,A,3} + 866940F_{4,A,6} + 199900CP_A^6 + 487674Pr_A + 207167t_{4,A}$
4. $84z(A^2,1,2,1) + x(A,1,2,1) = -64800(96P_{A-1}^3 - 7t_{4,A}$
5. $x(1,1,m+1,1) = 1728(2Pr_m + 1)^2(4F_{4,m,6} - 8P_m^5 - 8t_{3,m} - 1)$
6. $\mathrm{X(A(A+1),A,2m,m)}\left(-11\mathrm{CP_A^6} - 22\mathrm{t_{4,A}t_{3,A}}\right) - 10\mathrm{Y(A(A+1),A,2,m,m)(F_{4,A,6}} - \mathrm{t_{4,A}} = 12\mathrm{P_4^5(3Y(A(A+1),A,2m,m) - X(A(A+1),A,2m,m))}$

Pattern 3

In write 1 as

$1 = (i)^n(-i)^n$

Define (7.9) as the result of replacing (7.5) with factorizing

$$3p + i2q) = i^n(k(a^2 - b^2) - 2ab) + i(2kab + a^2 - b^2)$$

In the preceding equation, if we take a = 6A and b = 6B, we obtain a = 6B.

$$p = 12\left((k(A^2 - B^2) - 2AB)cos\frac{n\pi}{2} - (2kAB + A^2 - B^2)sin\frac{n\pi}{2}\right)$$

$$q = 18\left((k(A^2 - B^2) - 2AB)sin\frac{n\pi}{2} + (2kAB + A^2 - B^2)sin\frac{n\pi}{2}\right)$$

The non-zero distinct integral solutions of (7.1) may be obtained by substituting p, q

$$x(k,n,A,B) = 432\left(f_3cos\frac{n\pi}{2} - f_4sin\frac{n\pi}{2}\right)\left(f_3sin\frac{n\pi}{2} + f_4cos\frac{n\pi}{2}\right)$$

$$y(k,n,A,B) = 288\left(f_3cos\frac{n\pi}{2} - f_4sin\frac{n\pi}{2}\right)^2 - 324\left(f_3sin\frac{n\pi}{2} + f_4cos\frac{n\pi}{2}\right)^2$$

$$X(k,n,A,B) = 12\left(f_3cos\frac{n\pi}{2} - f_4sin\frac{n\pi}{2}\right)^2\left[144\left(f_3cos\frac{n\pi}{2} - f_4sin\frac{n\pi}{2}\right)^2 + 324\left(f_3sin\frac{n\pi}{2} + f_4cos\frac{n\pi}{2}\right)^2\right]$$

$$Y(k,n,A,B) = 18\left(f_3sin\frac{n\pi}{2} + f_4cos\frac{n\pi}{2}\right)^2\left[144\left(f_3cos\frac{n\pi}{2} - f_4sin\frac{n\pi}{2}\right)^2 + 324\left(f_3sin\frac{n\pi}{2} + f_4cos\frac{n\pi}{2}\right)^2\right]$$

$$z(A,B) = 36(A^2 + B^2)$$

Where

$$f_3 = f_3(k, A, B) = k(A^2 - B^2) - 2AB$$

$$f_4 = f_4(k, A, B) = 2kAB + A^2 - B^2$$

Properties

1. $x(3,1,A,2) + 2592\left(t_{3,A^2} + CP_A^{30}\right) + 5184P_{A-1}^3 + 11664t_{6,A} + 20736 = 1296\left(25Pr_A + 18t_{4,A}\right)$
2. $\mathrm{y}(3,1,\mathrm{A},1) + 2628\mathrm{t}_{4,\mathrm{A}}^2 - 44064\mathrm{P}_{\mathrm{A}-1}^3 \equiv -2628(\mathrm{mod}\ 14328)$
3. $x(k,1,A,2A) = 432(3k+4)(4k-3)(6F_{4,A,6} - 2CP_A^9 - Pr_A - t_{4,A}$
4. $437\left(\mathrm{Y}^2(1,1,\mathrm{A},2,\mathrm{A}) - \mathrm{X}^2(1,1,\mathrm{A},2\mathrm{A})\right)$ is a perfect square.

