New Applications of Quasi-Subordination for Bi-univalent Functions Title

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Abstract

In the present paper, we introduce and define new two subclasses $\mathcal{H}_{\Sigma}^{q}(\alpha,\beta,\gamma,\Pi)$ and $\mathcal{J}_{\Sigma}^{q}(\beta,\Pi)$ of the function class Σ of bi-univalent functions in the open unit disk U, by using quasi-subordination conditions and determine estimates of the coefficients $|a_2|$ and $|a_3|$ for functions of these subclasses. Also, consequences of the results are pointed out.

Keywords: Bi-univalent function, analytic function, coefficient estimate, quasi-subordination.

1. Introduction

Let $\mathbb{G}(\mathrm{U})$ be a class of analytic functions f defined in an open unit disk $\mathrm{U}=\{\mathrm{z}\in\mathbb{C}:|\mathrm{z}|<1\}$ and normalized by conditions $f(0)=0,\ f'(0)=1$ in U. An analytic function $f\in\mathbb{G}(U)$ has Taylor series expansion of the form:

$$f(z)=z+\sum_{j=2}^{\infty}a_j z^j \qquad ,(z\in U). \tag{1}$$

The well-known Koebe-One Quarter Theorem [12] state that the image of the open unit U under each univalent function in contains disk with the radius $\frac{1}{4}$. Thus every univalent function f has an inverse $f^{-1}$is satisfying such that:

$$f^{-1}(f(z))=z\ ,z\in U$$

and

$$f(f^{-1}(\omega))=\omega,\qquad \omega\in D_{r_0}=\left\{\omega\in C,:|\omega|<r_0(f),r_0(f)\geq\frac{1}{4}\right\}.$$

Let Σ denote the class of all bi-univalent functions in U. Since f in Σ has the form (1), a computation shows that the inverse $g=f^{-1}$ has the following expansion,

$$g(\omega)=f^{-1}(\omega)=\omega-a_2\omega^2+(2a_2^2-a_3)\omega^3-(5a_2^3-5a_2a_3+a_4)\omega^4+\cdots,\omega\in U$$

Let B be the class of all analytic and univalent functions in the unit open disk. These univalent functions are invertible but the inverse function may not be defined on the entire disk U, for f in $\mathbb{G}$(U). An analytic function f is called bi-univalent in U if both f and $f^{-1}$are univalent in U. The class of bi-univalent functions was introduced by Lewin [14] and proved that $|a_2| \geq 1.51$, for the function of the form (1). Subsequently, Brannan and Clunie[9] conjectured that $|a_2| \geq \sqrt{2}$. Later, Netanyahu in [17], on the other hand, showed that $\underset{f\in\Sigma}{max}|a_2| = \frac{4}{3}$. Also several authors studied classes of bi-univalent analytic functions and found estimates of the coefficients estimate problem for each of the following Taylor-Maclaurin coefficients $|a_2|\, and\, |a_3|$ for functions in these classes([1,2,3,4,5,6,7]). For functions $f \in \mathbb{G}(U)$ and h ∈ $\mathbb{G}$(U) of the form (1) in the following form:

$$h(z) = z + \sum_{j=2}^{\infty} b_j z^j \quad ,(z \in U). \tag{2}$$

in the year 1970, the concept of quasi-subordination was first mentioned by [19]. For two analytic functions $g\, and\, f$ in U, we say that the function f is quasi subordinate to g in U , if there exist analytic functions $\phi\, and\, F$, with $|\phi(z)| \leq 1, F(0) = 0\, and\, |F(z)| \leq 1$, such that $f(z) = \phi(z)g\big(F(z)\big)$, also denote this quasi-subordination by [13], as follows:

$$f(z) \prec_q g(z), \;\; z \in U. \tag{3}$$

Note that if $\phi(z) = 1$, then $f(z) = g\big(F(z)\big)$, hence $f(z) \prec g(z)$, [16]. Furthermore if $F(z) = z$, $f(z) = \phi(z)g(z)$ and this case f is majorized by g , written $f(z) \ll g(z)$ in U. Ma-Minda [15] defined a class of starlike function by using the method of subordination and studied classes $\mathcal{S}^*(\Pi)$ and $\mathcal{C}(\Pi)$ which is defined by:

$$\mathcal{S}^*(\Pi) = \left\{ f \in \mathbb{G}: \frac{zf'(z)}{f(z)} \prec \Pi(z), \qquad z \in U \right\},$$

and

$$\mathcal{C}(\Pi) = \left\{ f \in \mathbb{G}: 1 + \frac{zf''(z)}{f'(z)} \prec \Pi(z), \qquad z \in U \right\},$$

where

$$\phi(z) = \mathcal{B}_0 + \mathcal{B}_1 z + \mathcal{B}_2 z^2 + \cdots, \tag{4}$$

and,

$$\Pi(z) = 1 + C_1 z + C_2 z^2 + \cdots, C_1 > 0, \tag{5}$$

where Π(z) is an analytic and univalent function with positive real part in U, Π is symmetric with respect to the real axis and starlike with respect to $\Pi(0) = 1$ and $\Pi'(0) > 0$. A function $f \in \mathcal{S}^*(\Pi)$ is called starlike or convex of Ma-Minda type respectively [10] and [22].

We introduce and study here certain new subclasses of class Σ.

Brannan and Taha([10] and [11]) "obtained initial coefficient bounds for certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. Later Srivastava et al. [21] introduced and investigated subclasses of bi-univalent functions and obtained bounds for the initial coefficients. Recently, Ali et al. [3] obtained the coefficient bounds for bi-univalent Ma-Minda starlike and convex functions". Some more important results on coefficient inequalities can be found in, ([3],[8],[13],[20],[23],[24]).

We need the following Lemma in achieving results.

Lemma (1) [18]: If $p \in \mathrm{P}$, then $|p_i| \leq 2$ for each i, where P is the family of all analytic functions p, for which $Re\{p(z)\} > 0, z \in U$ where $p(z) = 1 + p_1 z + p_2 z^2 + \cdots, z \in U$.

2. The Subclass $\mathcal{H}_{\Sigma}^{q}(\alpha, \beta, \gamma, \Pi)$

Definition (1): Let a function $f \in \Sigma$, with $\beta \geq 0,\ \alpha \in \mathbb{C}\backslash\{0\}\ and\ 0 \leq \gamma \leq 1$, such that f belong to the class $\mathcal{H}_{\Sigma}^{q}(\alpha, \beta, \gamma, \Pi)$, if the following are holds:

$$\left(\frac{zf'(z)}{f(z)}\right)\left(\frac{f(z)}{z}\right)^{\beta} + \frac{1}{\alpha}\left(f'(z) + \gamma z f''(z) - 1\right) \prec_{q} (\Pi(z) - 1)$$

and,

$$\left(\frac{\omega g'(\omega)}{g(\omega)}\right)\left(\frac{g(\omega)}{\omega}\right)^{\beta} + \frac{1}{\alpha}\left(g'(\omega) + \gamma \omega g''(\omega) - 1\right) \prec_{q} (\Pi(\omega) - 1),$$

where $g = \mathrm{f}^{-1}$.

Theorem (1): If f is given by (1) belongs to the subclass $f \in \mathcal{H}_{\Sigma}^{q}(\alpha, \beta, \gamma, \Pi)$ (for $\beta \geq 0,\ \alpha \in \mathbb{C}\backslash\{0\}\ and\ 0 \leq \gamma \leq 1$), then:

$$|a_2| \leq min\left\{\frac{\alpha|\mathcal{B}_0 \mathrm{C}_1|}{\alpha(1+\beta) + 2(1+\gamma)}, \sqrt{\frac{2\alpha\mathcal{B}_0(\mathrm{C}_1 + |\mathrm{C}_2 - \mathrm{C}_1|)}{\alpha(\beta+1)(\beta+2) + 6(1+2\gamma)}}\right\}$$

