Generalized Eccentricity K^{th} Power Sum Energy of Graphs

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Abstract
Let G be a finite, simple and undirected graph with m points and n edges. For any integer $1 \leq k < \infty$, generalized eccentricity k^{th} power sum matrix of G is a $m \times m$ matrix with its $(r,s)^{th}$ entry as $e^k{}_r + e^k{}_s$ if $r \neq s$ and zero otherwise, where e_r is the eccentricity of the r^{th} vertex of a graph G. In this paper, the new energy of graph the under the name as generalized eccentricity k^{th} power sum energy of G ($EGE^kS(G)$) has been introduced. Generalized eccentricity k^{th} power sum energy $EGE^kS(G)$ of some standard graphs has been obtained.

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1. Introduction

Huckel theory proposed a concept on energy in a graph which deals with conjugated carbon molecule. π - electron energy which is evaluated, whose value agrees with the energy of a graph. In discrete structures, adjacency matrix has many graph polynomials based on matrices such as degree sum matrix, distance matrix, Laplacian matrix, adjacency matrix. In this paper, generalized eccentricity k^{th} power sum matrix of G has been newly introduced.

Let G = (V(G), E(G)) be a finite, simple and undirected graph with |V(G)| = m vertices and |E(G)| = n edges. Let the points of G be labeled as $v_1, v_2, \ldots, v_m$. The distance d(x, y) between any two vertices x and y in a graph G is the length of the shortest x – y path. Eccentricity of a vertex is defined as the maximum distance between a vertex to all other vertices. The adjacency matrix of G is a $m \times m$ matrix whose (s, t)-entry is equal to one if the vertex v_s is adjacent to v_t, or else it is equal to zero [7].

In 1978, the concept energy of a graph G originated by I. Gutman [6]. Let G be a graph which containing m points and n edges and $C(G) = (c_{ij}) = \begin{cases} 1, \text{if } v_i v_j \in E \\ 0, \text{otherwise.} \end{cases}$

In 2018, B. Basavanagoud, E.Chitra, the concept of Degree Square Sum (DSS) matrix had been defined. Let $u_1, u_2, \ldots, u_m$ be the points of a graph G and let $d_j = deg_G(u_j)$. The Degree

Square Sum (DSS) matrix of G is an $m \times m$ matrix represented by DSS(G) = [dss_{jk}] and whose elements are determined as $dss_{jk} = {d_j}^2 + {d_k}^2$, if $j \neq k$ and zero otherwise [3].

In 2020, D.S. Revankar, M.M. Patil, B.S.Durgi and S.R.Jog, have defined the eccentricity sum matrix. A simple graph G which containing m vertices labeled as v_1,v_2, …, v_m. Let e_j be the eccentricity of v_j , j = 1, 2, 3…, m and ES(G) = [a_{ij}] is called the eccentricity sum matrix of a graph G, $a_{ij} = e_i + e_j$, if $i \neq j$ and zero otherwise [9].

Motivated by these papers, the concept of the generalized eccentricity k^{th} power sum matrix $GE^kS(G)$ of G has been imported and obtained the characteristic equation $PGE^kS(G)$ (λ) of the generalized eccentricity k^{th} power sum matrix of G.

Let G be a finite, simple and undirected graph with n vertices and m edges. For any integer $1 \leq k < \infty$, a graph G whose matrix is denoted by $GE^kS(G) = [ge^ks_{ij}]$ is determined as

$$ge^ks_{ij} = \begin{cases} e^k(v_i) + e^k(v_j), \text{if } i \neq j \\ 0 \text{ , otherwise.} \end{cases}$$

