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Some Weighted Mean Inequalities

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Abstract: In this paper we study the nessesary as well as sufficient conditions for the boundedness of the Hardy operator and its dual between Banach function spaces X^q and L^p for the cases $1<q<p<\infty$ and $0<q<1<p<\infty$.

Key words and phrases: Banach function space, Hardy inequality, Level function.
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1 Introduction

In [6], Hardy proved the inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p x^\alpha dx \le \left(\frac{p}{p-1-\alpha}\right)^p \int_0^\infty f^p(x)x^\alpha dx,$$

where $p>1$ and $f\ge 0$ is a nonnegative function.

A plenty of work has been done on Hardy inequalities. For more details one can refer to books Opic and Kufner [12], Davies [2], Kufner, Persson and Samko [9], Edmunds and Evans [3], Ghoussoub and Moradifam [4], Balinsky [1].

In this paper we show the boundedness of the Hardy operator and its dual on a more general space. More specifically, we study these operators from L^p to X^q for the cases $1<q<p<\infty$ and $0<q<1<p<\infty$.

The paper is organized in the following way: Section 2 gives some definitions which are standard but will ease the reading of the paper. Section 3 deals with the case $1<q<p<\infty$ for the L^p - X^q boundedness of the operator H as well as its adjoint operator H^*. In Section 4, we study the case $0<q<1<p<\infty$ which requires the use of level functions and complimentary

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level functions. Finally, in Section 5, we give some remarks. Also, in this section, a conjecture is mentioned giving the X^p - X^q boundedness of the Hardy operator H.

2 Definitions

Luxemberg [11] introduced Banach function spaces which are more general than L^p-spaces.

A real normed linear space $X = \{f : \| f \|_X < \infty\}$ of measurable functions is called a Banach function space (BFS), if with the usual norm axioms, $\| f \|_X$ satisfies the following:

(1) $\| f \|_X = \| \, | f | \, \|_X$ for all $f \in X$;

(2) $0 \le f \le g$ a.e. $\Rightarrow \| f \|_X \le \| g \|_X$;

(3) $0 \le f_n \uparrow f$ a.e. $\Rightarrow \| f_n \|_X \uparrow \| f \|_X$;

(4) mes $E < \infty \Rightarrow \| \psi_E \|_X < \infty$

(5) mes $E < \infty \Rightarrow \int_E f(x) dx \le c_E \| \, | f | \, \|_X$

for some constant c_E depending upon E.

Given a BFS X, its associate space X' is defined by

$$X' = \left\{ g : \int_0^\infty | fg | < \infty \text{ for all } f \in X \right\},$$

and endowed with the associate norm

$$\| g \|_{X'} = \sup \left\{ \int_0^\infty | fg | : f \in X, \| f \|_X \le 1 \right\},$$

Examples of BFS are Lebesuge L^p-spaces, Lorentz spaces, Orlicz spaces etc.

Let X be a BFS and $-\infty < p < \infty$, $p \neq 0$. We define the space X^p to be the space of all measurable functions f for which

$$\| f \|_{X^p} := \| | f |^p \|_X^{1/p} < \infty .$$

For $1 < p < \infty$, X^p is a BFS. Note that for $X = L^1$, the space X^p coincides with L^p-space. These spaces have been studied and used in [13].

3 The Case: $1 < q < p < \infty$

The boundedness of the Hardy operator between general Banach function spaces X and Y has been proved by Lomakina and Stepanov in [10]. The boundedness from L^p to X^q is the following theorem which has been proved in [7].

Theorem 3.1. *Let* $1 < p \le q < \infty$ *and* u,v *be weight functions on* $(0,\infty)$. *Then the inequality*

$$\| (Hf)\cdot u^{\frac{1}{q}} \|_{X^q} \le C\left(\int_0^\infty f^p(x)v(x)dx\right)^{\frac{1}{p}} \tag{3.1}$$

holds for all measurable functions $f \ge 0$ if and only if

$$A := \sup_{t>0} \| \chi_{[t,\infty)} u^{\frac{1}{q}} \|_{X^q} \left(\int_0^t v^{1-p'}\right)^{\frac{1}{p'}} < \infty \tag{3.2}$$

Moreover, the best constant C *in* (3.1) *satisfies* $C \approx A$.

The above theorem gives Muckenhoupt-type L^p - X^q boundedness of H.

