Lattice Metric Space and Their Properties

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Abstract

In this paper, submitted defined of lattice metric space (LMS), and study basic properties to this space, after that provide set of new result about LMS and comparison with normed metric space.

Keywords: Metric space, lattice, vector lattice, complete, bounded, isomorphic, sequentially continuity, limit point, converge

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1-Introduction

Theory of lattice in the currentnotion was introduced be publishing Garrett Birkhoffs seminal book in 1940 since after that, it has been widely developed divisionthatis up to nowacquiescent new perceptions, applications and results. In its state as contemporary, there are several significanttheories of lattice applications, such as in algebraic non-classical logics semantics.

In [1]. He introduced the definition of lattice metric function, and tie axiom of this function, and he definition of open and closed ball. In [2]she introduced the definition of vector lattice.In[3,5,8].They presented lattice norm space. In[4,10].They presented Banach space[6,11]. They Introduced Symmetric function. In[7,12]. They presented operators on Bochner space. In [13]. He introduced the defintion of bounded lattice and the properties of bounded lattice and some theory about it. In [14] He Introduced definition of lattice and the axiom of lattice.The metric space was important concept in modern Mathematics and it is generalizatdion to concept distance and convergence in real number .The LMS is generalization to norm metric space.our paper, we provide definition,proposition, remarks and theorem and example in the context of metric space and linear metric space.

2-Vector lattice

In this section we proved concept of vector lattice and basic proprties related to it.

Definition (2.1) [14]:

A partially order set L is said to be lattice when each every element pair in Lhas infimum and supremum.

Definition(2.2) [14]:
Assume L is nonempty set closed under 2binarayoprationsnamed meet and goin defined,after that L is named lattice when the axiom as following hold in which
$x, y, z \epsilon L$.

1. Commutative law $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$
2. Associative law $(x \wedge y) \wedge z = x \wedge (y \wedge z)and(x \vee y) \vee z = x \vee (y \vee z)$
3. Absorption law $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$

Definition(2.3) [14]:
A lattice L is said to be complete lattice when eachL non-empty subset has an supremum and infimum.

Definition (2.4) [14]:A nonempty subset B of lattice L is known assub lattice when$inf\{x, y\}$, $\sup\{x, y\} \in B$ for wholex, $y \in B$.

Instance (2.5)
1. TheR set of all actual number withnormal relation ≤ is lattice, but not complete lattice. Since $\{x \ \epsilon \ R: x \geq \ 1\}$ is a subset of R which has no supremum.The N set of whole natural number and the Q set of whole relation number are sublattices of R. Since$inf \ \{x, y\}$,$sup\{x, y\} \epsilon \ N$ for all $x, y \ \epsilon \ N$, and $inf\{x, y\}, sup\{x, y\} \epsilon \ Q$.
2. Suppose p(x) is the set poweroff ofanonempty X set , so p(x) is the collection of wholeX subset , after that p(x) is partially ordered regarding the relation ⊆ if A,B ϵ p(x) , after that inf{A,B} A ∩ B ,sup{A , B} = A ∪ B ϵ P (x) . Hence p(x) is a lattice. F is the wholeactualset valued function defiened set X . After that F is partially ordered by relation.

3. let ≤ defiened by setting f ≤ g if f(x) ≤ g(x)for wholexϵX. When f, g ϵ F , after that inf {f, g} =min {f(x),g(x)} , sup{f, g} = max {f(x),g(x)} .Hence F is a lattice .

Definition(2.6) [14]:

Whicheverdual statement in alattice (L, ∧, ∨) is defiened to be a steatmentwhich is gainedthrough ∧ and ∨ interchanging

Example (2.7)

The $x \wedge (y \wedge z)$dual $= x \vee x \ is \ x \vee (y \vee z) = x \wedge x$

Defintion (2.8)[13]:

A lattice L is named bounded when has highest element 1 and asmallest amount element 0

Example (2.9)

1. Thep(s) as a set power under the intersection and union operation is a bounded lattice as long∅ is the smallest p(s)element, and the S set be the highest p(s)element.

2. The+ve integer $\mathbb{Z}^+$set under the ≤ as normal order is unbound lattice as long it possesselement I nonetheless the highest element is not existing .

Bounded lattice Properties

When L is a lattice being bounded,after that for whicheverx ϵ L element,the identities as follow weget:1.$x \vee 1 = x$

2.$x \wedge 1 = x$

3.$x \vee 0 = x$

4.$x \wedge 0 = 0$

Theorem(2.10)

Every finite lattice is bounded.

Proof:

Let L be any finite lattice , i.e $L=\{x_1 ,x_2 ,\dots x_n \}$ .Thus the highestlattice L element be $x_1 \vee x_2 \dots . \vee x_n$.Likewise, the minimumlattice L elementbe $x_1 \wedge x_2 \dots . \wedge x_n$. As long the highest and minimum elements occur for lattice as finite. Therefore L be bounded.

Definition(2.11) [13]:

Conisder anonempty a lattice L subset L_1, after that L_1 is namedA L sublattice when L_1 itself be alattice , i.e. The L operation such as$x \vee y \epsilon L_1$ and $x \wedge y \epsilon L_1$at any time $x , y \epsilon L_1$.