and

$$|a_3| \leq min\left\{\begin{array}{c}\dfrac{2\alpha\mathcal{B}_0(\mathrm{C}_1 + |\mathrm{C}_2 - \mathrm{C}_1|)}{\alpha(\beta+1)(\beta+2) + 6(1+2\gamma)} + \dfrac{\alpha|\mathrm{C}_1||\mathcal{B}_0 + \mathcal{B}_1|}{\alpha(2+\beta) + 3(1+2\gamma)}, \\ \\ \dfrac{|\alpha^3|(2-\beta-\beta^2)|\mathcal{B}_0|^2|\mathrm{C}_1|^2}{2(2\alpha + \alpha\beta + 3 + 6\gamma)\left(\alpha(1+\beta) + 2(1+\gamma)\right)^2} + \dfrac{|\alpha|(\mathrm{C}_1|\mathcal{B}_1 - \mathcal{B}_0| + \mathcal{B}_0|\mathrm{C}_2 - \mathrm{C}_1|)}{\alpha(2+\beta) + 3(1+2\gamma)}\end{array}\right\}.$$

Proof: Since $f \in \mathcal{H}_{\Sigma}^{q}(\alpha,\beta,\gamma,\Pi)$, then there exist an analytic functions $\mathfrak{r},\mathfrak{s}: \mathrm{U} \to \mathrm{U}$ with $\mathfrak{r}(0) = \mathfrak{s}(0) = 0, |\mathfrak{r}(\mathrm{z})| < 1$ and $|\mathfrak{s}(\mathrm{z})| < 1$, the function $\mathcal{P}$ defined by (2) satisfied:

$$\left(\frac{zf'(z)}{f(z)}\right)\left(\frac{f(z)}{z}\right)^{\beta} + \frac{1}{\alpha}(f'(z) + \gamma z f''(z) - 1) = \phi(z)[\Pi(\mathfrak{r}(z) - 1)] \tag{6}$$

and

$$\left(\frac{\omega g'(\omega)}{g(\omega)}\right)\left(\frac{g(\omega)}{\omega}\right)^{\beta} + \frac{1}{\alpha}(g'(\omega) + \gamma \omega g''(\omega) - 1) = \phi(\omega)[\Pi(\mathfrak{s}(\omega) - 1)]. \tag{7}$$

Define the functions u and v as an analytic and have positive real parts in U by:

$$u(z) = \frac{1 + \mathfrak{r}(z)}{1 - \mathfrak{r}(z)} = 1 + \mathcal{L}_1 z + \mathcal{L}_2 z^2 + \cdots \tag{8}$$

$$v(\omega) = \frac{1 + \mathfrak{s}(\omega)}{1 - \mathfrak{s}(\omega)} = 1 + k_1 \omega + k_2 \omega^2 + \cdots, \tag{9}$$

which are equivalently:

$$\mathfrak{r}(z) = \frac{u - 1}{u + 1} = \frac{1}{2}\left[\mathcal{L}_1 z + \left(\mathcal{L}_2 - \frac{{\mathcal{L}_1}^2}{2}\right) z^2 + \cdots\right] \tag{10}$$

and

$$\mathfrak{s}(\omega) = \frac{v - 1}{v + 1} = \frac{1}{2}\left[k_1 \omega + \left(k_2 - \frac{{k_1}^2}{2}\right) \omega^2 + \cdots\right]. \tag{11}$$

Now; in view of (6), (7), (10) and (11):

$$\left(\frac{zf'(z)}{f(z)}\right)\left(\frac{f(z)}{z}\right)^{\beta} + \frac{1}{\alpha}(f'(z) + \gamma z f''(z) - 1) = \phi(z)\left[\Pi\left(\left(\frac{u - 1}{u + 1}\right) - 1\right)\right] \tag{12}$$

and

$$\left(\frac{\omega g'(\omega)}{g(\omega)}\right)\left(\frac{g(\omega)}{\omega}\right)^{\beta} + \frac{1}{\alpha}(g'(\omega) + \gamma \omega g''(\omega) - 1) = \phi(\omega)\left[\Pi\left(\left(\frac{v - 1}{v + 1}\right) - 1\right)\right]. \tag{13}$$