Remark 3.2. While proving the necessity in Theorem 3.1 the order of p and q has no role to play. In fact, if the inequality (3.1) holds then the condition (3.2) is satisfied for all $0 < q < \infty$, $1 < p < \infty$.

Now we prove the first result of this section.

Theorem 3.3. *Let* $1 < q < p < \infty$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ *and* u,v *be weight functions defined on* $(0,\infty)$. *Then, the inequality* (3.1) *holds for all measurable functions* $f \ge 0$ *if and only if*

$$J := \left\{\int_0^\infty \| \chi_{[x,\infty)} u \|_X^{\frac{r}{q}} \left(\int_0^x v^{1-p'}(t)dt\right)^{\frac{r}{q'}} v^{1-p'}(x)dx\right\}^{\frac{1}{r}} < \infty. \tag{3.3}$$

Moreover, the best constant C in (3.1) has the estimate

$$q^{\frac{1}{q}}\left(\frac{qp'}{r}\right)^{\frac{1}{q'}} J \le C \le q^{\frac{1}{q}} (p')^{\frac{1}{q'}} J.$$

Proof. Let us first assume that $J < \infty$. We have by using Fubini's theorem and Hölder's inequality applied on the product of three functions (with exponents $\frac{p}{p-q}, p, \frac{p}{q-1}$)

$$\| (Hf)^q \cdot u \|_X^{\frac{1}{q}} = \sup_{h>0}\left\{\int_0^\infty \left(\int_0^x f(y)dy\right)^q u(x)h(x)dx : \| h \|_{X'} \le 1\right\}^{\frac{1}{q}}$$

$$= \sup_{h>0} q^{\frac{1}{q}} \left\{\int_0^\infty \left[\int_0^x \left(\int_0^y f(t)dt\right)^{q-1} f(y)dy\right] u(x)h(x)dx : \| h \|_{X'} \le 1\right\}^{\frac{1}{q}}$$

$$= \sup_{h>0} q^{\frac{1}{q}} \left\{ \int_0^\infty \left(\int_0^y f(t)dt \right)^{q-1} f(y) \left(\int_y^\infty u(x)h(x)dx : \| h \|_{X'} \le 1 \right) dy \right\}^{\frac{1}{q}}$$

$$= q^{\frac{1}{q}} \left\{ \int_0^\infty \left(\int_0^y f(t)dt \right)^{q-1} f(y) \| \chi_{[y,\infty)} u \|_X \; dy \right\}^{\frac{1}{q}}$$

$$= q^{\frac{1}{q}} (p-1)^{\frac{(1-q)}{p} \cdot \frac{1}{q}} \left\{ \int_0^\infty \| \chi_{[y,\infty)} u \|_X \left(\int_0^y v^{1-p'}(t)dt \right)^{q-1} \right.$$

$$\left. \times v^{(1-p')\frac{(p-q)}{p}}(y) f(y) v^{\frac{1}{p}}(y) (p-1)^{\frac{(q-1)}{p}} \left[\frac{Hf(y)}{\int_0^y v^{1-p'}(t)dt} \right]^{q-1} v^{\frac{(1-p')(q-1)}{p}}(y) dy \right\}^{\frac{1}{q}}$$

$$\le q^{\frac{1}{q}} (p-1)^{\frac{(1-q)}{pq}} \left\{ \int_0^\infty \| \chi_{[y,\infty)} u \|_X^{\frac{r}{q}} \left(\int_0^y v^{1-p'}(t)dt \right)^{\frac{r}{q'}} v^{1-p'}(y) dy \right\}^{\frac{1}{r}}$$

$$\times \left\{ \int_0^\infty f^p(y) v(y) dy \right\}^{\frac{1}{pq}} \left\{ \int_0^\infty H^p f(y) \tilde{u}(y) dy \right\}^{\frac{1}{pq'}}$$

$$= q^{\frac{1}{q}} (p-1)^{\frac{(1-q)}{pq}} J \left\{ \int_0^\infty f^p(y) v(y) dy \right\}^{\frac{1}{pq}} \left\{ \int_0^\infty H^p f(y) \tilde{u}(y) dy \right\}^{\frac{1}{pq'}}, \tag{3.4}$$

where $\tilde{u}(y) = (p-1) \left(\int_0^y v^{1-p'}(t)dt \right)^{-p} v^{1-p'}(y),\; y \in (0,\infty)$ is a new weight function.

