A Note on Square Free Detour Distance in Graphs

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Abstract

In this paper, we investigate the results on square free detour number of a simple, connected graph $G = (V, E)$ of order $n \geq 2$. It is proved that for any two vertices u and v in a connected graph G, $0 \leq d(u,v) \leq d_m(u,v) \leq D_{\square f}(u,v) \leq D(u,v) \leq n - 1$. The relationship between radious and diameter of various distance concepts is discussed. It is also shown that for each pair a, b of positive integers with $3 \leq a \leq b$, there exists a connected graph G with $rad_{\square f}(G) = a$ and $diam_{\square f}(G) = b$.

Keywords: distance; detour distance; triangle free detour distance; square free detour distance.

1 Introduction

For any vertices u and v in a finite undirected connected simple graph $G = (V, E)$, the distance $d(u, v)$ is the length of the shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called a $u - v$ geodesic. For a vertex v in a connected graph G, the eccentricity $e(v)$ of v is the distance between v and a vertex farthest from v in G. The minimum eccentricity among the vertices of G is its radius and the maximum eccentricity is its diameter, which are denoted by $rad(G)$ and $diam(G)$ respectively. Two vertices u and v of G are antipodal if $d(u, v) = diam(G)$. This geodesic concept was studied and extended to detour distance by Chartrand et. al. [2-5]. For two vertices u and v in a connected graph G, the detour distance $D(u, v)$ from u to v is defined as the length of a longest $u - v$ path in G. A $u - v$ path of length $D(u, v)$ is called a $u - v$ detour. The detour eccentricity $e_D(v)$ of v is the detour distance between the vertex v and a vertex farthest from v in G. The minimum detour eccentricity among the vertices of G is the detour radius $rad_D(G)$ of G and the maximum detour eccentricity is its detour diameter $diam_D(G)$ of G. This detour concept was further studied by Santhakumaran et. al. [11] For two vertices u and v in a connected graph G, a longest $u - v$ chordless path is called a $u - v$ detour monophonic. This detour monophonic distance was studied by Titus et. al. [10,11]. Further, the triangle free detour distance was introduced and studied by Keerthi Asir, Sethu Ramalingam and Athisayanathan [7-9]. The triangle free detour eccentricity $e_{\Delta f}(u)$ of a vertex u in G is the maximum triangle free detour distance from u to a vertex of G. The square free detour radius, $R_{\Delta f}$ of G is the minimum square free detour eccentricity among the vertices of G, while the triangle free detour diameter, $D_{\Delta f}$ of G is the maximum triangle free detour eccentricity among the vertices of G. In this paper, a similar concept of square free detour distance is introduced and investigated. For basic terminology refer to [1,6].

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2 Square free detour distance in a graph

Definition 2.1 Let G be a connected graph and u, v any two vertices in G. A $u - v$ path P is said to be a $u - v$ square free path if no three vertices of P induce a square, cycle C_4 in G. The square free detour distance $D_{\square f}(u, v)$, is the length of a longest $u - v$ square free path in G. A $u - v$ square free path of length $D_{\square f}(u, v)$, is called the $u - v$ square free detour.

Example 2.2 Consider the graph G given in Figure 2.1, the $v_3 - v_7$ path P: v_3, v_1, v_2, v_4, v_7 is a $v_3 - v_7$

square free detour path while the $v_3 - v_7$ paths P': v_3, v_5, v_6, v_7, P'': $v_3, v_1, v_2, v_5, v_6, v_7$ and P''': $v_3, v_5, v_1, v_2, v_4, v_7$ are not $v_3 - v_7$ square free detour paths. Here, $D_{\square f}(v_3, v_7) = 4$, $d(v_3, v_7) = 3$ and $D(v_3, v_7) = 5$. Thus the square free detour distance is different from the usual distance and detour distance in G. Also, P' is a $v_3 - v_7$ geodetic, P'' and P''' are the $v_3 - v_7$ detours and P is a $v_3 - v_7$ square free detour. Clearly, $v_3 - v_7$ geodesic, $v_3 - v_7$ square free detour and $v_3 - v_7$ detour are distinct.

