Mathematical Statistician and Engineering Applications
ISSN:2094-0343
2326-9865

The Daniell Integrals with Values in a Banach Space

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Article Info
Page Number: 856-863
Publication Issue:
Vol. 72 No. 1 (2023)

Article History
Article Received: *15 October 2022*
Revised: *24 November 2022*
Accepted: *22 November 2018*

Abstract
In this paper, we investigate the Daniell integral to an integral with values in an arbitrary Banach space. We start from a space of real valued functions on which an integral is defined and extend the integration to a complete space of functions with values in a Banach space.
Keywords: Daniell integral, Banach space, vector lattice, Daniell space, complete space.

Introduction

The history of definite integral is fascinating because the concept was developed to solve issues with estimating the lengths, areas, and volumes of curved geometric objects. Greek mathematicians originally tackled these issues, and they went through several stages before Riemann established the idea of integration over a period in 1868 and Lebesgue introduced the integral based on the idea of measurement in 1902. Known as "A general kind of integral," Daniell released his article in 1918 and described the integral as a nonnegative linear function defined on a Riesz space. Compared to the Riemann and Lebesgue integrals, the Daniell integral was more universal.

He established the definition of Daniell space in [1] and expanded it to demonstrate that it contains the first. He also defined the lower and upper Daniell integrals. He then dealt in [3] with the same definition in more detail. We observe in [2] that the researchers provided a different description for the functions in the Daniell space extension, as well as the concept of the full space, and demonstrated that the extension space is complete. In [4], he describes how to construct the Bochner integral on a Banach space and presents a straightforward restriction of the vector valued integral on abstract measure space. We describe the Daniell space as a whole as a Banach space.

In our paper, we presented the definition of the extension space as it was known in**[1]**and we showed that this space is complete based on the definition of the complete space in**[2]** also We adapt the method employed in the paper **[4]**. However the paper discusses the Bochner integral on a Banach space, but we discusses the complete Daniell space on a Banach space based on what was presented in **[4]**.

Fundamental Concepts

In this section the important and basic concepts are given to expression all the results that need it later**.**

Definition2.1,[1]:

LetΩ be an arbitrary set and h,$k$are real valued functions on Ωthen we defined
$h \vee k = \max\{h,k\} = max\{h-k,0\} + k$and
$h \wedge k = \min\{h,k\} = (h+k) - \max\{h,k\}$,where 0 is the zero function.

Definition 2.2,[7]:

Let L be a set of real valued function defined on Ω. we say that L is a lattice if $\max\{h,k\}$, $\min\{h,k\} \in L$ for all $h,k \in L$.The linear space of all real valued lattice functions on $\Omega$is called Riesz space

Remarks 2.3,[5]:

(1) Let L be a linear space of functions($h: \Omega \to \mathbb{R}$). Then S is a vector lattice(Riesz space) if $max\{h,0\} \in L$ for all h∈L.

(2) If h is a real valued function in Riesz space then $|h|$ is also in a Riesz space.

Definition2.4,[8]:

Let Ω be any set and $h : \Omega \to \mathbb{R}$ a function, we define the positive and negative parts $h^+$and h^- by
$h^+ = max\{h, 0\}$and $h^- = -min\{h, 0\} = max\{-h, 0\}$
where

$$h^+(x) = \begin{cases} h(x), & h(x) \geq 0 \\ 0, & h(x) \leq 0 \end{cases} \quad \text{and} \quad h^-(x) = \begin{cases} -h(x), & h(x) \leq 0 \\ 0, & h(x) > 0 \end{cases}$$

hold the following for h^+ and $h^-$are nonnegative such that

(1) $h = h^+ - h^-$and $|h| = h^+ + h^- = h^+ + (-h)^-$

(2) $h^+ = \frac{1}{2}(|h| + h)$ and $h^- = \frac{1}{2}(|h| - h)$

(3) $(-h)^+ = h^-$and $(-h)^- = h^+$

(4) If $\lambda > 0$, then $(\lambda h)^+ = \lambda h^+$and $(\lambda h)^- = \lambda h^-$.

Definition 2.5, [2]:

A triple (Ω, L, D) is a Daniell space if F is a nonempty set, L is a Riesz space of real valued functions on F, and $D: L \to \mathbb{R}$ is a Daniell functional.

Definition 2.6, [8]:

$D$is continuous under monotone limit if and only if $D(h_n) \downarrow 0$ whenever $h_n \downarrow 0$ and each $h_n \in L$.