Example (2.12)

Deliberate all positive integer Z^+ lattice under the of divisibilityoperation. The lattice D_n of wholen > 1 divisors is a Z^+sublattice. Governwhole the D_{30}sublattice which have aminimumof 4 elements $D_{30} =\{1,2,3,5,10,15,30\}$.

Solution:

The D_{30}sub-lattice which have a minimum of 4 elements as following1.{1,2,6,30} 2.{1,2,3,30} 3.{1,5,15,30} 4.{1,3,6,30} 5.{1,5,10,30} 6.{1,3,15,30} 7.{2,6,10,30}

Defintion (2.13)[14]:

Tow lattice L_1 and L_2 are named isomorphism lattice whena bijection is there from L_1 to L_2. Such as f: $L_1 \rightarrow L2$,thus $f (x \wedge y) = f (x) \wedge f(y)$and $f (x \vee y)=f(x) \vee f (y)$ for wholex,y ϵL_1

Defintion (2.14)[2]:

Let L be a linear space over field F we say that L is lattice if max {x,y}, min {x,y} ϵ L for wholex,y ϵ L , Alinear is named avector lattice.

Theorem (2.15)

Assume L is aspace as linear after thatL is avector lattice if max {x,0} ϵ L for whole X in L

Proof:

Supposex,y ϵ L , as long L is alinear space, after that x-y ϵ L, thus max {x –y ,0} ϵ L. Hence max {x,y} ϵ L .Similartyx+y ϵ L and max {x+y,0 } ϵ L some in {x,y} ϵ L

Definition(2.16) [13]:

Suppose Vis a lattice as vector over ℝ, and V^+ be its $+^{ve}$cone. Their functions expressed to V^+ from Vas following,for whichever x ϵ V,

1. $x^+ = x \vee 0$.

2. $x^- = (-x) \vee 0$.

3.$x\ | = (-x) \vee x$.

It is simple to observe suchroles are well–definite. Further down are few3 functions assets:

1. $x^+ = (-x)^-\ and\ x^- = (-x)^+$

2.$x = x^+ - x^-$as longx$x^+ - x^- = (x\vee 0) - (-x)\vee 0 = (x\vee 0) + (x\wedge 0) = x + 0 = x$.

3. $|x| = x^+ + x^-$,as long

$$x^+ + x^- = x{+}2x^- = x + (-2x)\vee 0 = (x - 2x)\vee(x + 0) = |x| .$$

4.$If\ 0 \le x$,Afterthat$x^+, x^- = 0$, andx $| = 0$. Also, $x \le 0$, implies $x^+ = 0$

$$x^- = -x \text{and } |x| = -x.$$

5. $|x\ | = 0$ iff x =0 . |x | =0 , after that (-x) ⋁ x = 0, thus x≤ 0 and –x ≤ 0 .Nonethelessafter that 0 ≤ x,

so x = 0.

6.$|rx| = |\ r||\ x|$for any rϵR. If 0≤ r , after that 0 ≤ r , after that

$|rx\ | = (-rx) \vee (rx) = r\big((-x) \vee x\big) = r\ |x\ | = |r\ |\ |x\ |$Conversely,when r ≤ 0 , after that

$$|rx\ | = (-rx) \vee (rx) = (-r)\big(x \vee (-x)\big) = -r\ |x\ | = |r\ |\ |x\ |.$$

7. $|x\ | + |y\ | = |x + y\ | \vee |x - y\ |$, as long LHS = (-x) ⋁ x + (-y) ⋁ y = (-x –y) ⋁ (z – y) ⋁ (x +y) = RHS.

8. $|x + y| \leq |x| + |y|$, since $|x + y| \leq |x + y| \vee |x - y| = |x| + |y|$

3-LMS

Forsuch part, we proved the LMSconcept, and some properties of LMS, and we study basic properties of LMS,and after that we proved new result related with concept of LMS.

Definition (3.1) [1]:

Assume X be anE set as non-empty isactualset numbers. A function

d :X× X →E

Is named a metric function whenfulfills the axioms as follows:

1.$d(x, y) \geq 0$for all $x, y\epsilon X$.

2.$d(x, y) = 0$ iff $x = y$ to the whole $x, y\epsilon X$

3.$d(x, y) = d(y, x)$for all $x, y\epsilon X$.

4.$d(x, z) \leq d(x, y) + d(y, z)$for all $x, y, z\epsilon X$.

(x,d,E) LMS.

Theorem(3.2) Each metric space is metric oflattice .

Evidence

Assume (x,d) is a space as metric.Define d :X xX → E by d(x,y)=x-y for wholex,y ϵ X

Let $x , y \epsilon X \Rightarrow x - y \epsilon X$ (becaue X is lattice vector space) .1

$\Rightarrow x - y \epsilon X \ d(x , y) \geq 0$.

2. Let $x , y \epsilon X$, after that $d(x , y) = 0 \Leftrightarrow x - y = 0 \Leftrightarrow x = y$

Suppose $x, y \epsilon X \Rightarrow d(x, y) = x - y = y - x = d(y, x)$.