Pattern 4

Write) as

$$9p^2 - k^2z^2 = z^2 - 4q^2$$

Which can be written as?

$$\frac{3p + kz}{z + 2q} = \frac{z - 2q}{3p - kz} = \frac{a}{b}, b \neq 0$$

A double equation system would be equivalent to the above.

$$3pb - 2qa + (kb - a)z = 0$$

$$-3pa - 2qb + (ka + b)z = 0$$

The values of p, q, and z may be found by using the cross-multiplication approach. When p and q are substituted, the non-zero integral solutions are shown to be

$$x(k, a, b) = 2g_1(k, a, b)g_2(k, a, b)$$

$$y(k, a, b) = 2g_1^2(k, a, b,) - g_2^2(k, a, b)$$

$$X(k, a, b) = g_1(k, a, b)(g_1^2(k, a, b) + g_2^2(k, a, b))$$

$$Y(k, a, b) = g_2(k, a, b)(g_1^2(k, a, b) + g_2^2(k, a, b))$$

$$z(a, b) = 6(a^2 + b^2)$$

Where

$$g_1(a, b) = 4ab - 2k(b^2 - a^2)$$

$$g_2(a, b) = 6kab + 3(b^2 - a^2)$$

Pattern 5

Equation can be written as

$$9p^2 - z^2 = k^2z^2 - 4q^2$$

Which can be written as?

$$\frac{3p+z}{kz-2q} = \frac{kz+2q}{3p-z} = \frac{a}{b}, b \neq 0$$

have non-zero unique integral solutions if we follow the pattern 4:

$$x(k,a,b) = 2h_1(k,a,b)h_2(k,a,b)$$

$$y(k,a,b) = 2h_1^2(k,a,b) - h_2^2(k,a,b)$$

$$X(k,a,b) = h_1(k,a,b)(h_1^2(k,a,b) + h_2^2(k,a,b))$$

$$Y(k,a,b) = h_2(k,a,b)(h_1^2(k,a,b) + h_2^2(k,a,b))$$

$$z(a,b) = 6(a^2 + b^2)$$

Where

$$h_1(a,b) = 2(a^2 - b^2) + 4kab$$

$$h_2(a,b) = 6ab - 3k(a^2 - b^2)$$

7.1 Integral Solutions of the Surd Equation $\sqrt[3]{x^2 - y^2} + \sqrt[3]{X^2 + Y^2} + 2\sqrt[3]{z^2 + w^2} = 6p^2$

With seven unknown's integral solutions is the transcendental surd equation to be solved for getting nonzero

$\sqrt[3]{x^2 - y^2} + \sqrt[3]{X^2 + Y^2} + 2\sqrt[3]{z^2 + w^2} = 6p^2$

Begin by reversing some of the transformations

$x = m(m^2 - n^2), y = n(m^2 - n^2), X = m(m^2 + n^2), Y = n(m^2 + n^2),$

$\mathrm{z = m^3 - 3mn^2, w = nm^2 - n^3}$

In leads to

$n^2 + 2m^2 = 3p^2$

Five alternative approaches are used to solve the given problem, resulting in five separate sets of solutions.

Pattern 1

In take

$$p = a^2 + 2b^2$$

Applying the method of factorization, define (n+im$\sqrt{2} = \left(1+i\sqrt{2}\right)(a+ib\sqrt{2})^2$