Definition 2.7, [2]:

Let L be a Riesz space of functions defined on F. A linear functional $D: \Omega \to \mathbb{R}$ is called

(1) Positive if $D(h) \geq 0$ whenever $h \in L$ and $f \geq 0$.

(2) continuous if and only if $D(h_n)\downarrow 0$ whenever $h_n \downarrow 0$ for each $h_n \in L$.

(3) Continuous under monotone limits if for every increasing sequence $\{h_n\}$ of functions in L and $h \in L$ such that $h(x) \leq \lim_{n\to\infty} f_n(x)$ for all $x \in \Omega$, then $D(f) \lim_{n\to\infty} = D(f_n)$.

(4) if D is positive, then $D(f) \leq D(g)$ for each $f \in L$ and $h \leq k$.

Then Daniell functional (Daniell integral) whenever D is positive and continuous under monotone limit.

Theorem 2.8, [1]:

Let L be a Riesz space on F. Suppose that D is a Daniell integral on S. Then $D(f) \leq \sum_{n=1}^{\infty} D(f_n)$ whenever $\{f_n\}$ is a sequence of nonnegative functions in L and $f \in L$ such that $f(x) \leq \sum_{n=1}^{\infty} f_n(x)$ for all $x \in \Omega$.

Definition 2.9, [2]:

If D be a Daniell functional, a function $f \in L$ is called a null function if $D(|f|) = 0$.

Example 2.10:

Let $\Omega = (0,1)$ define $f \in L$ by $f(x) = 0$if $x \notin \mathbb{Q}$, if $x = \frac{q}{l}$ then $f(x) = \frac{1}{l}$, also let $f(0) = f(1) = 0$ then this function is a null function thus $D(f) = f(0) - f(1) = 0$.

Remark 2.11, [3]:

If D be a Daniell functional and f is a null function and $|g| \leq |f|$ then g is a null function.
Since $0 \leq D(|g|) \leq D(|f|) = 0$.

Definition 2.12, [2]:

If D be a Daniell functional . A set $A \subseteq \Omega$ is called a null set if the characteristic function of A is a null function. That is $D(|I_A|) = 0$.

Example 2.13:

Let $\Omega = \mathbb{R}$and $A = \{f = 0\} \subseteq \mathbb{R}$, then A is a null set,
since$D(I_A) = D(0) = 0$.

Definition 2.14, [2]:

Let(Ω, L, D)be a Daniell space.A norm on L is a function $\|.\|: L \to \mathbb{R}$ which is defined by $\|f\| = D(|f|)$ having the following properties,

(1) $\|f\| \geq 0$ for all $f \in L$,

(2) $\|f\| = 0$ iff $f = 0$ a.e.,

(3) $\|\lambda f\| = |\lambda|\|f\|$ for all $f \in L$ and $\lambda \in \mathbb{R}$,

(4) $\|f + g\| \leq \|f\| + \|g\|$for all $f, g \in L$.

A Riesz space S together with $\|\cdot\|$ is called a normed space and it is denoted by $(L\ ,\|\cdot\|)$.

Remark 2.15:

Every subspace of a normed space is also normed space, that is $(L, \|.\|)$ be a normed space and$\|[f]\| = D(|f|)$.

Definition2.16:

A seminorm on L is a function $\mathfrak{p}: L \to \mathbb{R}$ which is defined by $\mathfrak{p}(f) = D(|f|)$, having the following properties,

(1) $\mathfrak{p}(\lambda f) = |\lambda|\mathfrak{p}(f)$ for all $f \in L$ and for all $\lambda \in \mathbb{R}$,

(2) $\mathfrak{p}(f + g) \leq \mathfrak{p}(f) + \mathfrak{p}(g)$ for all $f, g \in L$

A family $\mathcal{T}$ of seminorms on L is said to be separating if for each $f \neq 0$ corresponds at least one $\mathfrak{p} \in \mathcal{T}$ with $\mathfrak{p}(f) \neq 0$.

Theorem 2.17:

Suppose $\mathfrak{p}$ is a seminorm on , then:

(1) $\mathfrak{p}(0) = 0$,

(2) $\mathfrak{p}(-f) = \mathfrak{p}(f)$ for all $f \in L$,

(3) $\mathfrak{p}(f - g) = \mathfrak{p}(g - f)$ for all $f, g \in L$,

(4) $|\mathfrak{p}(f) - \mathfrak{p}(g)| \leq \mathfrak{p}(f - g)$ for all $f, g \in L$,

(5) $\mathfrak{p}(f) \geq 0$for all $f \in L$.

Proof: Let $f, g \in L$

(1) Let $f \in L$, Since $\mathfrak{p}(f) = D(|f|)$ then $\mathfrak{p}(0) = D(0) = 0$.

