Analysis of Batch Encouraged Arrival Markovian Model Due to a Secondary Optional Service, Break-Down and Numerous Vacations

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Abstract

In this study, we describe a batch encouraged arrival Markovian queuing model due to a secondary optional service, breakdown and numerous vacations. In this model, encouraged arrival is introduced, the server goes on vacation every time the system empties and the length of the vacation is predicted to follow an exponential distribution. If the server returns from vacation and there are no customers in line, he will take another vacation until there is at minimum one unit in the system. The steady-state findings have been derived and the time-dependent Probability-Generating- Functions (PGF) have been determined in terms of their Laplace transforms. In this model, we also discover the average queue length and average waiting time have been obtained.

Keywords: - Markovian model; break-down; encouraged arrival; numerous vacation; queue length

Introduction

In the literature on queuing theory, the study of queuing systems with server vacation has expanded in scope and interest. Server vacations are used to make better use of downtime. Vacation queuing models have been successfully used to a variety of situations, including production, financial services, telecommunication networks, internet technology, etc. Many researchers are interested in researching queuing models with various vacation rules, including single and multiple vacation policies.

In [1], we examine several elements of the M/G/1 queuing model with possible second service. We looked at a group arrival queuing system with an extra service channel in [2]. We investigated at an M/G/1 queuing system that included two phases of service and D-policy in [3]. We studied at an M/G/1 queue with two phases of service and numerous efforts at moving the queue into a steady state in [4]. In [5], we looked into a vacation queuing system with service interruptions. An M/G/1 Queuing model under the second optional service, general service time distribution was examined in [6]. We investigated batch arrivals queuing model with system failures, start and closed timings and vacation rules in [7].

We looked at the "optimal management policy" for heterogeneous arrival queuing model with system interruptions and vacations in [8]. We looked at the M^x $/G_1/G_2/1$ retrial queues for Bernoulli vacations times with repeating efforts and Initial breakdowns in [9]. In [10], we looked at bulk arrivals queuing model using service vacations. We investigated at a second optional services in an M/G/1 queuing model and servers vacation under Bernoulli timing in[11]. We studied at the possible re-service on M[x]/(G1;G2)/1 queuing model in [12]. We analyzed two M_x/M [a; b]/1 queuing systems having stochastic breakdown using steady state modeling in [13].
We investigated at a second optional services in an M/G/1 queuing along with server breakdowns in [14]. An M/M/1/N system including encouraged arrivals was investigated in [15]. Reduction of waiting time in an M/M/1/N encouraged arrival queue with feedback, balking and maintaining of reneged customers in [16]. We investigated M/M/C/N queue including encouraged arrivals, reneging, retention and Feedback consumers in [17]. We investigated the behaviour of M ([X])/G/1 using Second Optional Services, Multi Vacation, Breakdowns and Repairs in [18].

Main model premises

The following assumptions are follows,

- In an encouraged arrival procedure, customers enter the system in batches of various size, and they are serviced one by one according to the principle of "first come, first served."
- Let us consider, $\sum D_l dk\,, t = 1,2,3, \ldots$ be the 1st order likelihood a batch of encouraged 'l' customers enters the systems due to a period of time "$[\mathrm{k}, \mathrm{k} + \mathrm{dk}]$, where $0 \le D_l \le 1$ and $\Sigma D_l = 1$ and$\lambda * (1 + \vartheta) > 0$, is the average encouraged arrival process.
- Assume that t= $1,2,3, \ldots$ be the 1st order likelihood of encouraged arrival of ' k ' customers enter the systems due to a interval of time "$[\mathrm{k}, \mathrm{k} + \mathrm{dk}]$" where $0 \le D_l \le 1$ and $\Sigma D_l = 1$ and$\lambda * (1 + \vartheta) > 0$, is the average encouraged arrival process.
- There is a one server which givens the first important service to all encouraged arrival customers. Let us assume $I_1(\bar{v})\&i_1(\bar{v})$ being the initial service times distribution function and density function respectively.
- Let us the 1st services of a customer is finished, then will need for the 2nd services with our likelihood of customer$\dot{r}$ or will consider to depart from the systems with likelihood of customers $1 - \dot{r}$.
- The 2nd services periods as consider to be general distribution & the density functions $I_2(\bar{v})\&i_2(\bar{v})$. Then, Let us consider $\mu_b(y)dy$ using the conditional - density functions of b^{th} services finished due to the period of time $[\mathrm{y}, \mathrm{y} + \mathrm{dy}\,]$ produced that the service time is y.
- If no customers waiting in the line, server go away for a vacations. The vacations timings are identically distribution with average vacation period$\frac{1}{\varepsilon}$. On re- joining from the vacations if server repeats no customers in the line, then it is go -away for next vacations. So the servers serve the dual vacation.