Real and imaginary portions of this equation may be equated to give us

$$n = a^2 - 2b^2 - 4ab$$

$$m = a^2 - 2b^2 + 2ab$$

Substituting m, n, it offers the unique integral solutions of that are non-zero

$$x = f_1(a,b)(f_1^2(a.b) - g_1^2(a,b))$$

$$y = g_1(a,b)(f_1^2(a.b) - g_1^2(a,b))$$

$$X = f_1(a,b)(f_1^2(a.b) + g_1^2(a,b))$$

$$Y = g_1(a,b)(f_1^2(a.b) + g_1^2(a,b))$$

$$z = f_1(a,b)(f_1^2(a.b) + 3g_1^2(a,b))$$

$$w = g_1(a,b)(3f_1^2(a.b) - g_1^2(a,b))$$

$$p = a^2 + 2b^2$$

Where

$f_1(a,b) = a^2 - 2b^2 - 4ab$

$g_1(a,b) = a^2 - 2b^2 + 2ab$

Properties

1. $x(a,1) = 24(P_a^5\left(t_{4,a} - 6\right) + 4t_{3,a}$
2. $x(a,1) - y(a,1) - 1728Pt_a + 3027P_a^3 - 576Pr_a = 0$
3. $x(a,1) + y(a,1) = 12(240F_{5,a,3} - 144F_{4,a,6} - 8P_a^5 - 80t_{3,a})$
4. $X(2^n,1) = 2(j_{6n} + 60J_{3n} - 29) if\ n\ is\ odd = 2(j_{6n} + 60J_{3n} + 11) if\ n\ is\ even$
5. $Y(1,b) + z(1,b) = 12(480F_{5,b,3} - 576F_{4,b,6} + 68P_6^5 + 2t_{3,b} + 9t_{4,b}$
6. following of the Each is a nasty number

(i) $6(X^2 - x^2)$

(ii) $6(Y^2 - y^2)$

(iii) $6(x(a,a) + y(a,a) + X(a,a) - z(a,a)$

(iv) $12(x(a,a) - z(a,a))$

(v) $12(X(a,a) - x(a,a))$

Pattern 2

can be written as

$n^2 = 3p^2 - 2m^2$

Substitution of the linear transformations

$p = S - 2T, m = S - 3T$

In leads to

$n^2 + 6T^2 = S^2$

Whose solution is given by

$$T = 2ab, n = 6b^2 - a^2, S = 6b^2 + a^2$$

$$T = 2ab, n = 6b^2 - a^2, s = 6b^2 + a^2$$

As a result of substituting S, T, the values of both m p are as follows:

$$m = 6b^2 + a^2 - 6ab$$

$$p = 6b^2 + a^2 - 6ab$$

Substituting m, n, the non-zero distinct integral solutions of are given by

$$x = f_2(a,b)(f_2^2(a.b) - g_2^2(a,b))$$

$$y = g_2(a,b)(f_2^2(a.b) - g_2^2(a,b))$$

$$X = f_2(a,b)\left(f_2^2(a.b) + g_2^2(a,b)\right)$$

$$Y = g_2(a,b)(f_2^2(a.b) + g_2^2(a,b))$$

$$z = f_2(a,b)(f_2^2(a.b) - 3g_2^2(a,b))$$

$$w = g_2(a,b)(3f_2^2(a.b) - g_2^2(a,b))$$

$$p = 6b^2 + a^2 - 4ab$$

$$f_2(a,b) = 6b^2 + a^2 - 6ab$$

Where

$$g_2(a,b) = 6b^2 - a^2$$

$$f_2(a,b) = 6b^2 + a^2 - 6ab$$

Pattern 3

can be written as

$n^2 + 6T^2 = s^2 * 1$

Write 1 as

$$1 = \frac{\left(1 + i2\sqrt{6}\right)(1 - i2\sqrt{6}}{25}$$

Take

$S = a^2 + 6b^2$

Using the method of factorization, define

$$\left(n + i\sqrt{6}T\right) = (a + i\sqrt{6}b)^2 \frac{(1 + i2\sqrt{6})^2}{5}$$

Once real and imaginary portions of the equation are combined, one gets the following:

$n = \frac{a^2-6b^2-24ab}{5}$

$T = \frac{2a^2-12b^2+2ab}{5}$

Choices a = 5A, b = 5B in the values of S, n, T become

$$S = 25(A^2 + 6B^2)$$

$$n = 5(A^2 - 6B^2 - 24AB)$$

$$T = 5(2A^2 - 12B^2 + 2AB)$$

Substituting S, T and simplifying, the values of m and p are given by

$$m = 5(-A^2 + 66B^2 - 6AB)$$

$$p = 5(A^2 + 54B^2 - 4AB)$$

Unique non-zero integral solutions are obtained by substituting (m, n) into the equation

