Generalized Quotient Functions in Topological Spaces

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Abstract
Levine [2] offered a new and useful notion in General Topology, that is the notion of a generalized closed. A subset A of a topological space (X, τ)is called generalized closed (briefly g-closed) if cl(A) ⊆ U whenever A ⊆ U and U is open in (X,τ). This notion has been studied extensively in recent years by many topologists. The investigation of generalized closed sets had led to several new and interesting concepts. After the introduction of generalized closed there are many research papers which deal with different types of generalized closed. Recently Ravi and Ganesan [6] have introduced $\ddot{g}$-closed and studied their properties using sg-open set [1]. In this chapter we introduce $\ddot{g}$-quotient maps. Using these new types of maps, several characterizations and its properties have been obtained. Also the relationship between strong and weak forms of $\ddot{g}$-quotient maps have been established.

1. Introduction

Levine [2] offered a new and useful notion in General Topology, that is the notion of a generalized closed set. A subset A of a topological space (X, τ)is called generalized closed (briefly g-closed) if cl(A) ⊆ U whenever A ⊆ U and Uisopenin (X,τ). This notion has been studied extensively in recent years by many topologists. The investigation of generalized closed sets had led to several new and interesting concepts. After the introduction of generalized closed sets there are many research papers which deal with different types of generalized closed sets. Recently Ravi and Ganesan [6] have introduced $\ddot{g}_\alpha$-closed sets and studied their properties using sg-open set [1].In this chapter we introduce $\ddot{g}_\alpha$- quotient maps. Using these new types of maps, several characterizations and its properties have been obtained. Also the relationship between strong and weak forms of $\ddot{g}_\alpha$-quotient maps have been established.

2. Preliminaries

We recall the following definitions which are useful in the sequel.

Remark 4.2.1

A subset J of X is called $\ddot{g}_\alpha$-cld [7] if αcl(J) ⊆K whenever J ⊆K and K is sg-open in X. The complement of a $\ddot{g}_\alpha$-cld is $\ddot{g}_\alpha$-open.

The collection of all $\ddot{g}_\alpha$-open sets of X are denoted by $\ddot{G}_\alpha O(X)$.

Definition 4.2.2

A map f:X→Yiscalled

i. $\ddot{g}_\alpha$-continuous[7]iff⁻¹(P)isan $\ddot{g}_\alpha$-cldofXforanycldPofY.

ii. Strongly $\ddot{g}$-continuous[7]iff⁻¹(P)isacldofXforany $\ddot{g}$-cldPofY.

iii. α-continuous[3]iff−1(P)isanα-cldofXforanycldPofY.

iv. $\ddot{g}$-continuous[7]iff−1(P)isa $\ddot{g}$-cldofXforanycldPofY.

Definition 4.2.3

A map f:X→Yiscalled

(i) $\ddot{g}_\alpha$-irresolute[7]iff⁻¹(P)isan $\ddot{g}_\alpha$-open in X for any $\ddot{g}_\alpha$-openPofY.

(ii) α-irresolute[3]iff⁻¹(P)isanα-openinXforanyα-openPofY.

(iii) $\ddot{g}$-irresolute[7]iff⁻¹(P)isa $\ddot{g}$-open in Xforany $\ddot{g}$-openPofY.

Definition 4.2.4

Asurjectivemapf:X→Yissaidtobe

(i) a quotient map [35], provided a subset K of Y is open in Y iff f⁻¹(K) isopen in X.
(ii) anα-quotientmap[8]iffisα-continuousandf⁻¹(P)isopeninXimplies P is an α-open in Y.
(iii) anα*-quotientmap[8]iffisα-irresoluteandf⁻¹(P)isα-openinX implies P is an open in Y.

Remark4.2.5[6]

(i) Each cld is $\ddot{g}_\alpha$-cldbutnotreversed.
(ii) Eachα-cldis $\ddot{g}_\alpha$-cldbutnotreversed.
(iii) Each $\ddot{g}$-cld is $\ddot{g}_\alpha$-cld but not reversed.