(2) Let $f \in L, \mathfrak{p}(-f) = D(|-f|) = D(|f|) = \mathfrak{p}(f)$.

(3) Let$f, g \in L, \mathfrak{p}(f - g) = D(|f - g|) = D(|g - f|) = \mathfrak{p}(g - f)$

(4) Let$f, g \in L, |\mathfrak{p}(f) - \mathfrak{p}(g)| = |D(|f|) - D(|g|)| \geq |D(|f|)| - |D(|g|)| \leq D(|f|) - D(|g|) = D(|f - g|) = \mathfrak{p}(f - g)$.

(5) Let$f \in L$, since $D(|f|) \geq 0$ implies that $\mathfrak{p}(f) \geq 0$.

Remark 2.18:

(1) $\|\cdot\|$need not be a norm since if $\|f\| = 0$ need not to be $f = 0$ if and only if $f = 0\ a.e.$, such that if $f = 0$ a.e. then there is a set $A = \{x: f(x) \neq 0\}$, $A \subseteq F$, which is a null set, then $|f| = 0$ a.e. implies $D(f) = 0$. Therefore $\|f\| = 0$. Conversely, if $\|f\| = 0$, then $D(f) = 0$, since $|f| \geq 0$ then $f = 0$.

(2) A normed space or a seminormed which is complete in the metric induced by the norm is called a Banach space, that is every Cauchy sequences is convergent.

Complete Daniell Space

Definition 3.1, [5]:

Let L^* be the class of all extendedreal valued functions on Ωrepresented as a limit of a monotone non-decreasing sequences of functions in the vector lattice L.

That is (if$L$is a vector lattice, then $h \in L^*$if and only if$h: \Omega \to \overline{\mathbb{R}}$ a function and there exists a sequence $\{h_n\}$ of monotone increasing sequences of functions in S such that $h = \lim_{n\to\infty} h_n$).

Definition 3.2, [2]:

Let f be a real function on Ω. if there exist afunction $f_n \in L, n \in \mathbb{N}$, such that

(1) $\sum_{n=1}^{\infty} D(|f_n|) < \infty$

(2) $f(x) = \sum_{n=1}^{\infty} f_n(x)$for every $x \in \Omega$ and $\sum_{n=1}^{\infty}|f_n(x)| < \infty$, then we write $f = \sum_{n=1}^{\infty} f_n$.

Definition 3.3, [2]:

A Daniell space (Ω, L, D) is called complete if $f = \sum_{n=1}^{\infty} f_n$ for some $f_1, f_2, , \ldots \in L$, implies that $f \in L$.

Theorem 3.4, [8]:

(1) Let $\{f_n\}$ and $\{g_m\}$be a monotone increasing sequences such that f_n and g_m are in L for any$n, m \in \mathbb{N}$ and let $\lim_{n\to\infty} f_n \leq \lim_{m\to\infty} g_m$. Then $\lim_{n\to\infty} D(f_n) \leq \lim_{m\to\infty} D(g_m)$.

Furthermoreif f is in $L^*$and$f_n \uparrow f$,$g_m \uparrow f$then $\lim_{n\to\infty} D(f_n) = \lim_{m\to\infty} D(g_m)$.

(2) If f is in L^*, then there exist an increasing sequence$\{f_n\}$ such that f_n is in$L$ for all n and $f = \lim_{n\to\infty} f_n$. Then $D(f) = \lim_{n\to\infty} D(f_n)$.

(3) Let $f: \Omega \to \overline{\mathbb{R}}$be a function and $f(x) \geq 0 \ \forall \ x \in \Omega$then $f$is in L^* if and only if there exist a sequences $\{f_n\}$ of nonnegative functions in L with $f = \sum_{n=1}^{\infty} f_n$. Further, $D(f) = \sum_{n=1}^{\infty} D(f_n)$.

Theorem 3.5:

LetL^*be extended lattice set then a triple (Ω, L^*, D) is a complete Daniell space.

Proof :

i. First we have to prove that L^* is a Riesz space

Let $h, k \in L^*, \lambda, \beta \in \mathbb{R}$, then $h = \lim_{n\to\infty} h_n$and $k = \lim_{m\to\infty} k_m$, where$h_nandk_n$ are increasing sequences of functions in L, then, $\lambda h(x) + \beta k(x) = \lambda \lim_{n\to\infty} h_n(x) + \beta \lim_{m\to\infty} k_m(x) =$

$\lim_{n\to\infty} \lambda\, h(x) + \lim_{m\to\infty} \beta\, k_n(x) = \lim_{n,m\to\infty} (\lambda h_n(x) + \beta k_m(x))$ .there fore $\lambda h + \beta k$ is in L^*.