$$x = f_3(A,B)(f_3^2(A,B) - g_3^2(A,B))$$

$$y = g_3(A,B)(f_3^2(A,B) - g_3^2(A,B))$$

$$X = f_3(A,B)(f_3^2(A,B) + g_3^2(A,B))$$

$$Y = g_3(A,B)(f_3^2(A,B) + g_3^2(A,B))$$

$$z = f_3(A,B)(f_3^2(A,B) - 3g_3^2(A,B))$$

$$w = g_3(A,B)(3f_3^2(A,B) - g_3^2(A,B))$$

$= 5(A^2 + 54B^2 - 4AB)$

Where

$$f_3(A,B) = 5(-A^2 + 66B^2 - 6AB)$$

$$g_3(A,B) = 5(A^2 - 6B^2 - 24AB)$$

Pattern 4

In write 1 as

$$1=\frac{\left(5+i2\sqrt{6}\right)(5-i2\sqrt{6}}{49}$$

Pattern 3 can be followed in order to get the non-zero unique integral solutions to

$$x=f_4(A,B)(f_4^2(A,B)-g_4^2(A,B))$$

$$y=g_4(A,B)(f_4^2(A,B)-g_4^2(A,B))$$

$$X=f_4(A,B)(f_4^2(A,B)+g_4^2(A,B))$$

$$Y=g_4(A,B)(f_4^2(A,B)+g_4^2(A,B))$$

$$z=f_4(A,B)(f_4^2(A,B)-3g_4^2(A,B))$$

$$w=g_4(A,B)(3f_4^2(A,B)-g_4^2(A,B))$$

$$p=7(3A^2+66B^2-20AB)$$

Where

$$g_4(A,B)=7(5A^2-30B^2-24AB)$$

$$f_4(A,B)=7(A^2+78B^2-80AB$$

Pattern 5

can be written as

$$3p^2-n^2=2m^2$$

Take

$$=3a^2-b^2$$

Using the method of factorization, define

$$\sqrt{3}p+n=\left(\sqrt{3}+1\right)(\sqrt{3}a+b)^2$$

Equating rational and irrational parts of the above equation, we get

$p=3a^2+b^2+2ab$

$n=3a^2+b^2+6ab$

non-zero distinct integral solutions are obtained by substituting m, n.

$$x=f_5(a,b)(f_5^2(a,b)-g_5^2(a,b))$$

$$y=g_5(a,b)f_5^2(a,b)-g_5^2(a,b))$$

$$x=f_5(a,b)(f_5^2(a,b)+g_5^2(a,b))$$

$$Y=g_5(a,b)(f_5^2(a,b)-g_5^2(a,b))$$

$$z = f_5(a,b)(f_5^2(a,b) - 3g_5^2(a,b))$$

$$w = g_5(a,b)(3f_5^2(a,b) - g_5^2(a,b))$$

$$p = 3a^2 + b^2 + 2ab$$

Where

$$f_5(a,b) = 3a^2 - b^2$$

$$g_5(a,b) = 3a^2 - b^2$$

Remarkable Observation

Let (m_0, n_0, p_0) be the initial solution. Keeping n_0 fixed, the infinitely many values of n, p satisfying are given by

$$n_s = n_0, m_s = \frac{1}{2\sqrt{6}}\left(\sqrt{6}A_s m_0 + 3B_s p_0\right), p_s = \frac{1}{2\sqrt{6}}(2B_s m_0 + \sqrt{6}A_s p_0)$$

In which

$A_s = (5 + 2\sqrt{6})^s + (5 - 2\sqrt{6})^s$

$B_s = 5 + 2\sqrt{6})^S - (5 - 2\sqrt{6})^S$

The non-zero discrete integral solutions of are represented by.

$$x_s = m_s(m_s^2 - n_0^2), y_s = n_0(m_s^2 - n_0^2), X_s = m_s(m_s^2 + n_0^2), Y_s = n_0(m_s^2 + n_0^2)$$

$$z_s = m_s(m_s^2 - mn_0^2), w_s = n_0(3m_s^2 - n_0^2), P_s = \frac{1}{2\sqrt{6}}(2B_s m_0 + \sqrt{6}A_s p_0)$$

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