Gap Approximation Numbers and Compactness

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The gap is considered to be a meaningful metric on the space of all closed operators defined in a Hilbert space H. The gap between two closed operators S and T is defined as the gap between the graphs $G(S)$ and $G(T)$ of the operators, which are closed subspaces of the product space $H \times H$. Thus the study of the gap between subspaces has great impact on the gap between operators.
For a bounded operator A, the k^{th} approximation number $s_k(A)$ is defined by $s_k(A) = \inf\{ \| A - F \| / F \text{ isaboundedoperator}, \text{rank} F \leq k - 1\}$. Motivated by this definition, several other approximation numbers were introduced by many Mathematicians for bounded operators. The notion of approximation numbers was further generalized for unbounded operators using the notion of gap . They are called gap approximation numbers.
In this Paper we analyze connection between compactness and gap approximation numbers.

1 Intruduction

Let us state the definition of gap approximation numbers as given in [7].

Definition 1.1 For $A \in C(H)$, and $k = 1,2,3,\ldots$ let

$$b_k(A) = \inf\{ \hat{\delta}(A, F)/F \in B(H), \text{rank} F \leq k - 1, \| F \| \leq 1\}. \qquad (1)$$

The k^{th} gap approximation number β_k is defined by

$$\beta_k(A) = \frac{b_k(A)}{\sqrt{1 - b_k^2(A)}}. \qquad (2)$$

2 Gap Approximation Numbers and Compactness

For a bounded operator A,

$$\beta_1(A) = \| A \|$$

and

$$\beta_k(A) \leq \beta_1(A) = \| A \|, \quad \text{forall} \quad k.$$

Therefore,

$$\sup_k \{\beta_1(A), \beta_2(A), \ldots\} = \| A \|. \qquad (3)$$

Also,

$$\beta_k(A) \geq 0, \text{forall } k.$$

Now let us ask the following questions :

- What is the infimum of $\{\beta_k(A)/k = 1,2,3,\dots\}$?
- When does the infimum away from zero ?

Let us try to answer these questions, not only for the bounded case; for the unbounded case as well. For answering the above questions we require the following definition :

Definition 2.1 For $A \in C(H)$, define

$$b_0(A) = \inf\{\hat{\delta}(A,K)/K \in K(H)\}, \tag{4}$$

where $K(H)$ is the class of compact operators on H and

$$\beta_0(A) = \frac{b_0(A)}{\sqrt{1-b_0^2(A)}} \tag{5}$$

Lemma 2.2 Let $A \in C(H)$. Then $\inf\{\beta_1(A), \beta_2(A), \beta_3(A), \dots\} = \beta_0(A)$.

Proof. For any k,

$$b_k(A) = \inf\{\hat{\delta}(A,F)/F \in B(H), \text{rank} F \leq k\}$$

and

$$b_0(A) = \inf\{\hat{\delta}(A,K)/K \in K(H)\}.$$

We know that every finite rank operator is compact. Therefore,

$$b_k(A) = \inf\{\hat{\delta}(A,F)/F \in B(H), \text{rank} F \leq k\} \geq b_0(A), \text{forall } k. \tag{6}$$

Hence

$$\inf\{b_k(A)/k = 1,2,3,\dots\} \geq b_0(A). \tag{7}$$

To prove the converse inequality,

put $m = \inf\{b_k(A)/k = 1,2,3,\dots\}$.

Let $\varepsilon > 0$ and $K \in K(H)$.

As the set of finite rank operator is dense in K(H), there exists a sequence of finite rank operators $\{F_n\}$ such that $\| K - F_n \| \longrightarrow 0$ as $n \longrightarrow \infty$.

As the norm convergence is equivalent to the gap convergence for bounded operators, we have

$$\hat{\delta}(K, F_n) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

This implies

$$\hat{\delta}(K, F_{n_o}) < \varepsilon, \quad \text{forsomepositiveinteger} \quad n_0. \tag{8}$$

Now, $\hat{\delta}(A, F_{n_0}) \leq \hat{\delta}(A, K) + \hat{\delta}(K, F_{n_0})$, since $\hat{\delta}$ is a metric in the space of closed operators.

$$\text{So}, \hat{\delta}(A, F_{n_0}) < \hat{\delta}(A, K) + \varepsilon, \tag{9}$$

using (8).

Let $k_0 - 1$ be the rank of F_{n_0}.

Then,

$$b_{k_0}(A) \leq \hat{\delta}(A, F_{n_0}) < \hat{\delta}(A, K) + \varepsilon, \text{from}(9).$$

But $m = \inf\{b_k(A)/k = 1,2,3, \dots\}$.

So,

$$m < \hat{\delta}(A, K) + \varepsilon.$$

Hence, since $\varepsilon > 0$ is arbitrary , we get,

$$m \leq \hat{\delta}(A, K).$$

This holds good for every $K \in K(H)$ and so

$$m \leq \inf\{\hat{\delta}(A, K)/K \in K(H)\}.$$

That is,

$$m \leq b_0(A)$$

Thus,

$$\inf\{b_k(A)/k = 1,2,3, \dots\} \leq b_0(A) \tag{10}$$

From (7) and (10),we get,

$$\inf\{b_k(A)/k = 1,2,3, \dots\} = b_0(A)$$

For $A \in B(H), \{b_k(A)\}_{k=1}^{\infty}$ is a monotonic decreasing sequence, which is bounded below. Hence, the sequence $\{\beta_k(A)\}_{k=1}^{\infty}$ converges to its infimum.

We state it as a result:

Proposition 2.3 Let $A \in B(H)$. Then, $\lim_{k \to \infty} b_k(A) = b_0(A)$.