- If serves the systems will consider the break -down at irregular and break-downs are consider will occurs to a encouraged streams under the average break-down standard for $\Upsilon > 0$.

- The serves the systems ones breaks- down, it arrivals a repair processing instantly. The repairs periods are identically distribution with average repairs standard for $\delta > 0$

Different of stochastic model involving in serve the systems are separate of all others.

1. Governing of system of equations:

- $P_m^b(y, t.)$ = Probability at period ' $t.$ ' the serve the servers producing b^th services & the ' m ' customer in the line adding the one serve the customer & the delayed service period customer's in y.

- Let us $P_m^b(t.)$ = represents the likelihood that period 't.' and 'm' customers in the line neglect the customers in b^{th} serve the services of$y$.

- $\overline{V_m}(t.) =$ Probability that period at ' $t.$ ' and ' m' customer in the line and the serve the servers is on vacation of y.

- $\overline{R_m}(t.) =$ Probability that period at 't.' and the serve to be the servers not active during the break- down & the serve to the systems due to repairs are ' m ' customer in the line.

The governed differential-difference equations as follow:

Let us $\mathrm{MU}(\mathrm{t.})$ represent the line size at period t. and $\mathrm{Z}(\mathrm{t.})$ are as follows

1, if the serve the system is busy under 1st important service period at t.

2, if the serve the system is busy under 2nd serve the service period at t.

3, if the serve the system is on vacation at period t.

Let us length of(t) represent

$I_1^0(t.) =$ if $Z(t.) = 1$ Delayed service period for the 1st delayed service period at t.,

$I_2^0(t.) =$ if $Z(t.) = 1$ Delayed service period for the 2nd service period at t.,

$\bar{v}^0(t.) =$ if $Z(t.) = 1$ Delayed vacation period of the serve the servers at period at t.

The processing$\{M_U(t.),$ Length of $(\mathrm{t.})\}$. We represent the likelihood for $\mathrm{b} = 1,2$.

$$P_m^b(y, t.) = \text{Probability}\{M_U(t.) = m, Length\ of\ (t.) = \mathrm{I}_{\mathrm{i}}^0; y < I_b^0 \leq y + dy\}; y > 0, m > 0$$

The steady state solution we follows,

$P_m^b(y)dy = \lim_{t.\to\infty} P_m^b(\mathrm{y},\mathrm{t.}), \quad \mathrm{b} = 1{,}2\mathrm{y} > 0; m \geq 0$
$\bar{V}_m = \lim_{t.\to\infty} \bar{V}_m(t.); m \geq 0$
$\bar{R}_m = \lim_{t.\to\infty} \bar{R}_m(t.); m \geq 0$

Let us consider,

$\bar{V}_0(0) = 1, \bar{V}_m(0) = 0$, and for b = 1,2
$I_b(0), I_b(\infty) = 1$, Also $\bar{V}(\mathrm{y}) \& I_b(y)$ are all continuous at $y = 0$.
The governed differential-difference equation as follows that:

$$\frac{\partial}{\partial y} P_m^{(1)}(y,t.) + \frac{\partial}{\partial t} P_m^{(1)}(y,t.) + (\lambda * (1+\vartheta) + \mu_1(y) + \Upsilon) P_m^{(1)}(y,t.) = \lambda * (1+\vartheta) \sum_{l=1}^{\infty} D_l P_{m-l}^{(1)}(y,t.), m \geq 1 ----------(1)$$

$$\frac{\partial}{\partial y} P_0^{(1)}(y,t.) + \frac{\partial}{\partial t} P_m^{(1)}(y,t.) + (\lambda * (1+\vartheta) + \mu_1(y) + \Upsilon) P_0^{(1)}(y,t.) = 0 --- (2)$$

$$\frac{\partial}{\partial y} P_m^{(2)}(y,t.) + \frac{\partial}{\partial t} P_m^{(2)}(y,t.) + (\lambda * (1+\vartheta) + \mu_2(y) + \Upsilon) P_m^{(2)}(y,t.) = \lambda * (1+\vartheta) \textstyle\sum_{l=1}^{\infty} D_l P_{m-l}^{(2)}(y,t.), m \geq 1 ----------(3)$$

$$\frac{\partial}{\partial y} P_0^{(2)}(y,t.) + \frac{\partial}{\partial t} P_m^{(2)}(y,t.) + (\lambda * (1+\vartheta) + \mu_2(y) + \Upsilon) P_0^{(2)}(y,t.) = 0 --- (4)$$