It can be seen that for the weights $\tilde{u}$ and v,

$$\sup_{0<x<\infty} \left(\int_x^\infty \tilde{u}(t)dt \right)^{\frac{1}{p}} \left(\int_0^x v^{1-p'}(t)dt \right)^{\frac{1}{p'}} \le 1$$

and consequently in view of [12, Theorem 1.14 for $p = q$], the inequality

$$\left(\int_0^\infty H^p f(y) \tilde{u}(y) dy \right)^{\frac{1}{p}} \le p^{\frac{1}{p}} (p')^{\frac{1}{p'}} \left\{ \int_0^\infty f^p(x) v(x) dx \right\}^{\frac{1}{p}}$$

holds. By using this, (3.4) gives

$$\| (Hf) \cdot u^{\frac{1}{q}} \|_{X^q} \le q^{\frac{1}{q}} (p')^{\frac{1}{q'}} J \left\{ \int_0^\infty f^p(x) v(x) dx \right\}^{\frac{1}{p}}$$

which proves the sufficiency and the upper bound of the constant C in (3.1).

Conversely, assume that the inequality (3.1) holds. First we note, in view of Remark 3.2, that

$$\| \chi_{[x,\infty)} u \|_X < \infty, \left(\int_0^x v^{1-p'}(t) dt \right) < \infty, \tag{3.5}$$

for every $x \in (0, \infty)$.

Now, choose two sequences $\{a_n\}$, $\{b_n\}$ of real numbers such that $a_n \downarrow 0$, $b_n \uparrow \infty$ and for each $n \in \mathbf{N}$, $x \in (0, \infty)$ choose functions

$$f_n(x) = \| \chi_{[x,\infty)} u \|_X^{\frac{r}{pq}} \left(\int_{a_n}^x v^{1-p'}(t) dt \right)^{\frac{r}{pq'}} v^{1-p'}(x) \chi_{(a_n, b_n)}(x). \tag{3.6}$$

Clearly $f_n(x) > 0$ a.e. in (a_n, b_n) and therefore

$$\int_0^\infty f_n^p(x) v(x) dx = \int_{a_n}^{b_n} f_n^p(x) v(x) dx > 0$$

and if we define

$$J_n = \left\{ \int_{a_n}^{b_n} \| \chi_{[x,\infty)} u \|_X^{\frac{r}{q}} \left(\int_{a_n}^x v^{1-p'}(t) dt \right)^{\frac{r}{q'}} v^{1-p'}(x) dx \right\}^{\frac{1}{r}}$$

then we find, in view of (3.5), that

$$\int_0^\infty f_n^p(x) v(x) dx = J_n^r$$

$$\le \| \chi_{[a_n,\infty)} u \|_X^{\frac{r}{q}} \left\{ \int_{a_n}^{b_n} \left(\int_{a_n}^x v^{1-p'}(t) dt \right)^{\frac{r}{q'}} v^{1-p'}(x) dx \right\}$$

$$= \frac{p'}{r} \| \chi_{[a_n,\infty)} u \|_X^{\frac{r}{q}} \left(\int_{a_n}^{b_n} v^{1-p'}(t) dt \right)^{\frac{r}{p'}} < \infty. \tag{3.7}$$

Also, we have by Fubini theorem that

$$\| (Hf_n) \cdot u^{\frac{1}{q}} \|_{X^q} = \sup_{h>0} \left\{ \int_0^\infty \left(\int_0^x f_n(t)dt \right)^q u(x)h(x)dx : \| h \|_{X'} \le 1 \right\}^{\frac{1}{q}}$$

$$= \sup_{h>0} \left\{ q \int_0^\infty \left(\int_0^x \left(\int_0^y f_n(t)dt \right)^{q-1} f_n(y)dy \right) u(x)h(x)dx : \| h \|_{X'} \le 1 \right\}^{\frac{1}{q}}$$

$$= \sup_{h>0} q^{\frac{1}{q}} \left\{ \int_0^\infty \left(\int_0^y f_n(t)dt \right)^{q-1} f_n(y) \left(\int_y^\infty u(x)h(x)dx \right) dy : \| h \|_{X'} \le 1 \right\}^{\frac{1}{q}}$$

$$= q^{\frac{1}{q}} \left\{ \int_0^\infty \left(\int_0^y f_n(t)dt \right)^{q-1} f_n(y) \| \chi_{[y,\infty)} u \|_X \, dy \right\}^{\frac{1}{q}}$$

$$= q^{\frac{1}{q}} \left\{ \int_{a_n}^{b_n} \left(\int_{a_n}^y f_n(t)dt \right)^{q-1} f_n(y) \| \chi_{[y,\infty)} u \|_X \, dy \right\}^{\frac{1}{q}}.$$