3. Suppose $x, y, z \epsilon X$, after that $x - y = (x - z) + (z - y) \leq x - z \Rightarrow d(x, y) \leq d(x, z) + d(z, y)$

Example (3.3)

1. Assume X is a non-zero linear space,d is a discrete lattice metric function on X , i .e.

$$d(x,y)=\begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

2.Assume $d_u : X \times X \rightarrow E$ be a function define as $d_u(x,y)=|x-y|$ for wholex,y ϵ X after that $d_u$be a metric function E , and (E,d_u) named usual metric space .

3.Assumed:X × X →E defin as d(x,y) = |x-y|+1 for wholex,yϵ X , not to be ametric function on E .

[1]:(3.4)Definition

AssumeX is metric space

1.The ball as open of center $x_0 \epsilon$ X and radius r > 0 signifiedthrough

$\beta_r(x_0)$and define as$\beta_r(x_0) = \{x\epsilon X: d(x,x_0) < r\}$and the ball as closed is

$\beta_r(x_0) = \{x\epsilon X: d(x,x_0) \leq r\}$.

AnA subset of X is considered as being bounded if itbe present k >0 as |x| ≤ k for whole x ϵ A

Remarks

1.Every open ball and closed ball are nonempty sets because $x_0\epsilon$ β_r (x_0),

$x_0\epsilon$ βr (x_0).

2.$\beta_r(x_0) = x_0 + \beta_r(0) = x_0 + r\beta_i(0)$

Indeed

$$\beta_r(x_0) = \{x\epsilon X: d(x,x_0) < r\}\{x_0 + \text{y}: \ |\text{y}\ | < r\} = x_o + \{y: \ |y\ | < r\}$$

$$= x_0 + \beta_r(0)$$

$$\text{Also}\beta_r(0)\{x \in X: \ |x\ | < r\} = \left\{x \epsilon X: \frac{|\text{x}\ |}{r} < 1\right\} = r\{y: \ |y\ | < 1 = r\beta_1(0)$$

Let X be metric space. An A as subset is considered as an set as an openwhenspecifiedwhichever point, x ϵA , ithappens r > 0 so, $\beta_r(x) \subseteq$A , and A is named a closed set if A^c is set as open.

Definition(3.5)[1]:

Let τ say that τ is anon X Topology when it fulfills the axioms as follow:

.ϕ , $x\epsilon\tau$ 1

If$A_1, A_2, A_3, \dots, A_n \epsilon\tau$, after that $\bigcap_{i=1}^{n} A\ i\epsilon\tau$. 2.

.If$A_\lambda \epsilon\tau$ for all $\lambda\epsilon\ \wedge$, after that $\bigcup A_\lambda\ \epsilon\tau$.3

Remark

Since every metric lattice space is a spacetopology isnamed a metric topology on X , the space X is named the metric topological space.

Theorem(3.6)

Assume (x,d) be a space as metric

1. Allof X,∅besets beingopen in X.

$\bigcap_{i=1}^{n} Ai$will be set being open in Xafter that ,after thatset in X2.If $A_1, A_2..A_n$ be open ,after that$\bigcup A_\lambda$ will be set as open in x . .IfA_λ ,∀ λ ϵ ∧ set as an open in X3

Evidence:

1.Let ∅ not set as an open⇒ there exist $x\epsilon\emptyset \ni \beta_r(x) \subseteq \emptyset \; \forall r > 0$

and this impossible because Ø don't contain element ⇒Ø set as an open.

As longβ_r (x) ⊆ X ∀ x ϵ X , r > 0 ⇒ X set as an open .

2.Let $A_1, A_2 .....A_n$ set as open in X and let $x\epsilon \bigcap A_i \Rightarrow x\epsilon A_i \forall_i = 1,2, \dots, n$

Since A_i is set as an open in X $\forall_i$ =1,2….n⇒After thatexist $r_i > 0 \; \forall_i = 1,2..n$

$\beta_{ri}(x) \subseteq A_i \forall_i = 1,2 \dots$ So

put $r = \min\{r_1, r_2, .. r_n\} \Rightarrow \beta_r(x) \subseteq \beta_r(x) \forall i = 1,2 \dots \beta_r(x) \subseteq A_i$

$$\beta_r(x) \subseteq \cap A_i \Rightarrow \cap A_i \text{ set as an open in } X.$$

3.Let A_λ set as an open in X for all λ ϵ ∧ , and let $x\epsilon \bigcup_{\lambda\epsilon\wedge} A_\lambda \Rightarrow x\epsilon A_\lambda$for some λ ϵ ∧ .

Since A_λ set as an open in X ⇒ there exist r > 0 , so, β_r (x) ⊆A_λ⇒

$$\beta_r(x) \subseteq \cup A \Rightarrow \cup A_\lambda \text{ set as an open in} X.$$

Formula(3.7)

Suppose (x , d) isspace asmetric

1.X , Ø set as an closed in X .

2.if $A_1 , A_{2, ...} A_n$ are set as closed in X , after that$\bigcup A_i$ is set as an closed in X .

3.if $A_\lambda$for all λϵ∧ set as an closed in X , after that$\bigcap_{\lambda\epsilon\wedge} A_\lambda$ is set as closed in X .

Proof:

1.As long$Ø^c$ =x , and since x set as an open in x ⇒$Ø^c$ set as an open in x ⇒ Ø set as an closed in x, since x^c =Ø and Ø set as an open in x ⇒ x^c set as an open in x ⇒ x set as an closed in x .