$$\frac{d}{dt} \bar{V}_m(t.) + (\lambda * (1+\vartheta) + \varepsilon) \bar{V}_m(t.) = \lambda \sum_{i=1}^{\infty} D_l \bar{V}_{m-l}(t.), \quad m \geq 1 --- (5)$$

$$\frac{d}{dt} \bar{V}_0(t.) + (\lambda * (1+\vartheta) + \varepsilon) \bar{V}_0(t.) = \Upsilon \bar{V}_0(t) + (1-\dot{r}) \int_0^{\infty} P_0^1(y,t.) \mu_1(y) dy + \int_0^{\infty} P_0^2(y,t.) \mu_2(y) dy \quad --------- (6)$$

$$\frac{d}{dt} \bar{R}_0(t.) + (\lambda * (1+\vartheta) + \delta) \bar{R}_0(t.) = 0 ---------- (7)$$

$$\frac{d}{dt} \bar{R}_m(t.) + (\lambda * (1+\vartheta) + \delta) \bar{R}_m(t.) = \lambda * (1+\vartheta) \textstyle\sum_{l=1}^{\infty} D_l \bar{R}_{m-l}(t.) + \Upsilon \int_0^{\infty} P_{n-1}^{(1)}(x,t.) + \Upsilon \int_0^{\infty} P_{m-1}^{(2)}(y,t.), \quad m \geq 1 ---------- (8)$$

Following boundary conditions are:

$$P_0^{(1)}(0,t.) = \varepsilon \bar{V}_1(t.) + \delta \bar{R}_1(t.) + (1-\dot{r}) \int_0^{\infty} P_1^{(1)}(y,t.) \mu_2(y) dy + \int_0^{\infty} P_1^{(2)}(y,t.) \mu_2(y) dy -----------(9)$$

$$P_m^{(1)}(0) = \varepsilon \bar{V}_{m+1}(t.) + \delta \bar{R}_{m+1}(t.) + (1-\dot{r})\int_0^\infty P_{m+1}^{(1)}(y,t.)\mu_2(y)dy + \int_0^\infty P_{m+1}^{(2)}(y,t.)\mu_2(y)dy, m \geq 1 - - - - - - - - (10)$$

$$P_m^{(2)}(0) = \int_0^\infty P_m^{(1)}(y)\mu_1(y)dy\, , m \geq 0 - - - - - - - - - (11)$$

2. **Time varying result:**

To create function of queue line

Now, if we represent the P.G.F as following below,

$P^{(1)}(y,t.) = \sum_0^\infty P_m^{(1)}(y,\dot{z},t.)\dot{z}^m; P^{(1)}(\dot{z},t.) = \sum_0^\infty P_m^{(1)}(t)\dot{z}^m, |\dot{z}| \leq 1, y > 0$

$P^{(2)}(y,\dot{z},t.) = \sum_0^\infty P_m^{(2)}(y,t.)\dot{z}^n; P^{(2)}(\dot{z},t.) = \sum_0^\infty P_m^{(2)}(t)\dot{z}^m, |\dot{z}| \leq 1, y > 0$

$\bar{V}(\dot{z},t.) = \sum_0^\infty \dot{z}^m \bar{V}_m(t.) \quad ; \quad \bar{R}(\dot{z},t.) = \sum_0^\infty \dot{z}^m \bar{R}_m(t.) \quad ; \quad D(\dot{z}) = \sum_0^\infty D_m \dot{z}^m, |\dot{z}| \leq 1 - - - - (12)$

Taking the Laplace equations (1) to (11) as follows,

$$\frac{\partial}{\partial y}\bar{P}_m^{(1)}(y,H) + (H + \lambda * (1+\vartheta) + \mu_1(\text{y}) + \Upsilon)\bar{P}_m^{(1)}(y,H) = \lambda * (1+\vartheta)\sum_{l=1}^\infty D_l \bar{P}_{m-l}^{(1)}(y,H), \quad m \geq 1\text{-- } - - - - - - - - - - - - (13)$$

$$\frac{\partial}{\partial y}\bar{P}_0^{(1)}(y,H) + (H + \lambda * (1+\vartheta) + \mu_1(\text{y}) + \Upsilon)\bar{P}_0^{(1)}(y,H) = 0 - - - - - - - - - (14)$$