Since L is a Riesz space then $h_n \vee k_n$ is in L for all n then $\lim_{n\to\infty} (h_n \vee k_n)$ is in S^* to show that $h_n \vee k_n$ is monotone increasing, let $x \in \Omega$ then $h_n(x) \leq h_{n+1}(x) \leq (h_{n+1} \vee k_{n+1})(x)$

and$k_n(x) \leq k_{n+1}(x) \leq (h_{n+1} \vee k_{n+1})(x)$, so that $(h_n \vee k_n)(x) \leq (h_{n+1} \vee k_{n+1})(x)$.

Therefore $h_n \vee k_n$ is monotone increasing.If $\lim h_n(x) = \infty$ or $\lim g_n(x) = \infty$,

$\lim (h_n \vee k_n)(x) = \infty$ and$(h \vee k)(x) = \infty$ then $(h \vee k)(x) = \lim (h_n \vee k_n)(x)$.

Let $\lim h_n(x) \neq \infty$ and $\lim k_n(x) \neq \infty$, then $\lim (h_n \vee k_n)(x) \neq \infty$ and $(h \vee k)(x) \neq \infty$.

Now define $f_n = h_n \vee k_n$ for all n. Then f_n is in L, and $\lim f_n(x) = \lim (h_n \vee k_n)(x)$.

Let$f_n(x) > \max(\lim h_n(x), \lim k_n(x)) = \max(h(x), k(x))$.

Suppose that there exist $N \in \mathbb{Z}^+$such that $n \geq N$ implies that $f_n(x) > \max(h(x), k(x))$, sincef_n is monotone.

But $\{h_n\}, \{k_n\}$ monotone increasing implies $\lim k_n(x) \geq k_n(x)$ and $\lim h_n(x) \geq h_n(x)$ for all n.

Thus $f_n(x) > \max(h_i(x), k_i(x))$for all i and for all $n \geq N$ which is a contradication.

Therefore, $\lim f_n(x) \leq \max(h(x), k(x))$. But $f_n(x) \geq h_n(x)$ for all n implies $f_n(x) \geq h(x)$, and $f_n(x) \geq k_n(x)$ for all n implies $f_n(x) \geq k(x)$, thus $\lim f_n(x) \geq \max(h(x), k(x))$, so $\lim f_n(x) = \max(h(x), k(x))$.

Therefore, $h \vee k$ is in L^*, by the same way we can prove that $h \wedge k$ is in L^*. Clearly $L \subset L^*$, then L^* is a lattice contained L.

ii. Now to prove that $D: L^* \to \mathbb{R}$ is a positive linear function on L^*, let $h \in L^*, h \geq 0$, there is an increasing sequence$\{h_n\}$ such that h_n is in L for all $n = 1,2,3, \ldots$. and $0 \leq h = \lim_{n\to\infty} h_n$ implies $\lim_{n\to\infty} D(h_n) \geq D(0) = 0$.

Let $h, k \in L^*$ then $h = \lim_{n\to\infty} h_n$ and $k = \lim_{m\to\infty} k_m$, where$h_n$andk_n are monotone increasing sequences of function in L.

Suppose that $h \leq k$ implies that $\lim_{n\to\infty} h_n \leq \lim_{m\to\infty} k_m$ then

$D(h) = D(\lim_{n\to\infty} h_n) \leq D\left(\lim_{m\to\infty} k_m\right) = D(k)$implies$D(h) = \lim_{n\to\infty} D(h_n) \leq \lim_{m\to\infty} D(k_m) = D(k)$. There fore $D(h) \leq D(k)$.

Let $h, k \in L^*$ and $\alpha, \beta \in \mathbb{R}$ then $D(\alpha h + \beta k) = D(\alpha(\lim_{n\to\infty} h_n) + \beta(\lim_{m\to\infty} k_m)) =$

$\alpha D(\lim_{n\to\infty} h_n) + \beta D(\lim_{m\to\infty} k_m) = \alpha D(h) + \beta D(k)$.

iii. Now to prove that D is a Daniell functional on L^*.