2.Let $A_1, A_2 ….A_n$ set as an closed in x ⇒A_i^c set as an open in X for every i=1 , 2 ,…⇒

$\cap A_i^c$ set as an open in X$(\cap A_i^c)^c = \cup A_{An}$ set as an closed in X .

3.let A_λ set as an closed in X for each $\lambda \epsilon \wedge \Rightarrow A^c_\lambda$ set as an open in x for every

$$\lambda\epsilon \wedge\Rightarrow \cup^c A_\lambda$$

set as an open in X $\Rightarrow (\cup A_\lambda)^c = \cap A_\lambda$ set as an closed in X .

Theoreom (3.8)

Suppose (X , d_1) , (Y , d_2) isspace as metric , let f : X→ Yfunction as unbroken . When (z , d_z) is a subset from space as metric (x , d_1),after thatf_z the function be limited on Z will be unbroken.

proof

Let f_t : Z →Z the function is limited f on Z⇒f(x) = f_z(x) for whole x ϵ Z

Let G be set as an open in Y, as long f is continues function ⇒ f^{-1} (G) is set as an open in X .⇒ $z \cap f^{-1}(G)$ set as an open in Z

Since $f_z^{-1}(G) = z \cap f^{-1}(G)$from defin f_z ⇒ the set f_z^{-1} (G) open in Z⇒f_z is countinous

Definition(3.9)[1]:

Suppose (X,d_1) , (Y,d_2) be space as metric we say that the function

f :X → Y be a sequentially continuity at the x0 pointϵ x , if each sequence $\{x_n\}$ in X so$x_n \to x_0$, after that f (x_n) → f (x_0) in Y

Theorem (3.10)

Suppose (X, d1), (Y, d_2) isspace asmetric,after that thef function: X→ Yunbroken at the x0 ϵ X pointiff the function is sequentially continuity at the point

Proof

Suppose the f function is unbroken at the x0 point,assume $\{x_n\}$ is asequnce in X , so $x_n \to x_0$

We must prove f(x_n) → f (x_0) :

Let ϵ >0 , as long f is unbroken at the x0 point⇒It exist δ > 0 ,

So, every$x\epsilon X, d_1(x,x_0) < \delta \Rightarrow d_2(f(x), f(x_0) < \epsilon$

Since $x_n \to x_0$, δ > 0 ⇒ there exist k ϵ $\mathbb{Z}^+$ so d_1 (x_n , x_0) <δfor every

$$n > k, then\ d_2\big(f(x_n), f(x_0)\big) < \epsilon \forall\ n > k$$

From this we prove f(x_n) ⇒f(x_0) , that mean the ffunctionbe sequentially continuity at x0 pointϵ x .

Conversely,assume the f function be sequentially continuity at the x0 point ϵx and we prove f unbroken at x0 point.

now we prove by contradiction assume f is not unbroken at x0 point⇒ there exist ε>0 so, δ >0 , there exist x ϵ X and $d_1(x_n, x_0) < \delta \Rightarrow$

$$d_2\big(f(x_n), f(x_0)\big) \geq \varepsilon \Rightarrow \forall n \epsilon \mathbb{Z}^+, there exist x_n \epsilon X$$

$$such\ that\ d_1(x, x_0) < \frac{1}{n} \Rightarrow d_2\big(f(x), f(x_0)\big) \geq \varepsilon$$

so that mean$x_n \rightarrow x_0 in\ X\ but\ f(x_n) \nrightarrow f(x_0)$in Y and this contradiction ⇒ f *unbroken at* x_0.

Definition (3.11) [1]:

Alattice metric linear space X is considered as normablewhen the lattice metric function is induced by a metric.

Theorem(3.12)

Assume X isspace as metric

.Eachball as open in X is set as an open1 .

.2.Every ball as closed in X is closed

AnX subset is open iff it beopen balls family union3.

.Any finite X subset is closed4 .

Evidence:

. Suppose x_0 ϵ X and r >0 . To prove $\beta_r(x_0)$ is set as an open 1

$$\text{Let } x\epsilon\beta_r(x_0) \Rightarrow |\ \mathrm{x} - x_0| < r \Rightarrow r - |\ \mathrm{r} - x_0| > 0$$

$$\text{put} r_1 = r - |x - x_0| \Rightarrow r_1 > 0.\text{To prove} \beta_{r1}(x) \subseteq \beta_r(x_0)$$

$$Let\ y\epsilon\beta_1(x) \Rightarrow |y - x| < r_1 \Rightarrow |y - x| < r - |x - x_0| \Rightarrow |\mathrm{y} - \mathrm{x}| + |\mathrm{x} - x_0| \quad < r$$

$$Since\ |y - \ x_0| \leq |y - x| + |x - x_0| \Rightarrow d(y, x_0) < r$$

$$y\epsilon\beta_r(x_0) \Rightarrow \beta_r(x_0) is open\ set.$$

2.$Let\ A = (\beta_r(x_0))^c. To\ prove\ A\ is\ open$

$$since\ \beta_r(x_0) = \{x\epsilon\mathrm{X}:\ |\mathrm{x} - x_0\ | \leq \mathrm{r}\}, \text{after that A} = \{\mathrm{x}\epsilon\mathrm{X}:\ |x - x_0\ | > r\}$$

$$Letx\epsilon\mathrm{A} \Rightarrow\ |\mathrm{x} - x_0\ | > r. Put\ r_2 =\ |\mathrm{x} - x_0\ | - r \Rightarrow r_2 > 0.\text{To prove}$$