But for $y \in (a_n, b_n)$, (3.6) gives that

$$\int_{a_n}^y f_n(t)dt = \int_{a_n}^y \| \chi_{[t,\infty)} u \|_X^{\frac{r}{pq}} \left(\int_{a_n}^t v^{1-p'}(x)dx \right)^{\frac{r}{pq'}} v^{1-p'}(t)dt$$

$$\ge \frac{qp'}{r} \| \chi_{[y,\infty)} u \|_X^{\frac{r}{pq}} \left(\int_{a_n}^y v^{1-p'}(x)dx \right)^{\frac{r}{qp'}}$$

so that

$$\| (Hf_n)^q \cdot u \|_X^{\frac{1}{q}} \ \ge q^{\frac{1}{q}} \left(\frac{qp'}{r} \right)^{\frac{1}{q'}} \left\{ \int_{a_n}^{b_n} \| \chi_{[y,\infty)} u \|_X^{\frac{r}{q}} \left(\int_{a_n}^y v^{1-p'}(x)dx \right)^{\frac{r}{q'}} v^{1-p'}(y)dy \right\}^{\frac{1}{q}}$$

$$= q^{\frac{1}{q}} \left(\frac{qp'}{r} \right)^{\frac{1}{q'}} J_n^{\frac{r}{q}}.$$

Then the last estimate and (3.7) give

$$q^{\frac{1}{q}} \left(\frac{qp'}{r} \right)^{\frac{1}{q'}} J_n^{\frac{r}{q}} \le C J_n^{\frac{r}{p}}$$

that is

$$q^{\frac{1}{q}}\left(\frac{qp'}{r}\right)^{\frac{1}{q'}} J_n \le C.$$

Since $0 < J_n < \infty$, taking $n \to \infty$ we get

$$q^{\frac{1}{q}}\left(\frac{qp'}{r}\right)^{\frac{1}{q'}} J \le C$$

which prove the necessity and the lower bound of the constant.

Remark 3.4. If we choose $X = L^s$ and $X^q = L^{sq}$ then we choose s in such a way that $sq < p$.

Next, we prove the L^p-X^q boundedness of the adjoint Hardy operator $(H^* f)(x) = \int_x^\infty f(t)dt$. Since the dual space of X^q is not known, the boundedness of H^* can not be obtained by duality arguments or by the method of variable transformation. Thus, we have to work it out directly. However, the strategy is same as that for Theorem 3.3.

We state the following theorem which was proved by Jain, Gupta and Verma [7] and will be used while proving the boundedness of the adjoint operator.

Theorem 3.5. *Let* $1 < p \le q < \infty$, $s \in (1, p)$ *and* u, v *be weight functions on* $(0, \infty)$. $\tilde{V}(t) := \int_t^\infty v^{1-p'}(y)dy < \infty$, $0 < t < \infty$ *Then the inequality*

$$\| (H^* f) \cdot u^{\frac{1}{q}} \|_{X^q} \le C \left\{ \int_0^\infty f^p(x) v(x) dx \right\}^{\frac{1}{p}} \tag{3.8}$$

holds for all measurable functions $f \ge 0$ if and only if

$$B^* := \sup_{t>0} \tilde{V}^{(s-1)/p}(t) \| \tilde{V}^{(p-s)/p}(t) \cdot u^{\frac{1}{q}} \chi_{(0,t]} \|_{X^q} < \infty \tag{3.9}$$

Moreover, the best constant C in (3.8) has the estimate

$$\sup_{s\in(1,p)} \left(\frac{p}{p-s}\right)\left(\left(\frac{p}{p-s}\right)^p + \frac{1}{s-1}\right)^{-1/p} B^* \le C \le \inf_{s\in(1,p)} \left(\frac{p-1}{p-s}\right)^{1/p'} B^*$$

Lemma 3.6. Let $1 < p, q < \infty$,

$$B_1^* = \sup_{t>0} \| \chi_{(0,t]} u^{\frac{1}{q}} \|_{X^q} \left(\int_t^\infty v^{1-p'} \right)^{\frac{1}{p'}}.$$

Then $B_1^* \le B^*$.