It is clear that the series expansions for f and g given by (6) and (7) as follow:

$$\left(\frac{zf'(z)}{f(z)}\right)\left(\frac{f(z)}{z}\right)^{\beta} + \frac{1}{\alpha}(f'(z) + \gamma z f''(z) - 1) = 1 + \left(1 + \beta + \frac{2(1 + \gamma)}{\alpha}\right) a_2 z$$

$$+\left(\frac{(\beta+2)(\beta-1)}{2}a_2^2+\left(2+\beta+\frac{3(1+2\gamma)}{\alpha}\right)a_3\right)z^2+\cdots \qquad (14)$$

$$\left(\frac{\omega g'(\omega)}{g(\omega)}\right)\left(\frac{g(\omega)}{\omega}\right)^{\beta}+\frac{1}{\alpha}(g'(\omega)+\gamma\omega g''(\omega)-1)$$
$$=1-\left(1+\beta+\frac{2(1+\gamma)}{\alpha}\right)a_2\omega$$
$$+\left[\left(\frac{\beta^2+5\beta}{2}+\frac{6(1+2\gamma)}{\alpha}+3\right)a_2^2-\left(\frac{\alpha(2+\beta)+3(1+2\gamma)}{\alpha}\right)a_3\right]\omega^2-\ldots$$

.(15)

By using (8) and (9) with (4) and (5):

$$\phi(z)\left[\Pi\left(\left(\frac{u-1}{u+1}\right)-1\right)\right]$$
$$=\frac{1}{2}\mathcal{B}_0C_1\mathcal{L}_1z+\left[\frac{1}{2}\mathcal{B}_1C_1\mathcal{L}_1+\frac{1}{2}\mathcal{B}_0C_1\left(\mathcal{L}_2+\frac{\mathcal{L}_1{}^2}{2}\right)+\frac{1}{4}\mathcal{B}_0C_2\mathcal{L}_1{}^2\right]z^2+\cdots \qquad (16)$$

and

$$\phi(\omega)\left[\Pi\left(\left(\frac{v-1}{v+1}\right)-1\right)\right]$$
$$=\frac{1}{2}\mathcal{B}_0C_1k_1\omega+\left[\frac{1}{2}\mathcal{B}_1C_1k_1+\frac{1}{2}\mathcal{B}_0C_1\left(k_2+\frac{k_1{}^2}{2}\right)+\frac{1}{4}\mathcal{B}_0C_2k_1{}^2\right]\omega^2$$
$$+\cdots \ . \qquad (17)$$

Now equating (14) and (16) and comparing the coefficients to z and z^2:

$$\left(1+\beta+\frac{2(1+\gamma)}{\alpha}\right)a_2=\frac{1}{2}\mathcal{B}_0C_1\mathcal{L}_1 \qquad (18)$$

and

$$\frac{(\beta+2)(\beta-1)}{2}a_2^2+\left(2+\beta+\frac{3(1+2\gamma)}{\alpha}\right)a_3$$
$$=\frac{1}{2}\mathcal{B}_1C_1\mathcal{L}_1+\frac{1}{2}\mathcal{B}_0C_1\left(\mathcal{L}_2+\frac{\mathcal{L}_1{}^2}{2}\right)+\frac{1}{4}\mathcal{B}_0C_2\mathcal{L}_1{}^2. \qquad (19)$$

In the same steps (15) and (17)

$$-\left(1+\beta+\frac{2(1+\gamma)}{\alpha}\right)a_2=\frac{1}{2}\mathcal{B}_0C_1k_1 \qquad (20)$$

and

$$\left(\frac{\beta^2+5\beta}{2}+\frac{6(1+2\gamma)}{\alpha}+3\right)a_2^2-\left(\frac{\alpha(2+\beta)+3(1+2\gamma)}{\alpha}\right)a_3 = \frac{1}{2}\mathcal{B}_1 C_1 \mathcal{k}_1+\frac{1}{2}\mathcal{B}_0 C_1\left(\mathcal{k}_2+\frac{{\mathcal{k}_1}^2}{2}\right) + \frac{1}{4}\mathcal{B}_0 C_2 {\mathcal{k}_1}^2 . \quad (21)$$