(iv) Each cldis$\ddot{g}$-cld but not reversed.

Remark4.2.6[7]

(i) Each continuous map is $\ddot{g}_{\alpha}$-continuous but not reversed.
(ii) Each α-continuous map is $\ddot{g}_{\alpha}$-continuous but not reversed.
(iii) Each $\ddot{g}$-continuous map is $\ddot{g}_{\alpha}$-continuous but not reversed.
(iv) Each continuous map is $\ddot{g}$-continuous but not reversed.

Remark4.2.7[8]
Each quotient map is α-quotient but not reversed.

Definition4.2.8[**3**]
Amapf:X→Yiscalledα-openiff(P)isα-openinYforanyopenPofX.

Remark4.2.9

(i) Eachα-irresolutemapis$\ddot{g}_{\alpha}$-irresolutebutnotreversed[7].
(ii) Eachα-irresolutemapisα-continuousbutnotreversed[3].
(iii) Each$\ddot{g}_{\alpha}$-irresolutemapis$\ddot{g}_{\alpha}$-continuousbutnotreversed[7].
(iv) Each$\ddot{g}$-irresolutemapis$\ddot{g}$-continuousbutnotreversed[7].

Definition4.2.10[**4**]
Asurjectivemapf:X→Yissaidtobean$\ddot{g}_{\alpha}$-quotientmapiffis$\ddot{g}_{\alpha}$-continuous and $f^{-1}(P)$ is open in X implies P is $\ddot{g}_{\alpha}$-open in Y.
Definition4.2.11[**4**]
Let f :X → Y be a surjective map. Then f is called strongly $\ddot{g}_{\alpha}$-quotient map provided a set K of Y is open in Y if f $f^{-1}(K)$ is $\ddot{g}_{\alpha}$-open in X.

b. New Quotient Maps

Definition 4.3.1
A surjective map f: X→Y is said to be a $\ddot{g}_{\alpha}$-quotient map if f is $\ddot{g}_{\alpha}$-continuous and $f^{-1}(P)$ is open in X implies P is a $\ddot{g}_{\alpha}$-open in Y.
Example 4.3.2
Let X= {1,2,3,4},τ= {ϕ,X,{1},{1,2},{1,3,4}},Y={1,2,3}and σ= {ϕ,Y,{1}}.
We have $\ddot{G}_{\alpha}O(X)$={ϕ,X,{1},{1,2},{1,3,4}}and $\ddot{G}_{\alpha}O(Y)$= {ϕ,Y,{1}}.
If the map f is defined as f(a)=1,f (b)=3,f(c)=f(d)=2,then f is not $\ddot{g}_{\alpha}$-quotient.
Definition 4.3.3
A map f :X → Y is said to be $\ddot{g}_{\alpha}$-open if f(K) is $\ddot{g}_{\alpha}$-open in Y for any open K in X.
Definition 4.3.4
A map f:X→Y is said to be strongly $\ddot{g}_{\alpha}$-open iff (K) is $\ddot{g}_{\alpha}$-open in Y for any

$\ddot{g}_\alpha$-open K in X.

Proposition4.3.5

If a map f :X → Y is surjective, $\ddot{g}_\alpha$-continuous and $\ddot{g}_\alpha$-open, then f is a $\ddot{g}_\alpha$- quotient map.

Proof. Weonlyneedtoprovethatf^{-1}(P)is open in X implies P is a$\ddot{g}_\alpha$-open in Y. Let f^{-1}(V) be open in X. Then f (f^{-1}(P)) is a $\ddot{g}_\alpha$-open, since f is $\ddot{g}_\alpha$-open. Hence P is a$\ddot{g}_\alpha$-open, as fissurjective and f(f^{-1}(P))=P. Thus, f is a $\ddot{g}_\alpha$-quotient map.

Proposition4.3.6

Letf : (X, $\tau^{\ddot{g}}$) → (Y, $\sigma^{\ddot{g}}$) be a quotient map. Then f : X → Y is a $\ddot{g}_\alpha$-quotient map.