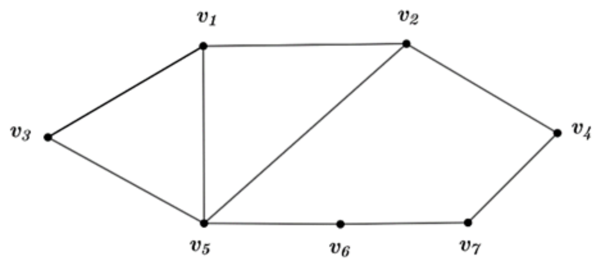

Figure 2.1: G

Remark 2.3 Though the usual distance d and the detour distance D are metrics on the vertex set of a connected graph, the square free detour distance $D_{\square f}$ is also not a metric on the vertex set of a connected graph. For the graph G given in Figure 2.2, $D_{\square f}(v_2, v_4) < D_{\square f}(v_2, v_3) + D_{\square f}(v_3, v_4)$. However for any two vertices u and v in a square free connected graph G, $D(u, v) = D_{\square f}(u, v)$. Therefore, the square free detour distance $D_{\square f}$ is a metric only on the vertex set of a square free connected graph.

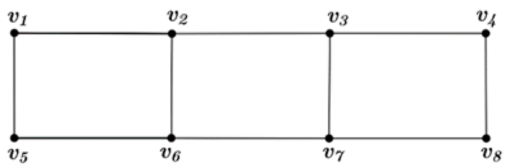

Figure 2.2: G

Titus et. al. [11] proved that $0 \leq d(u, v) \leq d_m(u, v) \leq D(u, v) \leq n - 1$ for any two vertices u and v in G, which yields the following theorem.

Theorem 2.4 For any two vertices u and v in a connected graph G, $0 \leq d(u, v) \leq d_m(u, v) \leq D_{\square f}(u, v) \leq D(u, v) \leq n - 1$.

Proof. It is enough to show that $d_m(u,v) \leq D_{\square f}(u,v)$ and $D_{\square f}(u,v) \leq D(u,v)$. Let P be any longest u − v path in G. Suppose that P does not contain a chord in G, then $d_m(u,v) = D_{\square f}(u,v) = D(u,v)$. Suppose that P contains a chord.

Case 1. If P does not induce a square in G, then $d(u,v) = d_m(u,v) = D_{\square f}(u,v)$. $D(u,v)$.

Case 2. If P induces a square in G, then $d_m(u,v) = D_{\square f}(u,v) < D(u,v)$.

Remark 2.5 The bounds in the Theorem 2.3 are sharp. If u = v, then $d(u,v) = d_m(u,v) = D_{\square f}(u,v) = 0$. Also, that if G is a path whose end vertices are u and v, then $d(u,v) = d_m(u,v) = D_{\square f}(u,v) = D(u,v)$. For the graph G given in Figure 2.1, $d(u,v) = d_m(u,v) = D_{\square f}(u,v) = D(u,v)$.

Theorem 2.6 For any two vertices u and v in a connected graph G, $(u,v) = d_m(u,v) = D_{\square f}(u,v) = D(u,v)$ if and only if G is a tree.

Proof. Let G be a connected graph and u, v any two vertices in G. Assume that G is a tree, then there is a unique square free path between u and v, so that $d(u,v) = d_m(u,v) = D_{\square f}(u,v) = D(u,v)$.

Conversely, consider that that $d(u,v) = d_m(u,v) = D_{\square f}(u,v) = D(u,v)$ for any two vertices u and v in G. To prove that G is a tree, it is enough to prove that G is acyclic. Suppose that G is cyclic. Then there exists atleast two vertices x and y in G such that the path P between x and y contains a cycle in G.

Case 1. Let P contain a cycle of length 4. Then $d(x,y) = d_m(x,y) = D_{\square f}(x,y) = D(x,y)$, which leads to a contradiction.

Case 2. Let P contain a cycle of length greater than 4. Then $d(x,y) = d_m(x,y) = D_{\square f}(x,y) = D(x,y)$, which is a contradiction.

Definition 2.7 The square free detour eccentricity of a vertex v in a connected graph G is defined by $e_{\square f}(v) = \max\{D_{\square f}(u,v) \mid u \in V\}$. The square free detour radius of G is defined by $rad_{\square f}(G) = \min\{e_{\square f}(v) \mid v \in V\}$ and the square free detour diameter of G is defined by $diam_{\square f}(G) = \max\{e_{\square f}(v) \mid v \in V\}$.

The following theorem is a consequence of Theorem 2.4. and Definition 2.7.

Theorem 2.8 Let G be a connected graph. Then

(i) $rad(G) \leq rad_m(G) \leq rad_{\square f}(G) \leq rad_D(G)$.

(ii) $diam(G) \leq diam_m(G) \leq diam_{\square f}(G) \leq diam_D(G)$

Now we have a realization theorem for the square free detour radius and the square free detour diameter of some connected graph.

Theorem 2.9 For each pair a, b of positive integers with $3 \leq a \leq b$, there exists a connected graph G with $rad_{\square f}(G) = a$ and $diam_{\square f}(G) = b$.