The characteristic polynomial of the generalized eccentricity k^{th} power sum matrix $GE^kS(G)$ is expressed by $PGE^kS(G)(\lambda)$ = det (λI_n – $GE^kS(G)$), where I_n is eccentricity n^{th} square sum unit matrix of order n×n and trace($GE^kS(G)$) = 0. The characteristic roots of $PGE^kS(G)(\lambda)$ are $\lambda_1, \lambda_2, \ldots, \lambda_n$ in a non-increasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ where λ_1 is largest and λ_n is smallest eigenvalues. If G has $\lambda_1, \lambda_2, \ldots, \lambda_n$, distinct eigenvalues related to multiplicities $m_1, m_2, \ldots, m_n$ then the spectrum can be written as Spectra(G) = $\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ m_1 & m_2 & \cdots & m_n \end{pmatrix}$. The generalized eccentricity k^{th} power sum energy of G is indicated by $EGE^kS(G)$ and it is determined as summing-up the absolute values of the characteristic roots of G, $EGE^kS(G) = \sum_{i=1}^{n} |\lambda_i|$. Generalized eccentricity k^{th} power sum energy of well-known graphs has been obtained.

2. Main Results

In this section, generalized eccentricity k^{th} power sum energy of some graphs has been obtained.

Theorem 2.1: If a connected graph G containing n points and $e(v_i) = e$, $1 \leq i \leq n$, then the characteristic roots of $GE^kS(G)$ are $-(2e)^k$ of multiplicity $(n-1)$ and $(n-1)(2e)^k$ of multiplicity 1 respectively, and $EGE^kS(G) = 2(n-1)(2e)^k$.

Proof: Let v_1,v_2, …, v_n be the vertices of a connected graph G and $e(v_i) = e$, $1 \leq i \leq n$.

Then, $ge^ks_{ij} = \begin{cases} e^k(v_i) + e^k(v_j), \text{if } i \neq j \\ 0 \text{ , otherwise.} \end{cases} = \begin{cases} (2e)^k, \text{if } i \neq j \\ 0 \text{ , otherwise.} \end{cases}$

Then $PGE^kS(G)(\lambda)$ = det (λI_n – $GE^kS(G)$

$$=(\lambda+(2e)^k)^{n-1}\begin{vmatrix} \lambda & -(2e)^k & \dots & -(2e)^k & -(2e)^k & -(2e)^k & \dots & -(2e)^k \\ -1 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ -1 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{vmatrix}$$

$PGE^kS(G)(\lambda) = (\lambda - (2e)^k(n-1))(\lambda + (2e)^k)^{n-1}$

The characteristic roots of $GE^kS(G)$ are $-(2e)^k$ of multiplicity $(n-1)$ and $(n-1)(2e)^k$ of multiplicity 1 respectively. Thus, $EGE^kS(G) = 2(n-1)(2e)^k$.

Hence, if a connected graph G containing n points and $e(v_i) = e$, $1 \le i \le n$, then the characteristic roots of $GE^kS(G)$ are $-(2e)^k$ of multiplicity $(n-1)$ and $(n-1)(2e)^k$ of multiplicity 1 respectively, and $EGE^kS(G) = 2(n-1)(2e)^k$.

Corollary 2.2: If a complete graph K_n($n\geq2$) then $EGE^kS(K_n) = 4(n-1)$.

Proof: Let K_n be the complete graph containing n vertices for all $n \geq 2$.

Since K_n is a connected graph with $e(v_i) = e = 1$, $1 \le i \le n$.

Then, $ge^ks_{ij} = \begin{cases} 1^k + 1^k, \text{if } i \neq j \\ 0 \text{ , otherwise.} \end{cases} = \begin{cases} 2(1)^k = 2, & \text{if } i \neq j \\ 0 & \text{, otherwise.} \end{cases}$

By theorem 2.1, The generalized eccentricity k^{th} power sum characteristic roots of K_n are -2 of multiplicity $(n-1)$ and $2(n-1)$ of multiplicity 1 respectively. Thus, $EGE^kS(K_n) = 4(n-1)$.

Hence, if a complete graph $K_n(n \geq 2)$ then $EGE^kS(K_n) = 4(n-1)$.