By (18) and (20) we obtain:

$$\mathcal{L}_1=-\mathcal{k}_1 \quad (22)$$

yields:

$$|a_2| \le \frac{\alpha|\mathcal{B}_0 C_1|}{\alpha(1+\beta)+2(1+\gamma)} \quad (23)$$

adding (19) and (21), we get:

$$a_2^2=\frac{\frac{1}{2}\mathcal{B}_0 C_1(\mathcal{L}_2+\mathcal{k}_2)+\frac{1}{4}\mathcal{B}_0|C_2-C_1|(\mathcal{L}_1^2+\mathcal{k}_1^2)}{\left((\beta^2+3\beta+2)+\frac{6(1+2\gamma)}{\alpha}\right)} \quad (24)$$

implies

$$a_2^2 \le \frac{2\alpha|\mathcal{B}_0|(C_1+|C_2-C_1|)}{\left(\alpha(\beta^2+3\beta+2)+6(1+2\gamma)\right)} . \quad (25)$$

Next, to find the upper bound for $|a_3|$, by subtracting (21) from (19) :

$$\left(\frac{\alpha(2+\beta)+3(1+2\gamma)}{\alpha}\right)2a_3=a_2^2+\frac{1}{2}\mathcal{B}_0 C_1(\mathcal{L}_2-\mathcal{k}_2)+\frac{1}{2}\mathcal{B}_1 C_1(\mathcal{L}_1-\mathcal{k}_1).$$

By using Lemma 1 and (24):

$$|a_3| \le \frac{2\alpha|\mathcal{B}_0|(C_1+|C_2-C_1|)}{\left(\alpha(\beta^2+3\beta+2)+6(1+2\gamma)\right)}+\frac{\alpha|C_1|(|\mathcal{B}_0+\mathcal{B}_1|)}{\alpha(2+\beta)+3(1+2\gamma)} . \quad (26)$$

Now by (18) and (19), we find:

$$|a_3| \le \frac{|\alpha|}{\alpha(2+\beta)+3(1+2\gamma)}\left[\frac{|\alpha^2|(2-\beta-\beta^2)|\mathcal{B}_0|^2|C_1|^2}{2\left(\alpha(1+\beta)+2(1+\gamma)\right)^2}+C_1|\mathcal{B}_1-\mathcal{B}_0|+\mathcal{B}_0|C_2-C_1|\right], \quad (27)$$

applying Lemma (1) in ((26) and (27)), we get the result. The proof is complete.

Taking $\beta = 1$ and $\gamma = 0$, in theorem (1), we get the following result.

Remark 1: If a function f is given by (1) belongs to the subclass $\mathcal{H}_{\Sigma}^{q}(\alpha, 1,0, \Pi)$, then:

$$\left(\frac{zf'(z)}{f(z)}\right)\left(\frac{f(z)}{z}\right) + \frac{1}{\alpha}(f'(z) - 1) \prec_{q} \Pi(\mathfrak{r}(z) - 1) \tag{28}$$

and,

$$\left(\frac{\omega g'(\omega)}{g(\omega)}\right)\left(\frac{g(\omega)}{\omega}\right) + \frac{1}{\alpha}(g'(\omega) - 1) \prec_{q} \Pi(\mathfrak{s}(\omega) - 1)\,. \tag{29}$$

Corollary 1: If a function f is given by (1) belongs to the subclass $\mathcal{H}_{\Sigma}^{q}(\alpha, 1,0, \Pi)$, then:

$$|a_2| \le min\left\{\frac{\alpha|\mathcal{B}_0 C_1|}{3\alpha + 2}, \sqrt{\frac{\alpha\mathcal{B}_0(C_1 + |C_2 - C_1|)}{3(\alpha + 1)}}\right\}$$

and

$$|a_3| \le min\left\{\frac{\alpha\mathcal{B}_0(C_1 + |C_2 - C_1|)}{3\alpha + 3} + \frac{\alpha|C_1||\mathcal{B}_0 + \mathcal{B}_1|}{3\alpha + 3}, \frac{|\alpha|(C_1|\mathcal{B}_1 - \mathcal{B}_0| + \mathcal{B}_0|C_2 - C_1|)}{3\alpha + 3}\right\}.$$