Suppose $A \in B(H)$.

Since

$$b_k(A) \longrightarrow b_0(A) \quad \text{as} \quad k \longrightarrow \infty$$

and

$$\beta_k(A) = \frac{b_k(A)}{\sqrt{1-b_k^2(A)}} \quad \text{and} \quad \beta_0(A) = \frac{b_0(A)}{\sqrt{1-b_0^2(A)}},$$

we have,

$$\lim_{k\to\infty} \beta_k(A) = \beta_0(A).$$

Since $\{\beta_k(A)\}_{k=1}^{\infty}$ is monotonic decreasing,

$$\lim_{n\to\infty} \beta_k(A) = \inf_k \beta_k(A).$$

Thus we have

Theorem 2.4 Let $A \in B(H)$.Then $\inf_k \beta_k(A) = \lim_{k\to\infty} \beta_k(A) = \beta_0(A)$

Theorem 2.5 Let $A \in B(H)$.Then $\beta_0(A) = 0$ if and only if A is compact.

Proof.

$$\beta_0(A) = 0 \quad \text{ifandonlyif} \quad b_0(A) = 0.$$

Hence, it is enough to prove that $b_0(A) = 0$ if and only if A is compact.

$$b_0(A) = \inf\{\hat{\delta}(A, K)/K \in K(H)\}.$$

Therefore, if A is compact, then obviously

$$b_0(A) = 0.$$

Conversely, assume that $b_0(A) = 0$.

That is,

$$\inf\{\hat{\delta}(A, K)/K \in K(H)\} = 0.$$

Then there exists a sequence $\{K_n\}$ in $K(H)$ such that

$$\hat{\delta}(A, K_n) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty,$$

which implies

$$\| A - K_n \| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

That is,

$$K_n \longrightarrow A \quad \text{in} \quad B(H).$$

But $K(H)$ is closed in $B(H)$. Hence A is compact.

We have established the following :

1. If A is compact, then $\lim_{n\to\infty}\beta_k(A) = \inf_k\beta_k(A) = 0$.
2. If A is a non-compact bounded operator, then $\inf_k \beta_k(A) > 0$.

Thus we have the result:

Proposition 2.6 Let $A \in B(H)$. Then A is compact if and only if $\inf_k \beta_k(A) = 0$.

Corollary 2.7 Let A be a compact operator, which is not of finite rank. Then 0 is a limit point of the set $\{\beta_1(A), \beta_2(A), \beta_3(A), \dots\}$.

Proof. Since A is compact , $\beta_0(A) = 0$.

Also,

$$\beta_k(T) = 0 \quad \text{ifandonlyifrank} T \leq k - 1.$$

Here, A is not of finite rank.

Therefore,

$$\beta_k(A) > 0, \quad \text{forall} \quad k.$$

But

$$\beta_k(A) \longrightarrow \beta_0(A) = 0 \quad \text{as} \quad k \longrightarrow \infty.$$

Hence, 0 is a limit point of $\{\beta_1(A), \beta_2(A), \beta_3(A), \dots\}$.

Proposition 2.8 β_0 is continuous with respect to the gap.

Proof. Let $A, A_n \in C(H)$, for $n = 1,2,3, \dots$

Let $\hat{\delta}(A_n, A) \longrightarrow 0$ as $n \longrightarrow \infty$.

For $k = 1,2,3, \dots$, we have, by (??),

$|b_k(A_n) - b_k(A)| \leq \hat{\delta}(A_n, A)$.

Fix n. Let $\varepsilon > 0$.

We have, by Proposition 4.1.9,

$$b_k(A_n) \longrightarrow b_0(A_n) \text{ as } k \longrightarrow \infty$$

and

$$b_k(A) \longrightarrow b_0(A) \text{ as } k \longrightarrow \infty.$$

Therefore there exists a positive integer k_0 such that

$$|b_k(A_n) - b_0(A_n)| < \frac{\varepsilon}{2} \text{ and } |b_k(A) - b_0(A)| < \frac{\varepsilon}{2}, \text{forall } k \geq k_0. \quad (11)$$

Choose $k \geq k_0$, Now,

$$|b_0(A_n) - b_0(A)| = |b_0(A_n) - b_k(A_n) + b_k(A_n) - b_k(A) + b_k(A) - b_0(A)|$$

$$\leq |b_0(A_n) - b_k(A_n)| + |b_k(A_n) - b_k(A)| + |b_k(A) - b_0(A)|$$

$$< \frac{\varepsilon}{2} + \hat{\delta}(A_n, A) + \frac{\varepsilon}{2}, \quad \text{using} \quad (??) \quad \text{and} \quad (??11)$$

$$\text{Thatis}, |b_0(A_n) - b_0(A)| < \hat{\delta}(A_n, A) + \varepsilon$$

Since $\varepsilon > 0$ ia arbitrary, we get,

$$|b_0(A_n) - b_0(A)| \leq \hat{\delta}(A_n, A). \tag{12}$$

$$\text{Let } \hat{\delta}(A_n, A) \longrightarrow 0$$

Then, (4.14) implies

$$b_0(A_n) - b_0(A)| \longrightarrow 0. \text{ So}, b_0(A_n) \longrightarrow b_0(A)$$

$$\text{Now}, \beta_0(A) = \frac{b_0(A)}{\sqrt{1 - b_0^2(A)}}$$

$$\text{Hence } \beta_0(A_n) \longrightarrow \beta_0(A).$$

Thus $A_n \longrightarrow A$ in gap implies $b_0(A_n) \longrightarrow b_0(A)$.

So, β_0 is continuous with respect to the gap.

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