$$\frac{\partial}{\partial y}\bar{P}_m^{(2)}(y,H) + (H + \lambda * (1+\vartheta) + \mu_1(\text{y}) + \Upsilon)\bar{P}_m^{(2)}(y,H) = \lambda * (1+\vartheta)\sum_{l=1}^\infty D_l \bar{P}_{m-l}^{(2)}(y,H), \quad m \geq 1 - - - - - - - - - - - - (15)$$

$$\frac{\partial}{\partial y}\bar{P}_0^{(2)}(y,H) + (H + \lambda * (1+\vartheta) + \mu_1(\text{y}) + \Upsilon)\bar{P}_0^{(2)}(y,H) = 0 - - - - - - - - - (16)$$

$$(\text{H} + \lambda * (1+\vartheta) + \text{v})\bar{\text{V}}_0(\text{s}) = 1 + (1-\dot{r})\int_0^\infty \bar{P}_0^{(1)}(y,H)\mu_2(y)dy + \int_0^\infty \bar{P}_0^{(2)}(y,H)\mu_2(y)dx + \overline{V *}\,\bar{\text{V}}_0(\text{H})$$

$$(H + \lambda * (1+\vartheta) + \varepsilon)\bar{V}_m(H) = \lambda * (1+\vartheta)(H), m \geq 1 - - - - - - - - - - - - (17)$$

$$(\text{H} + \lambda * (1+\vartheta) + \delta)\bar{\text{R}}_0(\text{H}) = \Upsilon\int_0^\infty \bar{P}_0^{(1)}(y,H)\mu_1(y)dy + \Upsilon\int_0^\infty \bar{P}_0^{(2)}(y,H)\mu_2(y)dy - - - (18)$$

$$(H + \lambda * (1+\vartheta) + \delta)\bar{R}_m(H) = \lambda * (1+\vartheta)\bar{R}_{m-1}(H), m \geq 1 - - - - - - - - - (19)$$

$$\bar{P}_0^{(1)}(o,H) = \text{X}\bar{V}_1(H) + \delta\bar{\text{R}}_1(H) + (1-\dot{r})\int_0^\infty \bar{P}_1^{(1)}(y,H)\mu_2(y)dy + u\int_0^\infty \bar{P}_1^{(2)}(y,H)\mu_2(y)dy - - - - - - - - - - - (20)$$

$$\bar{P}_m^{(1)}(o,H) = \varepsilon\bar{\text{V}}_1(H) + \delta\bar{\text{R}}_1(H) + (1-\dot{r})\int_0^\infty \bar{P}_{m+1}^{(1)}(y,H)\mu_1(y)dy + \int_0^\infty \bar{P}_{m+1}^{(2)}(y,H)\mu_2(y)dy, \quad m \geq 1 - - - - - - - - - - - - (21)$$

$$\bar{P}_m^{(2)}(o,H) = \int_0^\infty \bar{P}_m^{(1)}(y,H)\mu_1(y)dy, m \geq 0 - - - - - - - - - - - - (22)$$

We multiplication on two sides of the equations (13) and (14) by appropriate powers of $\dot{z}$, add over m and using equation (12) and explain ,algebraic formulations

$$\frac{\partial}{\partial y}\bar{P}^{(1)}(y,\dot{z},H)+[H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z})+\mu_1(y)+\Upsilon]\bar{P}^{(1)}(y,\dot{z},H)=0---(23)$$

Effecting then as well operation on the equations (15) and (16) and utilizing equation (12), We follows,

$$\frac{\partial}{\partial y}P^{(2)}(y,\dot{z},H)+[H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z})+\mu_1(y)+\Upsilon]P^{(2)}(y,\dot{z},H)=0---(24)$$

Consequently operations on the equations (17),(18),(19) and (4.9) produce.

$$[H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z})+\varepsilon]\bar{V}(\dot{z},H)$$
$$=1+(1-\dot{r})\int_0^\infty \bar{P}_0^{(1)}(y,H)\mu_2(y)dy+\int_0^\infty \bar{P}_0^{(2)}(y,H)\mu_2(y)dy+\varepsilon\bar{\mathrm{V}}_0(\mathrm{H}\varepsilon)\ ---(25)$$

$$[H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z})+\delta]\bar{R}(\dot{z},H)$$
$$=\Upsilon\dot{z}\int_0^\infty \bar{P}^{(1)}(y,\dot{z},H)dy+\Upsilon\dot{z}\int_0^\infty P^{(2)}(y,\dot{z},H)dy---(26)$$