Proof. We have

$$
\begin{aligned}
B^* &= \sup_{t>0} \tilde{V}^{\frac{(s-1)}{p}}(t) \| \tilde{V}^{\frac{(p-s)}{p}} \cdot u^{\frac{1}{q}} \chi_{(0,t]} \|_{X^q} \\
&= \sup_{t>0} \tilde{V}^{\frac{(s-1)}{p}}(t) \sup_{h>0} \left\{ \int_0^t \left(\int_x^\infty v^{1-p'} \right)^{\frac{(p-s)q}{p}} u(x)h(x)dx : \|h\|_{X'} \le 1 \right\}^{\frac{1}{q}} \\
&\ge \sup_{t>0} \tilde{V}^{\frac{(s-1)}{p}}(t) \sup_{h>0} \left(\int_t^\infty v^{1-p'} \right)^{\frac{(p-s)}{p}} \left\{ \int_0^t u(x)h(x)dx : \|h\|_{X'} \le 1 \right\}^{\frac{1}{q}} \\
&= \sup_{t>0} \tilde{V}^{\frac{1}{p'}}(t) \| \chi_{(0,t]} u \|_X^{\frac{1}{q}} \\
&= B_1^*.
\end{aligned}
$$

Now, we state our theorem for the dual Hardy operator:

Theorem 3.7. *Let* $1 < q < p < \infty$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ *and* u, v *be weight functions on* $(0, \infty)$. *Then the inequality* (3.8) *holds for all measurable functions* $f \ge 0$ *if and only if* $J^* < \infty$, *where*

$$
J^* = \left\{ \int_0^\infty \| \chi_{(0,x]} u \|_X^{\frac{r}{q}} \left(\int_x^\infty v^{1-p'}(t)dt \right)^{\frac{r}{q'}} v^{1-p'}(x)dx \right\}^{\frac{1}{r}}. \tag{3.10}
$$

Moreover, the best constant C in (3.8) has the estimate

$$
q^{\frac{1}{q}} \left(\frac{qp'}{r} \right)^{\frac{1}{q'}} J^* \le C \le q^{\frac{1}{q}} (p')^{\frac{1}{q'}} J^*.
$$

Proof. The proof follows on the same lines as that of Theorem 3.3. We make use of Lemma 3.6 in the necessary part.

4 The Case: $0 < q < 1 < p < \infty$

The case $0 < q < 1 < p < \infty$ requires different techniques than used earlier in Section 3. First we give the following result from [5].

Theorem 4.1. *Let* f *be a non-negative measurable function and* w *be a weight function both defined on* $(0, \infty)$ *such that*

$$
\int_0^\infty f(t)dt < \infty, \quad \int_0^\infty w(t)dt < \infty. \tag{4.1}
$$

Then there exists a non-negative measurable function f_0 satisfying

(i) $\int_0^x f(t)dt \le \int_0^x f_0(t)dt, \quad x\in(0,\infty);$ (4.2)

(ii) $\dfrac{f_0}{w}$ is non-increasing on $(0,\infty)$; (4.3)

(iii) $\left\|\dfrac{f_0}{w}\right\|_{p,(0,\infty),w} \le \left\|\dfrac{f}{w}\right\|_{p,(0,\infty),w}$ for every $p\in[1,\infty)$. (4.4)

Remark 4.2. The function f_0 in Theorem 4.1 is the level function corresponding to f which was introduced by Halperin [5] and used by Sinnamon [14, 15] to prove the L^p-L^q boundedness of Hardy operator for the case $0<q<1<p<\infty$. The corresponding L^p-X^q boundedness is proved in the following result.

Theorem 4.3. *Let* $0<q<1<p<\infty$ *and u, v be weight functions defined on* $(0,\infty)$ *such that* $v^{1-p'}\in L^1$. *Then the inequality* (3.1) *holds for all measurable functions* $f\ge 0$ *if and only if* (3.3) *holds. Also, the best constant* C *in* (3.1) *has the estimate*

$$q^{\frac{1}{q}}(p')^{\frac{1}{q'}}J \le C \le q^{\frac{1}{q}}J\,. \tag{4.5}$$

Proof. We prove the sufficiency first.