$\beta_{r2}(x) \subseteq A$

$Let\ y\ \ \beta_{r2}(x) \Rightarrow\ |y - x\ | < r_2 \Rightarrow\ |y - x\ | <\ |x - x_0\ | - \mathrm{r} \Rightarrow\ |\mathrm{x} - x_0\ | -$ $|y - x\ | > r$

Since $|x - x_0\ | \leq\ |x - y\ | +\ |x - x_0\ | -\ |y - x\ | \leq\ |y - x_0\ |$

$\Rightarrow\ |y - x_0\ | > r \Rightarrow y\epsilon\, A \Rightarrow \beta_{r2}(x) \subseteq A \Rightarrow A$ is an set as an open⇒A^C

$= \beta_r(x_0)$is set as an closed .

3. If A = Ø the proof ends . If A ≠ Ø.

Suppose A is open in X,after that for all x ϵ A , there is r_x>0 so,

$$B_{rx}(x) \subseteq A \Rightarrow A \subseteq \bigcup_{x\epsilon A} B_{rx}\ (x) \subset A \Rightarrow A = \bigcup_{x\epsilon A} B_{rx}\ (x) \Rightarrow A$$

is the open balls union.

Conversely,assume: A be the open balls union

Since every open ball is set as an open, and the set as an openunion is set as an open,after thatA is set as an open.

4. Let A ={a}. To prove A is closed

$Let\ x\epsilon A^c \Rightarrow x \neq a \Rightarrow |a - x| > 0. Put\ r = d(a,x) \Rightarrow r > 0$

As long $\mathrm{a - x \geq r \Rightarrow a} \notin \beta_r(x) \Rightarrow \beta_r(x) \cap A = \emptyset \Rightarrow \beta_r(x) \subset A^c \Rightarrow A^c$

Is set as an open, after that A is closed

Let B be a finite set if B = Ø , the end proof , if B ≠ Ø , after that

B = {b_1,….,b_n},

Since {b_1} is closed for everyi=1,2,….., n , after that B = ⋃{b_i} is closed.

Definition (3.13) [1]:

Assume Xisspace as metric, and let A ⊆ X

1. Wholesets as an openunion in X contained in A is named theA interior, signified by int (A).

i.e. $int(A) = \bigcup\{B \subseteq X : B\epsilon T, B \subseteq A\}$. Thus int (A) is the biggest set as an open contained in A.

And $int(A) \subseteq A.$ Hence $int\ (A) = \{x : \epsilon A : \exists r > 0, \beta_r(x) \subseteq A\}, (x\epsilon A : \exists r > 0,$

$x + rB_1(0) \subset A\}$

2-All the set as an closed s intersection have A is named the A closuresignified by Ā.

i.e.$\overline{A} = \cap\{B \subseteq X: B^c \epsilon\, T, A \subseteq B\}$. Thus Ā is the least set as an closed containing A , and A ⊆ Ā.

$$\text{Hence}\overline{A} = \{x\epsilon X: \forall\, r > 0, \exists\, y\epsilon A \ni |x - y| < r\}, \overline{A} = \bigcap_{r>0}(A + rB_1\,(0))$$

3.Anx ϵ X point isnamed a limit A point whenevery set as an open G in X so, x ϵ G and A ∩ (G |{x}) ≠ Ø . All limit A points set is denoted by Ά and isnamed the derived set of A .

$$Hence\ \acute{A} = \{x\epsilon X: \forall\, r > 0, \exists\, y\epsilon A \ni y \neq x, |x - y\,| < r\}$$

4. The boundary of a subset A is defined as the difference between the closure and the subset A interior, i.e. ð (A) = Ā ∩ (int (A))^c .Hence

$$ð(A) = \{x\epsilon X: \forall\, r > 0, \exists\, y\epsilon A\, , z\epsilon A^c \ni |x - y| < r\, , |x - z| < r\}$$

5.The A exterior is the complement of Ā and signified by ext (A) , i.e.

$$Ext\ (A) = (\overline{A})^c\ .$$

Theorem (3.14)

Suppose X be space asmetric .If M is a subspace of X , after that M is subspace of X.

Proof:

$$Since\ 0\ \epsilon\ M \Rightarrow M \subset M \Rightarrow 0\ \epsilon\ M, so\ M \neq \emptyset$$

Let $x, y\ \epsilon\ M$ and $\alpha, \beta\epsilon F$. To prove $\alpha\, x + \beta y\ \epsilon M$

Let $r > 0$

1.If $\alpha \neq 0$ and $\beta \neq 0$, after that $\frac{r}{2|\alpha|}$ and $\frac{r}{2|\beta|} > 0$

There exist a , b ϵ M so

$$|\, x - a\, | < \frac{r}{2|\alpha|} \text{and} |\, y - b| < \frac{r}{2|\beta|}$$