Corollary 2.3: If a complete bipartite graph $(K_{m,n})$ then $EGE^kS(K_{m,n}) = (2)^{2k+1}(m+n-1)$, for all $m, n \neq 1$.

Proof: Let G be the complete bipartite graph $(K_{m,n})$ which containing (m+n) vertices for all $m, n \neq 1$.

Since $K_{m,n}$ connected graph with $e(v_i) = e = 2$, $1 \le i \le m+n$.

Then, $ge^ks_{ij} = \begin{cases} 2^k + 2^k, \text{if } i \neq j \\ 0 \text{ , otherwise.} \end{cases} = \begin{cases} 2(2)^k = (4)^k, & \text{if } i \neq j \\ 0 & \text{, otherwise.} \end{cases}$

By theorem 2.1, The generalized eccentricity k^{th} power sum characteristic roots of $K_{m,n}$ are $-(4)^k$ of multiplicity $(m+n-1)$ and $(m+n-1)(4)^k$ of multiplicity 1 respectively, and $EGE^kS(K_{m,n}) = 2(m+n-1)(4)^k$. Thus, $EGE^kS(G) = (2)^{2k+1}(m+n-1)$.

Hence, if a complete bipartite graph $K_{m,n}$ then $EGE^kS(K_{m,n}) = (2)^{2k+1}(m+n-1)$, for all $m, n \neq 1$.

Theorem 2.4: If a connected graph G which containing n vertices and $e(v_1) = 1$, $e(v_i) = 2$, $2 \le i \le n$, then the generalized eccentricity k^{th} power sum eigenvalues of G are -2^{k+1}, $(n-2)2^k + \sqrt{(n^2-4n+4)2^{2k} + (n-1)(2^k+1)^2}$ and $(n-2)2^k - \sqrt{(n^2-4n+4)2^{2k} + (n-1)(2^k+1)^2}$ with multiplicities $(n-2)$, 1 and 1 respectively, and $EGE^kS(G) = (n-2)2^{k+1} + 2(n-2)2^k$.

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of a connected graph G and $e(v_1) = 1$, $e(v_i) = 2$, $2 \le i \le n$.

Then, $ge^k s_{ij} = \begin{cases} e^k(v_1) + e^k(v_j), \text{if } 1 \ne j \\ e^k(v_i) + e^k(v_j), \text{if } i \ne j \\ 0 \text{ , otherwise} \end{cases} = \begin{cases} 2^k + 1, \text{if } 1 \ne j \\ 2^{k+1}, \text{if } i \ne j \\ 0 \text{ , otherwise} \end{cases}$

Then $PGE^kS(G)(\lambda) = \det(\lambda I_n - GE^kS(G))$

$$=(\lambda + 2^{k+1})^{n-2} \begin{vmatrix} \lambda & -(2^k+1) & \cdots & -(2^k+1) & -(2^k+1) & -(2^k+1) & \cdots & -(2^k+1) \\ -1 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$= -(\lambda + 2^{k+1})^{n-2} (\lambda^2 - ((n-2)2^{k+1}\lambda - (2^k+1)^2(n-1))$$

$$= -(\lambda + 2^{k+1})^{n-2} \{ (\lambda + (n-2)2^k \pm \sqrt{(n^2-4n+4)2^{2k} + (n-1)(2^k+1)^2} \}$$

$$= -(\lambda + 2^{k+1})^{n-2} \{ (\lambda + (n-2)2^k + \sqrt{(n^2-4n+4)2^{2k} + (n-1)(2^k+1)^2} \}$$
$$\{ (\lambda + (n-2)2^k - \sqrt{(n^2-4n+4)2^{2k} + (n-1)(2^k+1)^2} \}$$

Thus, generalized eccentricity k^{th} power sum characteristic roots of G are -2^{k+1}, $(n-2)2^k + \sqrt{(n^2-4n+4)2^{2k} + (n-1)(2^k+1)^2}$ and $(n-2)2^k - \sqrt{(n^2-4n+4)2^{2k} + (n-1)(2^k+1)^2}$ with multiplicities $(n-2)$, 1 and 1 respectively.