Assume first that $f\in L^1$. Since f satisfies the conditions of Theorem 4.1, there exists a function f_0 satisfying (4.2)-(4.4). Therefore

$$\|(Hf)^q\cdot u\|_X^{\frac{1}{q}} \le \sup_{h>0}\left\{\int_0^\infty\left(\int_0^x f_0(t)dt\right)^q u(x)h(x)dx : \|h\|_{X'}\le 1\right\}^{\frac{1}{q}} \tag{4.6}$$

Also, for $w=v^{1-p'}$, (4.4) gives

$$\left\{\int_0^\infty f_0^p(x)v(x)dx\right\}^{\frac{1}{p}} \le \left\{\int_0^\infty f^p(x)v(x)dx\right\}^{\frac{1}{p}} \tag{4.7}$$

and consequently, in view of (4.6) and (4.7), it suffices to establish the inequality

$$\|(Hf_0)^q\cdot u\|_X^{\frac{1}{q}} \le q^{\frac{1}{q}}J\left\{\int_0^\infty f_0^p(x)v(x)dx\right\}^{\frac{1}{p}}.$$

Now, by applying Fubini's theorem, we get

$$\| (Hf_0)^q \cdot u \|_X^{\frac{1}{q}} = \sup_{h>0} \left\{ q \int_0^\infty \left[\int_0^x \left(\int_0^y f_0(t)dt \right)^{q-1} f_0(y)dy \right] u(x)h(x)dx : \| h \|_{X'} \le 1 \right\}^{\frac{1}{q}}$$

$$= \sup_{h>0} \left\{ q \int_0^\infty \left(\int_0^y f_0(t)dt \right)^{q-1} f_0(y) \left(\int_y^\infty u(x)h(x)dx : \| h \|_{X'} \le 1 \right) dy \right\}^{\frac{1}{q}}. \qquad (4.8)$$

Next, for a fixed $y \in (0,\infty)$, we have in view of (4.3) that for every $t \in (0, y)$

$$\frac{f_0(y)}{w(y)} \le \frac{f_0(t)}{w(t)}$$

and therefore

$$\left(\int_0^y f_0(t)dt \right)^{q-1} \le \left(\frac{f_0(y)}{w(y)} \right)^{q-1} \left(\int_0^y w(t)dt \right)^{q-1}$$

using which and applying Hölder's inequality with exponents $\frac{r}{q}\left(= \frac{p}{p-q} \right)$, $\frac{p}{q}$ in (4.8), we get

$$\| (Hf_0)^q \cdot u \|_X^{\frac{1}{q}} \le q^{\frac{1}{q}} \sup_{h>0} \left\{ \int_0^\infty \left(\frac{f_0(y)}{w(y)} \right)^{q-1} \left(\int_0^y w(t)dt \right)^{q-1} f_0(y) \left(\int_y^\infty u(x)h(x)dx : \| h \|_{X'} \le 1 \right) dy \right\}^{\frac{1}{q}}$$

$$= q^{\frac{1}{q}} \left\{ \int_0^\infty \| \chi_{[y,\infty)} u \|_X \left(\int_0^y w(x)dx \right)^{q-1} w^{\frac{q}{r}}(y)\, f_0^q(y) w^{1-q-\frac{q}{r}}(y)dy \right\}^{\frac{1}{q}}$$

$$\le q^{\frac{1}{q}} \left\{ \int_0^\infty \| \chi_{[y,\infty)} u \|_X^{\frac{r}{q}} \left(\int_0^y w(x)dx \right)^{\frac{r}{q'}} w(y)dy \right\}^{\frac{1}{r}} \left\{ \int_0^\infty f_0^p(y) w^{1-p}(y)dy \right\}^{\frac{1}{p}}$$

$$= q^{\frac{1}{q}} J \left\{ \int_0^\infty f_0^p(x) v(x)dx \right\}^{\frac{1}{p}}$$

and we are done.

In case, f is a general non-negative measurable function on $(0,\infty)$, we define

$$f_n(x) = \min\left(f(x), \frac{n}{x^2}, n \right).$$

Then $f_n \in L^1$ and consequently by the above arguments, the inequality (3.1) holds with f_n instead of f, i.e., the inequality

$$\| (Hf_n)^q \cdot u \|_X^{\frac{1}{q}} \leq C \left\{ \int_0^\infty f_n^p(x) v(x) dx \right\}^{\frac{1}{p}}$$

holds with $C = q^{\frac{1}{q}} J$. Letting $n \to \infty$, we obtain (3.1) by the monotone convergence theorem.