Now, we multiplication two sides of the equation (20) by $\dot{z}$, multiplication on two sides of the equations (21) by $\dot{z}^{m+1}$, add over m from 1to ∞, sum of the both solutions and using equation (12)&(17).Thus we obtain after mathematical adjustments

$$\dot{z}\bar{P}^{(1)}(0,\dot{z},H)=$$
$$(1-\dot{r})\int_0^\infty \bar{P}^{(1)}(y,\dot{z},H)\mu_1(y)dy+\int_0^\infty \bar{P}^{(2)}(y,\dot{z},H)\mu_2(y)dy+\varepsilon\bar{\mathrm{V}}(\dot{z},\mathrm{H})-(1-\dot{r})\int_0^\infty \bar{P}_0^{(1)}(y,H)\mu_1(y)dy-\int_0^\infty \bar{P}_0^{(2)}(y,H)\mu_2(y)dy+\delta\bar{R}(\dot{z},H)---(27)$$

$$\bar{P}^{(2)}(0,\dot{z},H)=\int_0^\infty \bar{P}^{(1)}(y,\dot{z},H)\mu_1(y)dy---(28)$$

Using (25) in (27), we obtain

$$\dot{z}\bar{P}^{(1)}(0,\dot{z},H)=(1-\dot{r})\int_0^\infty \bar{P}^{(1)}(y,\dot{z},H)\mu_1(y)dy+\int_0^\infty \bar{P}^{(2)}(y,\dot{z},H)\mu_2(y)dy+1-[H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z}]\bar{V}(\dot{z},H)+\delta\bar{R}(\dot{z},H)---(29)$$

$\int ing$ equations on (13), (14) & (15) in the middle of 0 and y, we obtain

$$\bar{P}^{(1)}(y,\dot{z},H)=\bar{P}^{(1)}(0,\dot{z},H)\quad e^{-(H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z})+\Upsilon)y-\int_0^\infty \mu_{1*(t)*dt}}---(30)$$
$$\bar{P}^{(2)}(y,\dot{z},H)=\bar{P}^{(2)}(0,\dot{z},H)e^{-(H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z})+\Upsilon)y-\int_0^\infty \mu_{2*(t)*dt}}---(31)$$

Again$\int ing$ equations (20) with respect to “y", we get

$$\bar{P}^{(1)}(\dot{z},H)=\bar{P}^{(1)}(0,\dot{z},H)\left[\frac{1-\bar{I}_1(H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z})+\Upsilon)}{(H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z})+\Upsilon)}\right]---(32)$$

Where $\bar{I}_1(H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z})+\Upsilon)=\int_0^\infty e^{-(H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z})+\Upsilon)y}d\bar{I}_1(y)---(33)$

is the Laplace equation of the 1st stage of service period.
Now, from the equation (20) afterwards few conspectus and using the equations (12) , we get

$$\int_0^\infty \bar{P}^{(1)}(y,\dot{z},H)\mu_1(y)dy = \bar{P}^{(1)}(0,\dot{z},H)\bar{I}_1(H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z})+\Upsilon) ---(34)$$

Again $\int ing$ equations (21) with respect to "y", we have

$$\bar{P}^{(2)}(\dot{z},H) = \bar{P}^{(2)}(0,\dot{z},H)\left[\frac{1-\bar{I}_2(H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z})+\Upsilon)}{(H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z})+\Upsilon)}\right] ---(35)$$

Where $\bar{I}_2(H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z})+\Upsilon) = \int_0^\infty e^{-(H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z})+\Upsilon)y}d\bar{I}_2(y) ---(36)$
is the Laplace equation of the 2st stage of service period..
Now, from the equation (21) afterwards few conspectus and using the equations (12) , we get

$$\int_0^\infty \bar{P}^{(2)}(y,\dot{z},H)\mu_2(y)dy = \bar{P}^{(2)}(0,\dot{z},H)\bar{I}_2(H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z})+\Upsilon) ---(37)$$

Using the equations (34) and (37) in (26) we have,

$$[H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z})+\delta]\bar{R}(\dot{z}) = \Upsilon\dot{z}\bar{P}^{(1)}(0,\dot{z},H)\frac{\left[\begin{matrix}1-\bar{I}_1(H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z})+\Upsilon)\bar{I}_2(H+\lambda*(1+\vartheta)- \\ \lambda*(1+\vartheta)D(\dot{z})+\Upsilon)\end{matrix}\right]}{(H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z})+\Upsilon)} ---(38)$$

Now, using the equations. (28) (31), (33),(34),(36) & (37) in the equation (29) and working for $\bar{P}^{(1)}(0,z)$ we obtain

$$\bar{P}^{(1)}(0,\dot{z},H) = \frac{f_1(\dot{z})f_2(\dot{z})[1-(H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z}))\bar{V}(\dot{z},H)]}{C\bar{R}} ---(39)$$