Thus, $EGE^kS(G) = (n-2)\, 2^{k+1} + 2(n-2)2^k$.

Hence, if a connected graph G which containing n vertices and $e(v_1) = 1$, $e(v_i) = 2$, $2 \le i \le n$, then the generalized eccentricity k^{th} power sum characteristic roots of G are -2^{k+1}, $(n-2)2^k + \sqrt{(n^2-4n+4)2^{2k} + (n-1)(2^k+1)^2}$ and $(n-2)2^k - \sqrt{(n^2-4n+4)2^{2k} + (n-1)(2^k+1)^2}$ with multiplicities $(n-2)$, 1 and 1 respectively, and $EGE^kS(G) = (n-2)2^{k+1} + 2(n-2)2^k$.

Corollary 2.5: If a star graph S_n $(n \ge 2)$ then $EGE^kS(S_n) = (n-2)2^{k+1} + 2(n-2)2^k$.

Proof: Let S_n be the star graph with n vertices for all $n \geq 2$.

Since S_n is connected graph with $e(v_1) = 1$, $e(v_i) = 2$, $2 \leq i \leq n$.

By theorem 2.4, The generalized eccentricity k^{th} power sum characteristic roots of S_n are -2^{k+1},

$(n-2)2^k + \sqrt{(n^2-4n+4)2^{2k}+(n-1)(2^k+1)^2}$ and $(n-2)2^k - \sqrt{(n^2-4n+4)2^{2k}+(n-1)(2^k+1)^2}$

with multiplicities $(n-2)$, 1 and 1 respectively. Thus, $EGE^kS(S_n) = (n-2)2^{k+1} + 2(n-2)2^k$.

Hence, if star graph S_n $(n \geq 2)$ then $EGE^kS(S_n) = (n-2)2^{k+1} + 2(n-2)2^k$.

Theorem 2.6: If C_n is a cycle, $n \geq 3$, then $EGE^kS(C_n) = \begin{cases} 4(n-1)\left(\frac{n-1}{2}\right)^k, & \text{if n is odd,} \\ 4(n-1)\left(\frac{n}{2}\right)^k, & \text{if n is even.} \end{cases}$

Proof: Let G be the cycle graph C_n with n vertices v_i, $1 \leq i \leq n$, $n \geq 3$.

Then $e(v_i) = \begin{cases} \frac{n-1}{2}, & \text{if n is odd}, 1 \leq i \leq n, \\ \frac{n}{2}, & \text{if n is even}, 1 \leq i \leq n. \end{cases}$

Case(i): when n is odd, $n \geq 3$.

$PGE^kS(G)(\lambda) = \det(\lambda I_n - GE^kS(G))$

$$= -\left(\lambda + \frac{(n-1)^k}{2^{k-1}}\right)^{n-1} \begin{vmatrix} \lambda & -\frac{(n-1)^k}{2^{k-1}} & \cdots & -\frac{(n-1)^k}{2^{k-1}} & -\frac{(n-1)^k}{2^{k-1}} & -\frac{(n-1)^k}{2^{k-1}} & \cdots & -\frac{(n-1)^k}{2^{k-1}} \\ -1 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$= -\left(\lambda + \frac{(n-1)^k}{2^{k-1}}\right)^{n-1} \left(\lambda - (n-1)\frac{(n-1)^k}{2^{k-1}}\right).$$

Thus, the characteristic roots of $GE^kS(C_n)$ are $-\frac{(n-1)^k}{2^{k-1}}$ of multiplicity $(n-1)$ and $(n-1)\frac{(n-1)^k}{2^{k-1}}$ of multiplicity 1 respectively.

Thus, the generalized eccentricity k^{th} power sum energy of the cycle C_n when n is odd is $EGE^kS(C_n) = 4(n-1)\left(\frac{n-1}{2}\right)^k$.