Proof. Let P be any open in Y. Then P is $\ddot{g}_\alpha$-open in Y and P ∈ $\sigma^{\ddot{g}}$. Then f^{-1}(P) is open in X, because f is a quotient map, that is , f^{-1}(P) is a $\ddot{g}_\alpha$-open in
X. Hence f is $\ddot{g}_\alpha$-continuous. Suppose f^{-1}(P) is open in X, that is, f^{-1}(P) ∈ $\tau^{\ddot{g}}$. Since f is a quotient map, P is open in Y and P is a $\ddot{g}_\alpha$-open in Y. This shows that f : X → Y is a $\ddot{g}_\alpha$-quotient map.

c. Stronger Form of $\ddot{g}_\alpha$ - Quotient Maps

Definition 4.4.1

Let f : X → Y be a surjective map.Then f is called strongly $\ddot{g}_\alpha$-quotient map provided a set K of Y is open in Y iff f^{-1}(K) is a $\ddot{g}_\alpha$-open in X.

Example4.4.2

LetX={1,2,3},τ={ϕ,X,{1},{2},{1,2},{2,3},{1,2,3}},Y={1,2,3}andσ={ϕ,Y,{1},{2},{1,2}}.
We have $\ddot{G}_\alpha$O(X)={ϕ,X,{1},{2},{1,2},{2,3},{1,2,3}} and
$\ddot{G}_\alpha$O(Y)={ϕ,Y,{1},{2},{1,2}}.If the map f is
Defined as f (a)=1,f(b)=2=f(c),f(d)=3,thenfisstrongly$\ddot{g}_\alpha$-quotient.

Theorem 4.4.3

Each open set strongly $\ddot{g}_\alpha$-quotient map is $\ddot{g}_\alpha$-open.

Proof. Let f:X→Y be a strongly $\ddot{g}_\alpha$-quotient map. Let P be an open in X. Since every open is $\ddot{g}_\alpha$-open and hence P is$\ddot{g}_\alpha$-openinX.Thatisf^{-1}(f(P)) is $\ddot{g}_\alpha$-open in X. Since f is strongly $\ddot{g}_\alpha$-quotient, f(P) is open and hence $\ddot{g}_\alpha$-open in
Y. This shows that f is a $\ddot{g}_\alpha$-open.

Remark4.4.4

The reverse of Theorem 4.4.3 need not be true.

Example4.4.5

Let X={1,2,3},τ={ϕ,X,{1},{2},{1,2}},Y={1,2,3}andσ={ϕ,Y,{1},{2},{1,2},{1,3}}.

We have $\ddot{G}\alpha O(X)$={ϕ,X,{1},{2},{1,2}}and $\ddot{G}\alpha O(Y)$={ϕ, Y,{1}, {2}, {1, 2}, {1,3}}.If the map f is defined as f(a)=1= f(b), f(c) = 3, f(d) = 2, then f is $\ddot{g}_\alpha$-open but not strongly $\ddot{g}_\alpha$-quotient, since f^{-1}({2}) = {d} is not $\ddot{g}_\alpha$-open in X.

Theorem 4.4.6

Each strongly $\ddot{g}_\alpha$-quotient map is strongly $\ddot{g}_\alpha$-open.

Proof. Let f:X→Y be a strongly $\ddot{g}_\alpha$-quotient map. Let P be a $\ddot{g}_\alpha$-open in X.Thatisf^{-1}(f(P))is$\ddot{g}_\alpha$-openinX.Sincefisstrongly$\ddot{g}_\alpha$-quotient,f(P)isopen and hence$\ddot{g}_\alpha$-open in Y. This shows that f is strongly $\ddot{g}_\alpha$-open.

Remark4.4.7

ThereverseofTheorem4.4.6neednotbetrue.

Example4.4.8

LetX={1,2,3},τ={ϕ,X,{1},{2},{1,2}},Y={1,2,3}andσ={ϕ,Y,{1,2}}.Wehave$\ddot{G}\alpha O(X)$={ϕ,X,{1},{2},{1,2}}and$\ddot{G}\alpha O(Y)$={ϕ,Y,{1},{2},{1,2}}. If the map f is defined as f(a)=1,f(b)=2,f(c)=3,thenfis strongly$\ddot{g}_\alpha$-openbutnotstrongly$\ddot{g}_\alpha$-quotientbecausef^{-1}({1})={1}is$\ddot{g}_\alpha$-open in X but {1} is not open in Y.