Where $C\bar{R} = f_1(\dot{z})f_2(\dot{z})\{\dot{z}-(1-\dot{r})\bar{I}_1(H+-\lambda*(1+\vartheta)*D(\dot{z})+\Upsilon)-\dot{r}\bar{I}_1(H+\lambda*(1+\vartheta)-$
$\lambda*(1+\vartheta)D(z)+\Upsilon)I2(H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(z)+\Upsilon-\Upsilon\delta z(1-1-rI1H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)Dz$
$+\Upsilon-rI1H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)Dz+\Upsilon I2H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)Dz+\Upsilon$

$f_1(\dot{z}) = \mathrm{H}+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)\mathrm{D}(\dot{z})+\Upsilon$ and $f_2(\dot{z}) = \mathrm{H}+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)\mathrm{D}(\dot{z})+\delta ---(40)$

Substitute value of the term $P^{(1)}(0,z)$ from the equations (32) into the equation (23), (26) & (28) we obtain $\bar{P}^{(1)}(\dot{z},H) = \frac{f_2(\dot{z})[1-\bar{I}_1(H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z})+\Upsilon)]}{C\bar{R}}[1-(H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\bar{z}))\mathrm{V}(\dot{z},H)] ---(41)$

$$\bar{P}^{(2)}(\dot{z},H) = \frac{f_2(\dot{z})\bar{I}_1(H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z})+\Upsilon)[1-\bar{I}_2(H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z})+\Upsilon)]}{C\bar{R}}[1-(H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z}))\mathrm{V}(\dot{z},H)] ---(42)$$

$$R(\dot{z},H)=\frac{\Upsilon\dot{z}\begin{bmatrix}1-(1-\dot{r})\bar{I_1}(H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z})+\Upsilon)-\\ \dot{r}\bar{I_1}(H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z})+\Upsilon)\\ \bar{I_2}(H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z})+\Upsilon)\end{bmatrix}}{C\bar{R}}[1-(H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z}))\mathrm{V}(\dot{z},H)]---(43)$$

In this part we have elaborate steady -state probability –distribution function for the queuing model. To description of the steady- state probability. By –using the Tauberian property,

$$\lim_{H\to 0}\bar{f}(H)=\lim_{t\to\infty}f(t.)$$

$$P^{(1)}(\dot{z})=\frac{f_2(\dot{z})[1-I_1(\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z})+\Upsilon)]}{C\bar{R}}\lambda*(1+\vartheta)(D(\dot{z})-1)\bar{V}(\dot{z})---(44)$$

$$P^{(2)}(\dot{z})=\frac{f_2(\dot{z})I_1(\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z})+\Upsilon)[1-I_2(\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z})+\Upsilon)]}{C\bar{R}}\lambda*(1+\vartheta)(C(\dot{z})-1)\bar{V}(\dot{z})\ ---(45)$$

$$\bar{R}(\dot{z})=\frac{\Upsilon\dot{z}\begin{bmatrix}1-(1-\dot{r})I_1(H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z})+\Upsilon)-\\ \dot{r}I_1(H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z})+\Upsilon)\\ I_2(H+\lambda*(1+\vartheta)-\lambda*(1+\vartheta)D(\dot{z})+\Upsilon)\end{bmatrix}}{C\bar{R}}*\lambda*(1+\vartheta)(D(\dot{z})-1)\bar{V}(\dot{z})---(46)$$

In sequence to define $P^{(1)}(z),P^{(2)}(z)$, R(z) entirely, we get to define the un-known $\bar{V}(1)$ which shows in the right- sides of the equations (44), (45) and (46). We have use the normalizing -condition.

$$P^{(1)}(1.)+P^{(2)}(1.)+\bar{V}(1)+\bar{R}(1)=1---(47)$$

$$P^{(1)}(1)=\frac{\lambda*(1+\vartheta)\delta d'(1)(1-I_1(\Upsilon))}{c\dot{r}}\bar{V}(1)---(48)$$

$$P^{(2)}(1)=\frac{\lambda*(1+\vartheta)\delta D'(1)I_1(\Upsilon)(1-I_2(\Upsilon))}{c\dot{r}}\bar{V}(1)---(49)$$

$$\bar{R}(1)=\frac{\lambda*(1+\vartheta)\Upsilon D'(1)(1-(1-\dot{r})I_1(\Upsilon)-\dot{r}I_1(\Upsilon)I_2(\Upsilon))}{c\dot{r}}\bar{V}(1)---(50)$$

where $c\dot{r}=\Upsilon\delta(1-\mathrm{C}\dot{r}I_1(\Upsilon)i_2(\Upsilon)-(1-I_1(\Upsilon)I_2(\Upsilon))\lambda*(1+\vartheta)D'(1)(\Upsilon+\delta)$