Since M is subspace and a ,b ϵ M , after that α a + β b ϵ M

$$(\alpha x + \beta y) - (\alpha a + \beta b)| = \alpha(x - a) + \beta(y - b)$$

$$|(\alpha x + \beta y) - (\alpha a + \beta b)| \leq |\,\alpha| \quad |\, x - a| + |\,\beta||\, y - b\,| < |\alpha\,| \frac{r}{2|\alpha|} + |\beta\,| \frac{r}{2|\beta|} = r$$

$$\Rightarrow \alpha x + \beta y\ \epsilon\ M.$$

-If $\alpha = 0$ and $\beta = 0,$ after that $\alpha x + \beta y = 0 \epsilon M \subset M2$

-If $\alpha = 0 \; or \; \beta = 0,$ after that $\alpha x + \beta y = \beta y \; or \; \alpha x + \beta y = \alpha x,$ so $\alpha x + \beta y \epsilon M$ 3

Hence M is subspace of X.

Definition(3.15) [1]:

Assume X is aset as non-empty. A series in X is any function from $\mathbb{N}$ (all natural numbers set) into X. When f is a series in X , the image f (n) of $n \in \mathbb{N}$ is usually , signified by x_n . It is ordinary to signify the classical sequence of the symbol $\{x_n\}$.So, $\{x_n\}$ the actual numbers sequence if X =E. Sometime, we write it as $\{x_1 ,x_2 ,\ldots\ldots,x_n,\ldots\ldots\}$.The n image x_n is named the nth term of the sequence .

Note that, there is difference between the sequence and its range . For example ,the rang of the order $\{(-1)^n\}$is $\{x_n : n \in \mathbb{N}\}=\{-1,1\}$but the sequence is $\{x_n\}=\{(-1)^n\} =\{1,-1,1,-1 \ldots.\}$.

Definition (3.16)[1]:

An$\{x_n\}$ sequence in space asX metric isconsidered as

1.Converge to the $x \in X$ point ,Whenlim d (x_n ,x) =0 ,such as. When for each $\varepsilon >0$, it be present as $k \in \mathbb{Z}^+$ so, d (x_n, x) $<\varepsilon$ for whole $n \geq k$. and we write lim$x_n \rightarrow x$ or $x_n = x$ as $n\rightarrow\infty$ it tracks that $x_n \rightarrow x$ iff $d(x_n , x)\rightarrow 0$

2. Sequence of Cauchyin X, when for every $\varepsilon > 0$

, it be present as $k \in \mathbb{Z}^+$ so, $d(x_n, x_m) < \varepsilon$ for all $n,m \geq k$

4. Bounded,when there is M > 0 so, $|x_n| \leq M$ for all n.

Theorem (3.17)

AssumeX is a lattice normed space and let $A \subseteq X$

1. Limit point of sequence is unique.

2. Every convergence lattice arrangement is sequence of Cauchy, nevertheless the converse not correct.

3. Eachsequence of Cauchy is bounded, lattice but the converse not true

4.$x \in \bar{A}$ when f it be present as anarrangement $\{x_n\}$ in A so $x_n\rightarrow x$

5. When a sequence of Cauchy in X has a convergent sub-sequence, after that the sequence is convergent.

6. When $\{y_n\}$ and $\{x_n\}$ are sequences of Cauchyin X,after that d (x_n-y_n) is convergent in E

Proof:

1.Assume $\{x_n\}$ is anX sequence so, $x_n \to x$ and $x_n \to y$

$$\text{If } x \neq y, then\ d(x-y) > 0. \text{put} |\ x-y| = \varepsilon \Rightarrow \varepsilon > 0$$

Since $x_n \to x$, after that it be present as $k_1 \in \mathbb{Z}^+$ so $|x_n - x| < \frac{\varepsilon}{2}$ for all $n > k_1$

Also since $x_n \to y$, after that it be present as $k_2 \in \mathbb{Z}^+$ so $|x_n - y| < \frac{\varepsilon}{2}$ for wholen $> k_2$

$|x_n - x| < \frac{\varepsilon}{2}$, $d(x_n , y) < \frac{\varepsilon}{2}$ Taking $k = \max\{k_1 , k_2\}$, we get

For whole $n > k$

$$\varepsilon = x - y \leq x_n - x + x_n - y < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

This contradiction . Hence $x = y$

2. Assume $\{x_n\}$ be a converge X sequence, after that it be present as $x \in X$ so

$x_n \to x$ suppose $\varepsilon > 0$

For whole $n > k$ Since $x_n \to x$, after that it be present as $k \in \mathbb{Z}^+$ so, $|x_n - x| < \frac{\varepsilon}{2}$

$$\text{If } n, m > k, \text{after that} |x_n - x| < \frac{\varepsilon}{2} , |x_m - x| < \frac{\varepsilon}{2}$$

$$|x_n - x_m| \leq |x_n - x| + |\ x_m - x\ | < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Therefore $\{x_n\}$ is sequence of Cauchyin X.