Let $\{h_n\}$ be an increasing sequence in L^* and h in L^* with $h \leq \lim_{n\to\infty} h_n$, let $k_n = h_n - h_1$, $k_n \geq h$, then $D(h_n) = D(k_n) + D(h_1)$ implies that $\lim_{n\to\infty} D(h_n) = \lim_{n\to\infty} D(k_n) + D(h_1)$, let $k = \lim_{n\to\infty} k_n + h_1 = \lim_{n\to\infty} h_n$ then k in L^*, then $h \leq \lim_{n\to\infty} h_n = k$ implies

$D(h) \leq D(k) = \lim_{n\to\infty} D(k_n) + D(h_1) = \lim_{n\to\infty} D(h_n)$. There fore $D(h) \leq \lim_{n\to\infty} D(h_n)$.

Hence (Ω, L^*, D) is a Daniell space.

iv. to prove that (Ω, L^*, D) is complete space.

Let $h_n \in L^*, h_n \geq 0$ for each $n = 1,2, \ldots$ and $h = \sum_{n=1}^{\infty} h_n$, we must prove that $h \in L^*$.

Since $h_n \in L^*$ then by (1.2.9 (3)) there exist a sequences of positive functions $\{g_{m,n}\}$ In L for each n such that $h_n = \sum_{m=1}^{\infty} k_{n,m}$, then $h = \sum_{n=1}^{\infty} h_n = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} k_{n,m} = \sum_{n,m=1}^{\infty} k_{n,m}$ then $h \in L^*$ since $L \subset L^*$.

Therefore (Ω, L^*, D) is a complete Daniell space.

Theorem3.6:

Let $(\Omega, L^*. D)$ be a complete Daniell space and $f_n \in L^*$ we have, $\|\sum_{n=1}^{\infty} f_n\| \leq D(|\sum f_n|)$

Proof:

Let $f_n \in L^*$, since $\|\sum_{n=1}^{\infty} f_n\| = \|f_1 + \cdots + f_n\| \leq \|f_1\| + \cdots + \|f_n\| = D(\|f_1\| + \cdots + \|f_n\|)$
$= D(|f_1 + \cdots + f_n|)$.

Theorem3.7: Let $(\Omega, L^*. D)$ be a complete Daniell space then,

(1) $|f| \in L^*$

(2) $D(|f|) = \lim_{n\to\infty} D(|f_1 + \cdots + f_n|)$,

(3) $\|D(f_1) + D(f_2) + \cdots\| \leq D(|f|)$.

Proof:

(1) Let $g_n = \sum_{n=1}^{\infty} f_n$ for $n \epsilon N$ and $h_1 = |f_1|$ and $h_n = |g_n| - |g_{n-1}|$ for $n \geq 2$ we will show that $|f| = h_1 + |f_1| - |f_1| + h_2 + |f_2| - |f_2| + \cdots$, since $|h_n| = ||g_n| - |g_{n-1}|| \leq |g_n - g_{n-1}| \leq |f_n$ I we have, $\|h_1\| + \|f_1\| + \|f_1\| + \|h_2\| + \|f_2\| + \|f_2\| + \cdots \leq 3\sum_{n=1}^{\infty} \|f_n\| < \infty$.
If $|h_1(x)| + \|f_1(x)\| + \|f_1(x)\| + |h_2(x)| + \|f_2(x)\| + \|f_2(x)\| + \cdots < \infty$ for some $x \in \Omega$, then $\sum_{n-1}^{\infty} \|f_n(x)\|_1 < \infty$ and consequently $\sum_{n-1}^{\infty} f_n(x) = f(x)$. Hence $\sum_{n=1}^{m} h_n(x) = \|g_m(x)\| = \|\sum_{n=1}^{m} f_n(x)\| \to \| f(x) \|$ as $m \to \infty$. we obtain that $|f| \in L^*$.

(2) Since $D|f| = \mathrm{D}(h_1) + \mathrm{D}|f_1| - \mathrm{D}|f_1| + \mathrm{D}(h_2) + \mathrm{D}|f_2| - \mathrm{D}|f_2| + \cdots = \lim_{n\to\infty} D(h_1 + \cdots + h_n) = \lim_{n\to\infty} D|g_n| = \lim_{n\to\infty} \mathrm{D}|f_1 + \cdots + f_n|$.

(3) Since $\| \sum_{n=1}^{m} Df_n \| \leq \mathrm{D}|\sum_{n=1}^{m} f_n| = \mathrm{D}|g_m| = \mathrm{D}|f_1 + \cdots + f_m|$,
we have $\|Df_1 + \mathrm{D}f_2 + \cdots\| \leq \lim_{n\to\infty} \mathrm{D}|f_1 + \cdots + f_n| = \mathrm{D}|f|$.