$P^{(1)}(1),P^{(2)}(1)\&R(1)$ represent the steady- state probability, that the -server is giving 1st&2nd stage of the service and the server during a repair without neglect to the no. of customers in the line. Now, we using the -equations (48), (49), (50) into, the normalized -condition (47) and streamlining, we get

$$\bar{V}(1) = \left[1 - \frac{\lambda * (1+\vartheta)D'(1)}{\delta[(1-\dot{r})I_1(\Upsilon) + \dot{r}I_1(\Upsilon)I_2(\Upsilon)]} - \frac{\lambda * (1+\vartheta)D'(1)}{\Upsilon[(1-\dot{r})I_1(\Upsilon) + \dot{r}I_1(\Upsilon)I_2(\Upsilon)]} + \frac{\lambda * (1+\vartheta)D'(1)}{\delta} + \frac{\lambda * (1+\vartheta)D'(1)}{\Upsilon}\right] --- (51)$$

and hence, using factor rho of the- system is provided by

$$rho = \left[\frac{\lambda * (1+\vartheta)d'(1)}{\delta[(1-\dot{r})I_1(\Upsilon) + \dot{r}I_1(\Upsilon)I_2(\Upsilon)]} + \frac{\lambda * (1+\vartheta)d'(1)}{\Upsilon[(1-\dot{r})I_1(\Upsilon) + \dot{r}I_1(\Upsilon)I_2(\Upsilon)]} - \frac{\lambda * (1+\vartheta)d'(1)}{\delta} - \frac{\lambda * (1+\vartheta)d'(1)}{\Upsilon}\right] --- (52)$$

Where $rho < 1$ is the stability- condition due to the steady- states condition exits.

3. The average queue line & systems size:

Let $P_u(\dot{z})$ represent the P.G.F of the queue line in especial of the -server state. Then sum of the equations (34), (35) and (36) we get

$$P_u(\dot{z}) = P^{(1)}(\dot{z}) + P^{(2)}(\dot{z}) + \bar{R}(\dot{z})P_u(\dot{z}) = \frac{\mathrm{M}(\dot{z})}{\mathrm{C}(\dot{z})} --- (53)$$

$$M(\dot{z}) = (\lambda * (1+\vartheta)D(\dot{z}) - 1)\begin{pmatrix} 1 - (1-\dot{r})\bar{I}_1(\lambda * (1+\vartheta) - \lambda * (1+\vartheta)D(\dot{z}) + \Upsilon) \\ -\dot{r}\bar{I}_1(\lambda * (1+\vartheta) - \lambda * (1+\vartheta)D(\dot{z}) + \Upsilon)\bar{I}_2\begin{pmatrix}\lambda * (1+\vartheta) - \\ D(\dot{z}) + \Upsilon\end{pmatrix}\end{pmatrix}$$

$$(\Upsilon\dot{z} + f_2(\dot{z}))\bar{V}(\dot{z})$$

$$C(\dot{z}) = f_1(\dot{z})f_2(\dot{z})\begin{Bmatrix} \dot{z} - (1-\dot{r})\bar{I}_1(-\lambda * (1+\vartheta)D(\dot{z}) + \Upsilon) - \dot{r}\bar{I}_1(\lambda * (1+\vartheta) \\ -\lambda * (1+\vartheta)D(\dot{z}) + \Upsilon)\bar{I}_2(-\lambda * (1+\vartheta)D(\dot{z}) + \Upsilon)\end{Bmatrix}$$

$$-\Upsilon\delta\dot{z}\left(1 - (1-\dot{r})\bar{I}_1(\lambda * (1+\vartheta) - \lambda * (1+\vartheta)D(z) + \Upsilon) - \dot{r}\bar{I}_1\begin{pmatrix}\lambda * (1+\vartheta) - \\ \mathrm{I}_2(-\lambda * (1+\vartheta)D(\dot{z}) + \Upsilon)\end{pmatrix}\right)$$