Case(ii): when n is even, $n \geq 4$.

$PGE^kS(G)(\lambda) = \det(\lambda I_n - GE^kS(G))$

$$=(\lambda+\frac{n^k}{2^{k-1}})^{n-1}\begin{vmatrix} \lambda & -\frac{n^k}{2^{k-1}} & \dots & -\frac{n^k}{2^{k-1}} & -\frac{n^k}{2^{k-1}} & -\frac{n^k}{2^{k-1}} & \dots & -\frac{n^k}{2^{k-1}} \\ -1 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ -1 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{vmatrix}$$

$$=(\lambda+\frac{n^k}{2^{k-1}})^{n-1}(\lambda-(n-1)\frac{n^k}{2^{k-1}}).$$

Thus, the generalized eccentricity k^{th} power sum characteristic roots of C_n are $-\frac{(n-1)^k}{2^{k-1}}$ of multiplicity $(n-1)$ and $(n-1)\frac{(n-1)^k}{2^{k-1}}$ of multiplicity 1 respectively. .

Hence, the generalized eccentricity k^{th} power sum energy of the cycle C_n when n is even is $EGE^kS(C_n) = 4(n-1)(\frac{n}{2})^k$.

Hence, if C_n is a cycle, $n \geq 3$, then $EGE^kS(C_n) = \begin{cases} 4(n-1)(\frac{n-1}{2})^k, & \text{if n is odd,} \\ 4(n-1)(\frac{n}{2})^k, & \text{if n is even.} \end{cases}$

Theorem 2.7: If W_n is a wheel graph, $n \geq 4$, then $EGE^kS(W_n) = \begin{cases} 12, & \text{if } n = 4, \\ (n-2)2^{k+1} + 2(n-2)2^k, & \text{if } n \geq 5. \end{cases}$

Proof: Let $W_n = (V(G),E(G))$ be a wheel graph with n vertices where $V(G) = \{v_i : 1 \leq i \leq n\}$.

Case (i): when n = 4.

Let W_4 be a wheel graph with four vertices $\{v_1, v_2, v_3, v_4\}$.

Since W_4 is a connected graph with $e(v_i) = 1$, $1 \leq i \leq 4$.

By theorem 2.1, The generalized eccentricity k^{th} power sum characteristic roots of W_4 are -2 and 6 with multiplicities 3 and 1 respectively. Thus, $EGE^kS(G) = 12$.

Case (ii): when ≥ 5.

Since W_n, is connected graph $e(v_1) = 1$, $e(v_i) = 2$, $2 \leq i \leq n, n \geq 5$.

By theorem 2.4, The generalized eccentricity k^{th} power sum characteristic roots of S_n are -2^{k+1},

$(n-2)2^k + \sqrt{(n^2-4n+4)2^{2k}+(n-1)(2^k+1)^2}$ and $(n-2)2^k - \sqrt{(n^2-4n+4)2^{2k}+(n-1)(2^k+1)^2}$

with multiplicities $(n-2)$, 1 and 1 respectively. Thus, $EGE^kS(W_n) = (n-2)2^{k+1} + 2(n-2)2^k$.

Hence, if W_n is a wheel graph, $n \geq 4$, then $EGE^kS(W_n) = \begin{cases} 12, & \text{if } n = 4, \\ (n-2)2^{k+1} + 2(n-2)2^k, & \text{if } n \geq 5. \end{cases}$

3. Conclusion

In this paper, generalized eccentricity k^{th} power sum energy of a graph G has been newly defined. Generalized eccentricity k^{th} power sum energy of some standard graphs has been attained. Eccentricity sum energy [9] and degree square sum energy [3] of graph G have been introduced and some results have been proved for k = 1, 2 which has been extended to the $GE^kS(G)$ for $1 \leq k < \infty$. Analogous work can be also carried for other families of graphs.

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