3. Let $\{x_n\}$ be a sequence of Cauchy in X

Let $\varepsilon = 1$, after that it be present as $k \in \mathbb{Z}^+$ so, $|x_n - x_m| < 1$ for whole $n , m > k$

After that, for all $n \geq k$, $|x_n| = |x_n + x_k| + x_k \leq |x_n - x_k| + |\ x_k\ | < 1 + |x_k\ |$

Take $r = \max\{|\ x_1| , |x_2| , \ldots, |x_{k-1}| , 1 + |x_k|\}, then\ |x_n| \leq r\ for\ all\ n.$

Therefore $\{x_n\}$ is a sequence being bounded in X

. Suppose $x \in \bar{A}$4

Since $\bar{A} = A \cup A'$, after that $x \in A \cup A'$,either $x \in A$ or $x \in A'$

If $x \in A$, after that $\{x\}$ is a sequence in A so x

Or $x \in A'$,after that $A \cap (B_r(x)\ \ |\{x\}\ \neq \emptyset$

$\mathbb{Z}^+$ For whole n (x)| {x}X_n ϵ A ∩ (β

After that {x_n} is a sequence in A .TO prove $x_n \to x$

x ϵ Ā When f it be present as a sequence {x_n} so, $x_n \to x$

< ε Let ε > 0 , after that there is k ϵ $\mathbb{Z}^+$so$\frac{1}{k}$

$$\text{Since} x_n \in \beta\ (x)|x_n - x\ | < \frac{1}{n} \Rightarrow \text{for whole } n \epsilon \mathbb{Z}^+$$

Let $n > k \Rightarrow \frac{1}{n} < \frac{1}{k} \Rightarrow |\ x_n - x| \ < \frac{1}{k} < \ \varepsilon \Rightarrow x_n \to x$

Conversely,consider {x_n} is a sequence in A so, $x_n \to x$

To prove x ϵ Ā , i.e.x ϵ A ∪A'.If x ϵ A after that x ϵ Ā

Or x A , Iet G be set as open in X so, x ϵ G , after that there is r >0

So B_r (x) ⊆ G

Since r >0 and $x_n \to x$ after that there is k ϵ $\mathbb{Z}^+$so|x_n-x| <r for whole n > k

⇒x_n ϵ B_r (x)for all n >k , Since x_n ϵ A for whole n ϵ $\mathbb{Z}^+$⇒A ∩(B_r (x) |{x}) ≠ ∅

$$\text{Since} B_r(x) \subset G \Rightarrow A \cap (G|\ \{x\}) \neq \emptyset \Rightarrow x \epsilon A' \Rightarrow x \epsilon \overline{A}$$

5. Suppose {x_n} is a sequence of Cauchy in a space as metric (X , d) and suppose {$x_{i\ n}$}be a subsequence of {x_n}converging to $x_0 \epsilon\ X, i.e. x_{in} \to 0$ .To prove $x_n \to x_0$

Let ε >0 , since {x_{in} }is converge ⇒{x_{in} }is a sequence of Cauchy , after that there is

$k \epsilon \mathbb{Z}^+$so$|\ x_{in} - x_{im}| < \varepsilon$ for all n , m ≥ k

Since {i_m } is an increasing strictly positive integers sequence

Making m→∞ , we have $d(x_n, x_0) < \varepsilon\ for\ all\ n \geq k \Rightarrow x \to x_0$

Definition(3.18)[1]:

A space as metric X is complete if eachsequence of Cauchy is converges the point x ϵ X

Theorem (3.19)

Eachcomplete space metric subspace is complete when f is closed

Evidence

Assume M is a sub-space of a complete space as metric X

Suppose that M is complete. To prove M is closed

Suppose x ϵ M,after that a {x $_n$}sequence in M so, $x_n \rightarrow x$, hence {x_n} be

a sequence of Cauchy in M , since M is complete , thus is y ϵ M as long $x_n \rightarrow y$, but the limit is unique ,$y = x \Rightarrow x \epsilon M \Rightarrow M \subseteq M$ but $M \subseteq M$,after thatM = M . Hence M isclosed.

Conversely, suppose that Mbe closed

Assume {x_n} be a sequence of Cauchy in M , as long$M \subseteq X \Rightarrow \{X_n\}$be a sequence of Cauchy in X as long X is complete , after that there is x ϵ X so $x_n \rightarrow x$,

since x_n ϵ M , after that x ϵ M Since M is closed , $then$ $M = M \Rightarrow x\epsilon M \Rightarrow \{X_n\}$

converge in M , after that M is complete space .

Theorem (3.20)

Suppose{y_n} and {x_n} are 2 sequences in space as metric X so, $x_n \rightarrow x$ and $y_n \rightarrow y$

1.$x_n + y_n \rightarrow x + y$

2.$\lambda x_n \rightarrow \lambda x$ for all λ ϵ F

3.$| x_n| \rightarrow | x|$

4.$| x_n - y_n| \rightarrow | x - y|$

5.If $\{\lambda_n\}$is a sequence in F so$\lambda_n \rightarrow \lambda$, after that$\lambda_n x_n \rightarrow \lambda x$

proof:

1.$|(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)| \leq |y_n - y| \leq |x_n - x| + | y_n - y|$
2.$\lambda x_n \rightarrow \lambda x$
$|\lambda x_n| = | \lambda| \, | x_n|$, $since| x_n|$ →0 and n→∞,
after that λ |xn| → 0 as n→∞

$$| \lambda x_n| \rightarrow |\lambda x |$$

$$\lambda x_n \rightarrow \lambda x$$

3.Since| $|x_n|$−|x| | ≤ $|x_n - x|$ and$| x_n - x|$ →0 as n→∞, after that
$| |x_n| - |x| |$ →0 as n→∞,i.e. $| x_n|$ → $| x|$
4. $| |x_n - y_n| - |x - y| | \leq | (x_n - y_n| - (x - y) \leq |x_n - x| + |y_n - y|$
Since $|x_n - x| \rightarrow 0$ and $|y_n - y| \rightarrow 0$ as n → ∞,after that| $|x_n - y_n|$
$-| x - y| \rightarrow 0$ as n → ∞

5. $|\lambda_n x_n - \lambda x| = |\lambda_n x_n - \lambda_{nx} - \lambda x| = |\lambda_n (x_n - x) + (\lambda_n - \lambda) x| \leq |\lambda_n|$

$$|x_n - x| + |\lambda_n - y| |x|$$

Since $|x_n - x| \to 0$ and $|\lambda_n - \lambda| \to 0$as n $\to\infty$,after that| $\lambda_n x_n - \lambda x$| as n$\to\infty$.

Refrences

1. Al- Mayahi. N.F, Introduction in Mathematical analysis, Al- Qadisiyah University, (2015).
2. M.m Noor .R.hameed , Daniell Integration for functions on locally compact space . Al-Qadisiyah University . Iraq . 2022 .
3. Nariman Abasov. lattice normed space, Ann Funct. Anal. 7 (2016), no. 4, 646-655.
4. J. Lindenstrauss and L. Tzafriri, Classical Banach Spaces, Vol. 2 : Function Spaces, Ergeb. Math. Grenzgeb. (3) 97, Springer, Berlin, 1977.
5. Zb1 0403.46022. MR0540367. 647
6. M.Pliev, Narrow operators on lattice-normed spaces, Cent. Eur. J. Math. 9 (2011), no. 6, 1276-1287. Zb1 1253.47024. MR2836720. DOI 10.2478/s11533-011-0090-3. 646, 648, 650, 653
7. A. M. Plichko and M. M. Popov, Symmetric function spaces on atomless probability space, Dissertationes Math. (Rozprawy Mat.) 306 (1990), 1-85. Zb1 0715.46011. MR1082412. 646
8. Boyko., Kadets V. Narrow Operators On Bochner L_1-Spaces // Journal of Math. Physics, Analysis, Geometry (46), 2002. (2), p. 421-441.
9. Mazumdar, N., Sharma, J. K., Shavkatovich, S. N., Uike, D., Kadam, S., Verma, M., ... & Al Ansari, M. S. (2023). Application of distinct multi criteria decision analysis techniques in the manufacturing sector: A comprehensive review. *Materials Today: Proceedings*.
10. Bilik D., Kadets V., Shvidkov R., Sirotkin G., werner D. Narrow Operators On Vector-Valued Sup-Normed Spaces // Ilinois Journal of Math. (2), 2006. (4), P. 358-371.
11. Chawla, R., Konda, H. K., Deshmukh, A. A., Sagar, K. D., Al Ansari, M. S., &Taloba, A. I. (2022). A Hybrid Optimization Approach with Deep Learning Technique for the Classification of Dental Caries. *International Journal of Advanced Computer Science and Applications*, *13*(12).
12. AbdAlgani, Y. M., Caro, O. J. M., Bravo, L. M. R., Kaur, C., Al Ansari, M. S., &Bala, B. K. (2023). Leaf disease identification and classification using optimized deep learning. *Measurement: Sensors*, *25*, 100643.
13. Ghoussoub N., Rosental H. P.: Martingales $G_\delta$embeddings and quotients of L_1 // Math. Ann., 264(9), 1983. P. 321-332.
14. Naidu, K. B., Prasad, B. R., Hassen, S. M., Kaur, C., Al Ansari, M. S., Vinod, R., ... &Bala, B. K. (2022). Analysis of Hadoop log file in an environment for dynamic detection of threats using machine learning. *Measurement: Sensors*, *24*, 100545.
15. Johnson W. B., Maurey B., Schechtman G., Tzafriri L. Symmetric structures in Ba-nach spaces // Memories Amer. Math. Soc. (19), 1979.
16. Kantorovich L. V.: Some classes functional equations // Doklads Russian Academy of Science Math. Ann., 4(5), 1936. P. 211-216.
17. Zainal, A. G., Kaur, C., Al Ansari, M. S., Borda, R. F. C., Nageswaran, A., & Abd El-Aziz, R. M. (2022). Recognition of Copy Move Forgeries in Digital Images using Hybrid Optimization and Convolutional Neural Network Algorithm. *International Journal of Advanced Computer Science and Applications*, *13*(12).

18. Zhu, J., Jiang, X., Rong, Y., Wei, W., Wu, S., Jiao, T., & Chen, Q. (2023). Label-free detection of trace level zearalenone in corn oil by surface-enhanced Raman spectroscopy (SERS) coupled with deep learning models. *Food Chemistry*, *414*, 135705.
19. Kevin A. Reqresentation of compact and weakly compact operators on the space of Bochnerintegrable functions // Pacif. J. Math. 92(2), 1981. P. 257-267.
20. Garret Birkhoff, lattice theory, Harvard University,American mathematical society.581 west 116^{TH} street New York city ,1948.