For the necessity, assume that the inequality (3.1) holds. Then according to Remark 3.2, for $0 < q < \infty$, $p > 1$,

$$\| \chi_{[x,\infty)} u \|_X < \infty, \int_0^x v^{1-p'} < \infty, \text{ for } x \in (0,\infty).$$

Choose two sequences $\{a_n\}$, $\{b_n\}$ in $(0,\infty)$ such that $a_n \downarrow 0$, $b_n \uparrow \infty$ and for $n \in \mathbf{N}$ define

$$J_n = \left\{ \int_{a_n}^{b_n} \| \chi_{[x,\infty)} u \|_X^{\frac{r}{q}} \left(\int_{a_n}^x v^{1-p'}(t) dt \right)^{\frac{r}{q'}} v^{1-p'}(x) dx \right\}^{\frac{1}{r}}. \tag{4.9}$$

Clearly for each n, $J_n < \infty$. Indeed

$$J_n \leq \| \chi_{[a_n,\infty)} u \|_X^{\frac{1}{q}} \left\{ \int_{a_n}^{b_n} \left(\int_{a_n}^x v^{1-p'}(t) dt \right)^{\frac{r}{q'}} v^{1-p'}(x) dx \right\}^{\frac{1}{r}}$$

$$= \left(\frac{p'}{r} \right)^{\frac{1}{r}} \| \chi_{[a_n,\infty)} u \|_X^{\frac{1}{q}} \left(\int_{a_n}^{b_n} v^{1-p'}(t) dt \right)^{\frac{1}{p'}} < \infty.$$

Define

$$f_n(x) = \| \chi_{[x,\infty)} u \|_X^{\frac{r}{p}} \left(\int_{a_n}^x v^{1-p'}(t) dt \right)^{\frac{r}{q'}} v^{1-p'}(x) \chi_{(a_n,b_n)}(x).$$

Clearly, each f_n is non-negative measurable function and from (4.9) we have

$$\left\{ \int_0^\infty f_n(x) \| \chi_{[x,\infty)} u \|_X \, dx \right\}^{\frac{1}{r}} = J_n .$$

Note that

$$\int_0^x f_n(t)dt = \left[\left(\int_0^x f_n(t)dt\right)^{\frac{1}{q}}\right]^q$$

$$= \left\{\frac{1}{q}\int_0^x \left(\int_0^t f_n(s)ds\right)^{\frac{-1}{q'}} f_n(t)dt\right\}^q .$$

Therefore by applying Fubini theorem, we get

$$J_n^{\frac{r}{q}} = \sup_{h>0}\left\{\int_0^\infty u(t)h(t)\left(\int_0^t f_n(x)dx\right)dt : \| h \|_{X'} \le 1\right\}^{\frac{1}{q}}$$

$$= \frac{1}{q}\sup_{h>0}\left\{\int_0^\infty \left[\int_0^x \left(\int_0^t f_n(s)ds\right)^{\frac{-1}{q'}} f_n(t)dt\right]^q h(x)u(x)dx : \| h \|_{X'} \le 1\right\}^{\frac{1}{q}}$$

$$= \frac{1}{q}\sup_{h>0}\left\{\int_0^\infty (Hg_n)^q(x)u(x)h(x)dx : \| h \|_{X'} \le 1\right\}^{\frac{1}{q}},$$

where $g_n(t) = \left(\int_0^t f_n(s)ds\right)^{\frac{-1}{q'}} f_n(t)$.

The inequality (3.1) holds also for g_n, therefore by an application of Hölder's inequality with exponents $\frac{1}{q}, \frac{1}{1-q}$, we obtain

$$qJ_n^{\frac{r}{q}} \le C\left\{\int_0^\infty g_n^p(x)v(x)dx\right\}^{\frac{1}{p}}$$

$$= C\left\{\int_{a_n}^{b_n} f_n^p(x)v^{p'}(x)\left(\int_{a_n}^x v^{1-p'}(s)ds\right)^{p(1-q)} v^{(1-p')q}(x)\right.$$

$$\left.\times\left(\int_{a_n}^x f_n(s)ds\right)^{\frac{-p}{q'}}\left(\int_{a_n}^x v^{1-p'}(s)ds\right)^{p(q-1)} v^{(1-p')(1-q)}(x)dx\right\}^{\frac{1}{p}}$$

$$\le CJ_n^{\frac{rq}{p}}\left\{\int_{a_n}^{b_n}\left(\int_{a_n}^x f_n(s)ds\right)^{\frac{p}{q}}\tilde{u}(x)dx\right\}^{\frac{1-q}{p}},$$

where $\tilde{u}(x)=\left(\int_{a_n}^{x} v^{1-p'}(s)ds\right)^{-p} v^{1-p'}(x)$, $x \in (a_n, b_n)$.