Theorem 3.8:

The integral is a linear operator from the complete Daniell space to the Daniell functional.
That is, $\|D(f)\| \leq D(|f|)$ for all $f \in L^*$

Proof:

Linearity follows easily from the fact that, if $= \sum_{n=1}^{\infty} f_n$, $g = \sum_{n=1}^{\infty} g_n$ and $\gamma \in \mathbb{R}$, then
$f + g = f_1 + g_1 + f_2 + g_2$ and $\gamma f = \sum_{n=1}^{\infty} \gamma f_n$.
By part (3)of theorem 3.7,since $\|D(f)\| = \|D(\sum_{n=1}^{\infty} f_n)\| = \|D(f_1) + D(f_2) + \cdots\| \leq D(|f|)$.

The Complete Daniell Space as a Banach Space

in this section we proved that the normed space method can be applied to Daniellintegrable functions and show that the complete Daniell space is complete with respect to the norm .
we will start this section with the following definition.

Definition 4,1, [2]:

Let (Ω, L, D) be a Daniell space and let $f, f_n \in L, n \in \mathbb{N}$, we say that,

(1) f_n converges in norm to f, denoted by $f_n \xrightarrow{i.n.} f$, if $\|f_n - f\| \to 0$ as $n \to \infty$,

(2) $\{f_n\}$ is a cauchy in norm , denoted by f_n Cauchy i.n., if
$\|f_n - f_m\| \to 0$ as $n, m \to \infty$.

Example 4.2:

Let $F = [0,1]$ define $h_m = I_{[0,\frac{1}{n}]}$ then $h_m \xrightarrow{i.n.} 0$, since $\|h_m - 0\| = \|h_m\| = D\left(I_{[0,\frac{1}{n}]}\right) = \frac{1}{n} \to 0$ as $n \to \infty$.

Theorem 4.3:

Let (Ω, L, D) be a Daniell space and let $f \in L$ and $f = \lim_{n\to\infty} f_n$, then $f_n \xrightarrow{i.n.} f$.

Proof:

Let $\varepsilon > 0$, since $f = \lim_{n\to\infty} f_n$ there is $k \in \mathbb{Z}^+$, such that $|f_n - f| < \varepsilon$ for all $n \geq k$, then $D(|f_n - f|) < \varepsilon$ for all $n \geq k$ There fore $f_n \xrightarrow{i.n.} f$.

Remark 4.4:

If (Ω, L, D) be a Daniell space. We will denoted to the space of equivalent class in L by $\mathcal{L}$ and $[f]$ be the quivalence class of $f \in L$ such that $[f] = \{g \in L: D(|f - g|) = 0\}$.

To prove that $\sim_f$ be an equivalent relation on $\mathcal{S}$ we have to show that $\sim_f$ is,

(1) Reflexive: Let $\in S, |f-f| = |0| = 0$, then $D(|f-f|) = D(0) = 0$, so $D(|f-f|) = 0$. Therefore $f \sim_f f$.

(2) Symmetric: Let $f, g \in L$ and $f \sim_f g$ then $D(|f-g|) = 0 = D(|g-f|)$hence $g \sim_f f$.

(3) Transitive: Let $f, g, h \in L$ with $f \sim_f g$ and $g \sim h$ then $|f-h| = |f-h+g-g| \leq |f-g| + |g-h|$, so $|f-h| \leq |f-g| + |g-h|$, then $D(|f-h|) \leq D(|f-g| + |g-h|) = D(|f-g|) + D(|g-h|) = 0$, then $D(|f-h|) \leq 0$ and since $|f-h| \geq 0$, then $D(|f-h|) \geq D(0) = 0$, implies $D(|f-h|) \geq 0$, and hence $D(|f-h|) = 0$. Therefore $f \sim_f h$.

Theorem 4.5:

The space of equivalent class $(\Omega, \mathcal{L}, D)$ is a subspace of (Ω, L, D)

Proof:

It is clear that $\mathcal{L} \subseteq L$,

(1) Let$[f], [g] \in \mathcal{L}$ then$[f] + [g] = \{h \in L: D(|f-h|) = 0)\} + \{j \in L: D(|g-j|) = 0\} = \{h + j \in L: D(|f-h|) + D(|g-j|) = 0\} = \{h + j \in L: D(|f-h| + |g-j|) = 0\} = \{h + j \in L: D(|f+g| - |h+j|) = 0\} = [f+g]$. Therefore $[f] + [g] \in \mathcal{L}$

(2) Let$[f] \in \mathcal{L}$ and $\lambda \in \mathbb{R}$, then $\lambda[f] = \lambda\{g \in L: I^D(|f-g| = 0)\} = \{\lambda g \in L: \lambda D(|f-g|) = 0)\} = \{h = \lambda g \in L: D(|\lambda f - h| = 0)\} = \{h \in L: D(|\lambda f - h| = 0)\} = [\lambda f]$.Therefore $\lambda[f] \in \mathcal{L}$.