Corollary 2: : If a function f is given by (1) belongs to the subclass $\mathcal{H}_{\Sigma}^{q}(1,0,0, \Pi)$, then:

$$|a_2| \le min\left\{\frac{|\mathcal{B}_0 C_1|}{3}, \sqrt{\frac{\mathcal{B}_0(C_1 + |C_2 - C_1|)}{4}}\right\}$$

and

$$|a_3| \le min\left\{\begin{array}{c}\frac{\mathcal{B}_0(C_1 + |C_2 - C_1|)}{4} + \frac{|C_1||\mathcal{B}_0 + \mathcal{B}_1|}{5}, \\ \frac{|\mathcal{B}_0|^2|C_1|^2}{45} + \frac{(C_1|\mathcal{B}_1 - \mathcal{B}_0| + \mathcal{B}_0|C_2 - C_1|)}{5}\end{array}\right\}.$$

3. The Subclass $\mathcal{J}_{\Sigma}^{q}(\beta, \Pi)$

Definition 2: Let a function $f \in \mathbb{G}$, is said to be in the class $\mathcal{J}_{\Sigma}^{q}(\beta, \Pi)$, $(0 \le \beta \le 1)$, if it satisfies the following quasi-subordination:

$$(1 - \beta)\left(\frac{zf''(z)}{f'(z)} + 1\right) + \beta\left(\frac{zf''(z)}{f'(z)} + z^2 f'''(z)\right) \prec_{q} (\Pi(z) - 1), \tag{30}$$

and,

$$(1-\beta)\left(\frac{\omega g''(\omega)}{g'(\omega)}+1\right)+\beta\left(\frac{\omega g''(\omega)}{g'(\omega)}+\omega^2 g'''(\omega)\right)\prec_q (\Pi(\omega)-1), \tag{31}$$

where $g = f^{-1}$

Theorem 2: If f is given by (1) belong to the subclass $\mathcal{J}_{\Sigma}^{q}(\beta,\Pi)$, where $0 \le \beta \le 1$, then:

$$|a_2| \le min\left\{\frac{|\mathcal{B}_0 C_1|}{2}, \sqrt{\frac{\mathcal{B}_0(C_1+|C_2-C_1|)}{2(1-3\beta)}}\right\} \tag{32}$$

and

$$|a_3| \le min\left\{\frac{\mathcal{B}_0(C_1+|C_2-C_1|)}{2(1-3\beta)}+\frac{|C_1||\mathcal{B}_0+\mathcal{B}_1|}{6(1-\beta)}, \frac{|\mathcal{B}_0||C_1|}{3(1-\beta)}+\frac{C_1|\mathcal{B}_1-\mathcal{B}_0|+\mathcal{B}_0|C_2-C_1|}{6(1-\beta)}\right\}. \tag{33}$$

Proof: Let $f \in \mathcal{J}_{\Sigma}^{q}(\beta,\Pi)$, and $g = f^{-1}$, then there are analytic functions $\mathfrak{r},\mathfrak{s}: U \to U$ with $\mathfrak{r}(0) = \mathfrak{s}(0) = 0, |\mathfrak{r}(z)| < 1$ and $|\mathfrak{s}(z)| < 1$, the function ϕ defined by (4) verify:

$$(1-\beta)\left(\frac{zf''(z)}{f'(z)}+1\right)+\beta\left(\frac{zf''(z)}{f'(z)}+z^2 f'''(z)\right)=\mathcal{P}(z)[\Pi(\mathfrak{r}(z)-1)] \tag{34}$$

and

$$(1-\beta)\left(\frac{\omega g''(\omega)}{g'(\omega)}+1\right)+\beta\left(\frac{\omega g''(\omega)}{g'(\omega)}+\omega^2 g'''\right)=\mathcal{P}(\omega)[\Pi(\mathfrak{s}(\omega)-1)]\,. \tag{35}$$