As in [12, p. 18], we obtain the estimate

$$\left\{\int_{a_n}^{b_n}\left(\int_{a_n}^{x} f_n(s)ds\right)^{\frac{p}{q}}\tilde{u}(x)dx\right\}^{\frac{q}{p}} \le k\left(\frac{p}{q},\frac{p}{q}\right)\left\{\int_{a_n}^{b_n} f_n^{\frac{p}{q}}(x)\tilde{v}(x)dx\right\}^{\frac{q}{p}}$$

with

$$\tilde{v}(x)=\left(\frac{p}{q}-1\right)^{\frac{p}{q}-1}[\tilde{w}(x)]^{1-\frac{p}{q}}\left(\int_{x}^{b_n}\tilde{w}(t)dt\right)^{\frac{p}{q}},\ x\in(a_n,b_n)$$

and can show that the RHS of this estimate is not greater than

$$C\left(\frac{p'}{q}\right)^{\frac{-1}{q'}} J_n^{\frac{r}{p}}.$$

Combining, we get

$$q\left(\frac{p'}{q}\right)^{\frac{1}{q'}} J_n \le C$$

and Fatou Lemma yields (4.5).

Finally, we prove inequality (3.8) for the case $0<q<1<p<\infty$. For this, we will make use of the complementary level intervals and complementary level functions which were introduced by Jain, Jain and Gupta [8]. We mention the following theorem from [8].

Theorem 4.4. *Let* f *be a non-negative measurable function and* w *be a weight function both defined on* $(0,\infty)$ *such that*

$$\int_0^\infty f(x)<\infty,\ \int_0^\infty w(x)dx<\infty.$$

Then there exist a non-negative measurable function f_0 satisfying

(i) $\int_x^b f(t)dt \le \int_x^b f_0(t)dt,$ for $x \in (0,\infty);$

(ii) $\frac{f_0}{w}$ is increasing on $(0,\infty);$

(iii) $\left\| \frac{f_0}{w} \right\|_{p,(0,\infty),w} \le \left\| \frac{f}{w} \right\|_{p,(0,\infty),w}$, for every $p \in [1,\infty)$.

Now, we have the following

Theorem 4.5. *Let* $0 < q < 1 < p < \infty$ *and* u,v *be weight functions on* $(0,\infty)$ *such that* $v^{1-p'} \in L^1(0,\infty)$. *Then the inequality* (3.8) *holds for all measurable functions* $f \ge 0$ *if and only if* (3.9) *holds. Moreover, the constant* C *in* (3.8) *has the estimate*

$$q^{\frac{1}{q}}(p')^{\frac{1}{q'}} J^* \le C \le q^{\frac{1}{q}} J^* .$$

Proof. The proof follows the same idea as that of Theorem 4.3.

5 Final Remarks

Remark 5.1. In this paper, we have considered the interval $(0,\infty)$. However, all the results are also valid in a more general interval (a,b), $-\infty \le a < b \le \infty$.

Remark 5.2. In the present paper, the L^p - X^q boundedness of the Hardy operator H and its conjugate operator H^* for the case $1 < q < p < \infty$ and $0 < q < 1 < p < \infty$ has been studied. These operators can also be studied for boundedness from X^p - X^q. Our results could motivate to study other important operators as well such as fractional order Hardy operator, maximal operator etc. in the context of X^p-spaces. For example the X^p - X^q boundedness of the Hardy operator H for the case $1 < p \le q < \infty$ should have the following form:

Conjecture 5.3. Let $1 < p \le q < \infty$, u,v be weight functions defined on $(0,\infty)$ and $s \in (1,p)$. Then the following statements are equivalent:

(a) The inequality $\| (Hf) \cdot u^{\frac{1}{q}} \|_{X^q} \le C \| f \cdot v^{\frac{1}{p}} \|_{X^p}$ holds for all measurable functions $f \ge 0$.

(b) $\tilde{B}_1 := \sup_{t>0} \| \chi_{[t,\infty)} u \|_X^{\frac{1}{q}} \, \| \chi_{(0,t]} v^{1-p'} \|_X^{\frac{1}{p'}} < \infty$.

(c) $\tilde{B}_2 := \sup_{t>0} \| \chi_{(0,t]} v^{1-p'} \|_X^{\frac{(s-1)}{p}} \left\| \| \chi_{(0,t]} v^{1-p'} \|_X^{\frac{(p-s)}{p}} u^{\frac{1}{q}} \chi_{[t,\infty)} \right\|_{X^q} < \infty.$

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