Let L_u represent the average no. of customer in the line with steady -state solution. Thus we obtain

$$L_u = \frac{d}{d\dot{z}}[P_u(\dot{z})] \text{ at } \dot{z} = 1$$

$$L_u = \lim_{\dot{z}\to 1}\frac{C'(1)M''(1) - M'(1)C''(1)}{2C'(1)^2} --- (54)$$

Where primes & dual primes in the equation (46) represent 1st and 2 nd derivative at $\dot{z} = 1$, respectfully. Calling out the derivation at $\dot{z} = 1$ we get

$$M^{'}(1) = \lambda * (1+\vartheta) D^{'}(1)(\Upsilon+\delta)\bar{V}(1)[1-(1-\dot{r})\bar{I}_1(\Upsilon) - \dot{r}\bar{I}_1(\Upsilon)\bar{I}_2(\Upsilon)] --- (55)$$

$$M^{''}(1) = [1-(1-\dot{r})\bar{I}_1(\Upsilon) - \dot{r}\bar{I}_1(\Upsilon)\bar{I}_2(\Upsilon)] \begin{cases} C^{''}(1)(\alpha+\beta)V(1) - 2\left(\lambda * (1+\vartheta)D^{'}(1)\right)^2 \bar{V}(1) \\ \quad +2\lambda * (1+\vartheta)D^{'}(1)\Upsilon\bar{V}(1) + \end{cases}$$

$$2\lambda * (1+\vartheta)D^{'}(1)(\Upsilon+\delta)\bar{V}^{'}(1)\} - 2\lambda * (1+\vartheta)^3 \left(D^{'}(1)\right)^3 (\Upsilon+\delta)\bar{V}(1) \begin{bmatrix} (1-\dot{r})\bar{I}_2^{'}(\Upsilon) \\ +\dot{r}\bar{I}_1(\Upsilon)\bar{I}_2^{'}(\Upsilon) + \dot{r}\bar{I}_2(\Upsilon)\bar{I}_1^{'}(\Upsilon) \end{bmatrix} - (56)$$

$$C^{'}(1) = \Upsilon\delta[(1-\dot{r})\bar{I}_1^{'}(\Upsilon) + \dot{r}\bar{I}_1(\Upsilon)\bar{I}_2^{'}(\Upsilon)] - \begin{pmatrix} 1-(1-\dot{r})\bar{I}_1^{'}(\Upsilon) + \\ \dot{r}\bar{I}_1(\Upsilon)\bar{I}_2(\Upsilon) \end{pmatrix} [(\Upsilon+\delta)\lambda * (1+\vartheta)D^{'}(1)]$$

$$---(57)$$

$$C^{''}(1) = 2\Upsilon\delta(1-\dot{r})\bar{I}_1^{'}(\Upsilon) + \dot{r}(\bar{I}_1^{'}(\Upsilon)\bar{I}_2^{'}(\Upsilon) + \bar{I}_2(\Upsilon)\bar{I}_2^{'}(\Upsilon))] - (\Upsilon+\delta)\lambda * (1+\vartheta)D^{''}(1)(1-(1-\dot{r})\bar{I}_1(\Upsilon)$$
$$- \dot{r}\bar{I}_1(\Upsilon)\bar{I}_2(\Upsilon) - 2(\Upsilon+\delta)\lambda * (1+\vartheta)D^{'}(1)[1-(1-\dot{r})\bar{I}_1^{'}(\Upsilon) - \dot{r}(\bar{I}_1(\Upsilon)\bar{I}_2^{'}(\Upsilon) + \bar{I}_2(\Upsilon)\bar{I}_2^{'}(\Upsilon) --(58)$$

Then. If substitute the values of from the equation (55), (56), (57)& (58) into (54) we produce L_u in the- closed form. Then we find the average system size "L "using Little's law formula. Thus we get

$$LENGTH = L_u + \left[\frac{\lambda * (1+\vartheta)d^{'}(1)}{\delta[(1-\dot{r})I_1(\Upsilon) + \dot{r}I_1(\Upsilon)I_2(\Upsilon)]} + \frac{\lambda * (1+\vartheta)d^{'}(1)}{\Upsilon[(1-\dot{r})I_1(\Upsilon) + \dot{r}I_1(\Upsilon)I_2(\Upsilon)]} - \frac{\lambda * (1+\vartheta)d^{'}(1)}{\delta} - \frac{\lambda * (1+\vartheta)d^{'}(1)}{\Upsilon}\right]$$

where L_u have been established by the equation (54) and rho is produced from the equation (45).

6. Conclusion:

The encouraged arrival is very beneficial for a variety of organizations in terms of managing operations, planning, implementing and developing services for customers and other areas. In this study, a batch encouraged arrival Markovian queuing model due to a secondary optional- service, break-down and numerous vacations.The P.G.F in the line established due to the arbitrary variable methods. This method will be very much of the used in fabricate and tele-communication networks. It's comparatively Poisson arrival is encouraged arrival more effective.

Conflict of interest

No conflict of interest

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