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Proof.

Case 1. a = b. Let G = C_{a+1} : $u_1, u_2, \ldots, u_{a+1}, u_1$ be a cycle of order a + 1 . It is easy to verify that every vertex x in G with $e_{\square f}(x) = a$. Thus $rad_{\square f}(G) = a$ and $diam_{\square f}(G) = b$ as a = b.

Case 2. $3 \leq a < b \leq 2a$. Let C_{a+1} : $u_1, u_2, \ldots, u_{a+1}, u_1$ be a cycle of order a + 1 and P_{b-a+1} : $v_1, v_2, \ldots, v_{b-a+1}$ be a path of order b − a + 1. We construct the graph G of order b + 1 by identifying the vertex u_1 of C_{a+1} and v_1 of P_{b-a+1} as shown in Figure. 2.3.

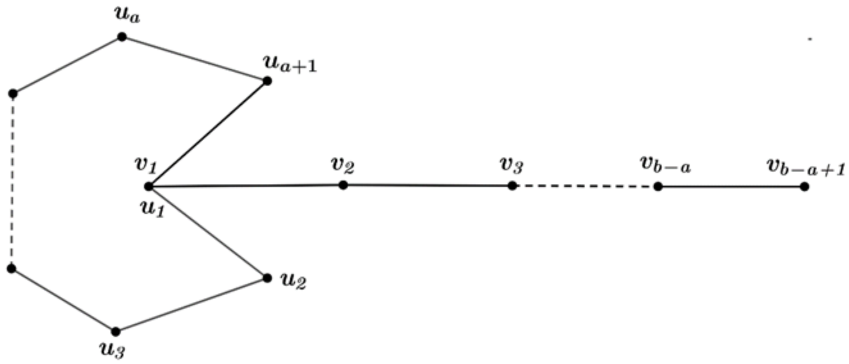

Figure 2.3: G

It is easy to verify that

$e_{\square f}(u_i) = e_{\square f}(v_i) = a$ for i=1

$$e_{\square f}(u_i) = b - i + 2 \text{ for } 2 \leq i \leq \left\lceil \frac{a+1}{2} \right\rceil$$

$$e_{\square f}(u_i) = b - a + i - 1 \text{ for } \left\lceil \frac{a+1}{2} \right\rceil \leq i \leq a + 1$$

$$e_{\square f}(v_i) = a + i - 1 \text{ for } 2 \leq i \leq b - a + 1$$

Particularly, $e_{\square f}(u_2) = e_{\square f}(u_{a+1}) = e_{\square f}(v_{b-a+1}) = b$. It is easy to verify that there is no vertex x in G with $e_{\square f}(x) < a$ and there is no vertex y in G with $e_{\square f}(y) > b$. Thus $rad_{\square f}(G) = a$ and $diam_{\square f}(G) = b$ as $a < b \leq 2a$.

Case 3. b > 2a.

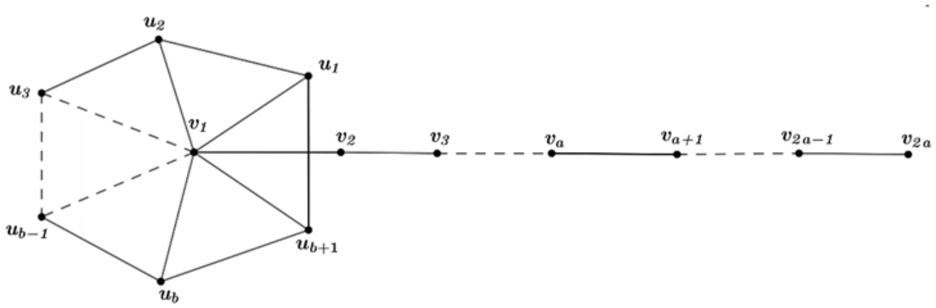

Figure 2.4: G

Let G be a graph of order b + 2a + 1 obtained by identifying the central vertex of the wheel $W_{b+2} = K_1 + C_{b+1}$ and an end vertex of the path P_{2a} as shown in Figure 2.4., where $K_1 = v_1$, C_{b+1} : $u_1, u_2, \ldots, u_{b+1}, u_1$ and P_{2a} : $v_1, v_2, \ldots, v_{2a}$.

We can easily verify that there is no vertex x in G with $e_{\Box f}(x) < a$ and there is no vertex y in G with $e_{\Box f}(y) > b$. Thus $rad_{\Box f}(G) = a$ and $diam_{\Box f}(G) = b$ as $b > 2a$.

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