Define the functions u and v as an analytic and have positive real parts in U by:

$$u(z)=\frac{1+\mathfrak{r}(z)}{1-\mathfrak{r}(z)}=1+\mathcal{L}_1 z+\mathcal{L}_2 z^2+\cdots$$

$$v(\omega)=\frac{1+\mathfrak{s}(\omega)}{1-\mathfrak{s}(\omega)}=1+k_1\omega+k_2\omega^2+\cdots,$$

which are equivalently:

$$\mathfrak{r}(z)=\frac{u-1}{u+1}=\frac{1}{2}\left[\mathcal{L}_1 z+\left(\mathcal{L}_2-\frac{{\mathcal{L}_1}^2}{2}\right)z^2+\cdots\right]$$

and

$$\mathfrak{s}(\omega)=\frac{\mathcal{v}-1}{\mathcal{v}+1}=\frac{1}{2}\left[\mathcal{k}_1\omega+\left(\mathcal{k}_2-\frac{{\mathcal{k}_1}^2}{2}\right)\omega^2+\cdots\right].$$

Now; in view of (6), (7), (10) and (11):

$$(1-\beta)\left(\frac{zf''}{f'}+1\right)+\beta\left(\frac{zf''}{f'}+z^2f'''\right)=\mathcal{P}(z)\left[\Pi\left(\left(\frac{u-1}{u+1}\right)-1\right)\right] \tag{36}$$

and

$$(1-\beta)\left(\frac{\omega g''(\omega)}{g'(\omega)}+1\right)+\beta\left(\frac{\omega g''(\omega)}{g'(\omega)}+\omega^2g'''\right)=\mathcal{P}(\omega)\left[\Pi\left(\left(\frac{\mathcal{v}-1}{\mathcal{v}+1}\right)-1\right)\right]. \tag{37}$$

It is clear that the series expansions for $f\ and\ g$ given by (6) and (7) as follow:

$$(1-\beta)\left(\frac{zf''}{f'}+1\right)+\beta\left(\frac{zf''}{f'}+z^2f'''\right)$$
$$=(1-\beta)+2a_2z+(6(1-\beta)a_3-4a_2^2)z^2+\cdots \tag{38}$$

$$(1-\beta)\left(\frac{\omega g''(\omega)}{g'(\omega)}+1\right)+\beta\left(\frac{\omega g''(\omega)}{g'(\omega)}+\omega^2g'''\right)$$
$$=(1-\beta)-2a_2\omega$$
$$+[4(2-3\beta)a_2^2-6(1-\beta)a_3]\omega^2-\ldots\ . \tag{39}$$

By using (8) and (9) with (4) and (5):

$$\phi(z)\left[\Pi\left(\left(\frac{u-1}{u+1}\right)-1\right)\right]$$
$$=\frac{1}{2}\mathcal{B}_0\mathrm{C}_1\mathcal{L}_1z+\left[\frac{1}{2}\mathcal{B}_1\mathrm{C}_1\mathcal{L}_1+\frac{1}{2}\mathcal{B}_0\mathrm{C}_1\left(\mathcal{L}_2+\frac{{\mathcal{L}_1}^2}{2}\right)+\frac{1}{4}\mathcal{B}_0\mathrm{C}_2{\mathcal{L}_1}^2\right]z^2+\cdots \tag{40}$$

and

$$\phi(\omega)\left[\Pi\left(\left(\frac{\mathcal{v}-1}{\mathcal{v}+1}\right)-1\right)\right]$$
$$=\frac{1}{2}\mathcal{B}_0\mathrm{C}_1\mathcal{k}_1\omega+\left[\frac{1}{2}\mathcal{B}_1\mathrm{C}_1\mathcal{k}_1+\frac{1}{2}\mathcal{B}_0\mathrm{C}_1\left(\mathcal{k}_2+\frac{{\mathcal{k}_1}^2}{2}\right)+\frac{1}{4}\mathcal{B}_0\mathrm{C}_2{\mathcal{k}_1}^2\right]\omega^2$$
$$+\cdots\ . \tag{41}$$