Theorem 4.6:

The normed space $(\mathcal{L}, \|\cdot\|)$is a Banach space.

Proof:

Let $\{f_n\}$be a Cauchy sequencein $\mathcal{L}$ then for every $\varepsilon > 0$ there exist $s \in \mathbb{Z}^+$ such that $\|f_n - f_m\| < \varepsilon$ for all $n, m \geq s$, suppose that $f_n \to f$ we must prove that f is in $\mathcal{S}$, since $f = \lim_{n\to\infty} f_n$**by (4.3)** we have $f_n \xrightarrow{i.n.} f$.

Theorem 4.7:

Let(Ω, L, D) be a Daniell space and let $f, f_n, g, g_n \in S, n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, such that $f_n \xrightarrow{i.n.} f$and $g_n \xrightarrow{i.n.} g$ then,

(1) f_n Cauchy a.u,

(2) $\lambda f_n \xrightarrow{i.n.} \lambda f$,

(3) $f_n + g_n \xrightarrow{i.n.} f + g$,

(4) $|f_n| \xrightarrow{i.n.} |f|$,

(5) $D(f_n) \xrightarrow{i.n.} D(f)$.

Proof:

(1) Since $f_n \xrightarrow{i.n.} f$ then $\|f_n - f\| = D(|f_n - f|) \to 0$ as $n \to \infty$implies that f_n is a Cauchy sequence in norm.

(2) Since $f_n \xrightarrow{i.n.} f$ then $\|f_n - f\| = D(|f_n - f|) \to 0$ as $n \to \infty$, $\lambda\|f_n - f\| = \lambda D(|f_n - f|) = D\big(\lambda(|f_n - f|)\big) = D(|\lambda f_n - \lambda f|) = \|\lambda f_n - \lambda f\| \to 0$ as $n \to \infty$, therefore$\lambda f_n \xrightarrow{i.n.} \lambda f$.

(3) Since $f_n \xrightarrow{i.n.} f$ then $\|f_n - f\| = D(|f_n - f|) \to 0$ as $n \to \infty$ and since $g_n \xrightarrow{i.n.} g$ then $\|g_n - g\| = D(|g_n - g|) \to 0$ as $n \to \infty$therefore
$\|(f_n + g_n) - (f + g)\| = D(|(f_n + g_n) - (f + g)|)$
$= D(|(f_n - f) + (g_n - g)|) \leq D(|f_n - f|) + D(|g_n - g|) \to 0 \; as \; n \to \infty$,then
$\|(f_n + g_n) - (f + g)\| \to 0 \; as \; n \to \infty$. Therefore $f_n + g_n \xrightarrow{i.n.} f + g$.

(4) Since $f_n \xrightarrow{i.n.} f$ then $\|f_n - f\| = D(|f_n - f|) \to 0$ as $n \to \infty$, then $\||f_n| - |f|\| = D\big(\big||f_n| - |f|\big|\big) \leq D(|f_n - f|) \to 0 \; as \; n \to \infty$then $\||f_n| - |f|\| \to 0$ as $n \to \infty$Therefore$|f_n| \xrightarrow{i.n.} |f|$.

(5) Since $f_n \xrightarrow{i.n.} f$ then $\|f_n - f\| = D(|f_n - f|) \to 0$ as $n \to \infty$, then $\|D(f_n) - D(f)\| = |D(f_n) - D(f)| = |D(f_n - f)| \leq D(|f_n - f|) \to 0$ as $n \to \infty$. Therefore $D(f_n) \xrightarrow{i.n.} D(f)$.

Theorem 4.8:
Let $(\Omega, L^{*}.D)$ be a complete Daniell space then for every $\epsilon > 0$ there exist a sequence of functions $\{f_n\}$ such that $f = \sum_{n=1}^{\infty} f_n$ and $\sum_{n=1}^{\infty}(|f_n| \leq D(|f|) + \epsilon$.

Proof:
Let $f = g_1 + g_2 + \cdots$ be an arbitary expansion of f. Then there exists an $\alpha_0 \in \mathbb{N}$ such that $\sum_{\alpha=\alpha_0+1} \mathrm{D}(|g_n|) < \frac{\varepsilon}{2}$. Define $f_1 = g_1 + \cdots + g_{\alpha_0}$ and $f_n = g_{\alpha+\alpha_0-1}$ for $n \geq 2$.
Then obviously $f = f_1 + f_2 + \cdots$, since$\mathrm{D}(|f_1|) - \mathrm{D}(|f|) \leq \mathrm{D}(|f_1 - f|)$ and $f - f_1 = f_2 + f_3 + \cdots$, we get $D(|f_1|) - \mathrm{D}(|f) \leq \sum_{n=2}^{\infty} \mathrm{D}(|f_n|)$ and hence, $D(|f_1|) - \sum_{n=2}^{\infty} \mathrm{D}(|f_n|) \leq \mathrm{D}(|f|)$
Consequently,$\sum_{n=1}^{\infty} D(f_n) = \mathrm{D}(|f_1|) + \sum_{n=2}^{\infty} D(f_n) = D(|f_1|) - \sum_{n=2}^{\infty} D(f_n) + 2\sum_{n=2}^{\infty} D(f_n) \leq D(|f|) + 2\sum_{n=2}^{\infty} D(|f_n|) = \mathrm{D}(|f|) + 2\sum_{n=n_0+1}^{\infty} \mathrm{D}(|g_n|) < \mathrm{D}(|f|) + \varepsilon$.

Theorem 4.9:
let$(\Omega, L^{*}.D)$ be a complete Daniell space, then $f_1 + f_2 + \cdots . = f i.n.$, and$D(f) = D(f_1) + D(f_2) + \cdots$

proof:
Let $\varepsilon > 0$ be arbitrary and let $\varepsilon_1 + \varepsilon_2 + \cdots$ be a series of positive numbers whose sum is ε. By theorem 4.8, we can choose expansions $f_i = f_{11} + f_{12} + \cdots, \quad (i = 1,2, \ldots)$,where $f_{ij} \in L^{*}$, such that $D(f_{\mathrm{i}1}) + D(f_{\mathrm{i}2}) + \cdots < D(f_{\mathrm{i}}) + \varepsilon_{\mathrm{i}1}$for all $i \in \mathbb{N}$. Let $g_1 + g_2 + \cdots$be a series of functions in L^{*}which is composed of all the series in (4.1). Then from (4.2) we obtain $D(g_1) + D(g_2) + \cdots < M + \varepsilon_1 + \varepsilon_2 + \cdots$,where $M = \mathrm{D}(f_1) + \mathrm{D}(f_2) + \cdots$, Moreover, if the series (4.3) converges absolutely at a point $x \in \Omega$, then esch of the series in (4.1) converges absolutely at that point, and consequently $g_1(x) + g_2(x) + \cdots = f_1(x) + f_2(x) + \cdots = f(x)$at that x. This proves that f is Daniellintegrableand $D(f) = D(g_1) + \mathrm{D}(g_2) + \cdots = \mathrm{D}(f_1) + \mathrm{D}(f_2) + \cdots$, Moreover, since for every $n \in \mathbb{N}$,$f - f_1 - \cdots - f_n = f_{n+1} + f_{n+2} + \cdots$, we have $\|f - f_1 - \cdots - f_n\|_1 \leq \sum_{k=n+1}^{\infty} \|f_k\|_1 \to 0$as $n \to \infty$, which means that $f_1 + f_2 + \cdots = f$ i.n.

Theorem 4.10
Let $(\Omega, L^{*}.D)$be a complete Daniell space and $f_1, f_2, \ldots \in L^{*}$ and $\sum_{n=1}^{\infty} \|(f_n)\| < \infty$, then there exists $f \in L^{*}$ such that $f = \sum_{n=1}^{\infty} f_n$.

proof:
Let $f_1, f_2, \ldots \in L^{*}$ and $\sum_{n=1}^{\infty} \|(f_n)\| < \infty$. Define $f(x) = \begin{cases} \sum_{n=1}^{\infty} f_n(x) & \\ 0 & o.w. \end{cases}$, $\sum_{n=1}^{\infty} |f_n(x)| < \infty$, then $f = \sum_{n=1}^{\infty} f_n$ implies that $f \in L^{*}$.

Theorem 4.11:
the space $((\Omega, L^{*}.D), \|\cdot\|)$ is complete

Proof:
We will prove that every absolutely convergent series converges in norm.
If $\sum_{n=1}^{\infty} \|(f_n)\| < \infty$, for some$f_n \in L^{*}$, then by theorem 4.10, there exist$f \in L^{*}$such that$f = \sum_{n=1}^{\infty} f_n$, by theorem 4.9, that the series $\sum_{n=1}^{\infty} f_n$convergesto f in norm.

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