Now equating (38) and (40) and comparing the coefficients to z and z^2:

$$2a_2=\frac{1}{2}\mathcal{B}_0\mathrm{C}_1\mathcal{L}_1 \tag{42}$$

and

$$6(1-\beta)a_3-4a_2^2=\frac{1}{2}\mathcal{B}_1\mathrm{C}_1\mathcal{L}_1+\frac{1}{2}\mathcal{B}_0\mathrm{C}_1\left(\mathcal{L}_2+\frac{{\mathcal{L}_1}^2}{2}\right)+\frac{1}{4}\mathcal{B}_0\mathrm{C}_2{\mathcal{L}_1}^2\,. \tag{43}$$

In the same steps (39) and (41)

$$-2a_2=\frac{1}{2}\mathcal{B}_0\mathrm{C}_1\mathcal{k}_1 \tag{44}$$

and

$$4(2-3\beta)a_2^2-6(1-\beta)a_3=\frac{1}{2}\mathcal{B}_1\mathrm{C}_1\mathcal{k}_1+\frac{1}{2}\mathcal{B}_0\mathrm{C}_1\left(\mathcal{k}_2+\frac{{\mathcal{k}_1}^2}{2}\right)+\frac{1}{4}\mathcal{B}_0\mathrm{C}_2{\mathcal{k}_1}^2\,. \tag{45}$$

By (42) and (44) we obtain:

$$\mathcal{L}_1=-\mathcal{k}_1 \tag{46}$$

yields:

$$|a_2|\le\frac{|\mathcal{B}_0\mathrm{C}_1|}{2} \tag{47}$$

adding (43) and (45), we get:

$$a_2^2=\frac{\frac{1}{2}\mathcal{B}_0\mathrm{C}_1(\mathcal{L}_2+\mathcal{k}_2)+\frac{1}{4}\mathcal{B}_0(|\mathrm{C}_2-\mathrm{C}_1|)(\mathcal{L}_1^2+\mathcal{k}_1^2)}{4(1-3\beta)} \tag{48}$$

implies

$$a_2^2\le\frac{\mathcal{B}_0(\mathrm{C}_1+|\mathrm{C}_2-\mathrm{C}_1|)}{2(1-3\beta)}\;\;. \tag{49}$$

Next, to find the upper bound for $|a_3|$, by subtracting (45) from (43) :

$$12(1-\beta)a_3-4(3-3\beta)a_2^2=\frac{1}{2}\mathcal{B}_0\mathrm{C}_1(\mathcal{L}_2-\mathcal{k}_2)+\frac{1}{2}\mathcal{B}_1\mathrm{C}_1(\mathcal{L}_1-\mathcal{k}_1).$$

By using Lemma 1 and (48):

$$|a_3|\le\frac{\mathcal{B}_0(\mathrm{C}_1+|\mathrm{C}_2-\mathrm{C}_1|)}{2(1-3\beta)}+\frac{|\mathrm{C}_1||\mathcal{B}_0+\mathcal{B}_1|}{6(1-\beta)}\;\;. \tag{50}$$

Now by (42) and (43) we find:

$$|a_3|\le\frac{|\mathcal{B}_0||\mathrm{C}_1|}{3(1-\beta)}+\frac{\mathrm{C}_1|\mathcal{B}_1-\mathcal{B}_0|+\mathcal{B}_0|\mathrm{C}_2-\mathrm{C}_1|}{6(1-\beta)} \tag{51}$$

This the proof is complete■

Corollary 3: If f is given by (1) belongs to the class $J_{\Sigma}^{q}(0,\Pi)$, where $0 \leq \beta \leq 1$, then, we have :

$$|a_2| \leq min\left\{\frac{|\mathcal{B}_0 C_1|}{2}, \sqrt{\frac{\mathcal{B}_0(C_1 + |C_2 - C_1|)}{2}}\right\}$$

and

$$|a_3| \leq min\left\{\frac{\mathcal{B}_0(C_1 + |C_2 - C_1|)}{2} + \frac{|C_1||\mathcal{B}_0 + \mathcal{B}_1|}{6}, \frac{|\mathcal{B}_0||C_1|}{3} + \frac{C_1|\mathcal{B}_1 - \mathcal{B}_0| + \mathcal{B}_0|C_2 - C_1|}{6}\right\}.$$

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