Definition 4.4.9

Letf:X→Ybeasurjectivemap.Thenfiscalled$\ddot{g}_\alpha$*-quotientmapiffis $\ddot{g}_\alpha$-irresoluteandf^{-1}(K)isa$\ddot{g}_\alpha$-openinXimpliesKisopeninY.

Example4.4.10

Let X= {1,2,3,4},τ= {ϕ,X,{1},{1,2},{1,3,4}},Y={1,2,3}and σ= {ϕ,Y,{1},{1,2},{1,3}}.Wehave$\ddot{G}\alpha O(X)$={ϕ,X,{1},{1,2},{1,3,4}} and $\ddot{G}\alpha O(Y)$= {ϕ, Y, {1}, {1, 2}, {1, 3}}.If the map f is defined as f(a) = 1, f(b) = 3, f(c) = f(d) = 2, then f is $\ddot{g}_\alpha$*-quotient.

Proposition4.4.11

each$\ddot{g}_\alpha$*-quotientmapis$\ddot{g}_\alpha$-irresolute.

Proof.ItfollowsfromDefinition4.4.9.

Remark4.4.12

ThereverseofProposition4.4.11neednotbetrue.

Example4.4.13

content...LetX={1,2,3,4},τ={ϕ,X,{1},{1,2},{1,3,4}},Y={1,2,3} andσ={ϕ,Y,{1},{1,2}}.Wehave$\ddot{G}\alpha O(X)$={ϕ,X,{1},{1,2},{1,3,4}} and $\ddot{G}\alpha O(Y)$= {ϕ, Y, {1}, {1, 2}}.If the map f is defined as f(a) = 1, f(b) = 3, f(c)=f(d)=2thenfis$\ddot{g}_\alpha$-irresolutebutnot$\ddot{g}_\alpha$*-quotientbecausef^{-1}({1,3})=

{1,2}isg̈α-openinXbut{1,3}isnotopeninY.

Theorem 4.4.14

Eachg̈α*-quotientmapisstronglyg̈α-open.

Proof. Letf: X→Ybeag̈α*-quotientmap. LetPbeag̈α-openinX.Then $f^{-1}(f(P))$ is g̈α-open in X. Since f is g̈α*-quotient, this implies that f(P) is open inYandthusg̈α-openinY.Hencefisstronglyg̈α-open.

Remark4.4.15

ThereverseofTheorem4.4.14neednotbetrue.

Example4.4.16

LetX={1,2,3},τ={ϕ,X,{1},{2},{1,2}},Y={1,2,3}andσ={ϕ,Y,{1,2}}.WehaveG̈αO(X)={ϕ,X,{1},{2},{1,2}}andG̈αO(Y)={ϕ,Y,{1},{2},{1,2}}. Ifthemapfisdefinedasf(a)=1,f(b)=2,f(c)=3,thenfis stronglyg̈α-openbutnotg̈α*-quotientbecausef^{-1}({1})={1}isg̈α-openinX but {1} is not open in Y.

Proposition4.4.17

eachg̈-irresolute(α-irresolutemap)isg̈α-irresolute.

Proof.LetKbeag̈-cld(α-cld)inY.Sincefisg̈-irresolute(α-irresolute), $f^{-1}(K)$ is g̈α-cld (α-cld) which is g̈-cld and hence g̈α-cld in X.

d. Comparison

Proposition4.5.1

i. Eachquotientmapisag̈α-quotientmap.

ii. Eachα-quotientmapisag̈α-quotientmap.

Proof.Sinceeachcontinuous(α-continuous)mapisg̈α-continuousandeach open(α-open)isg̈α-open,theprooffollowsfromRemark4.2.5andDefinition 4.2.10.

Remark4.5.2

TheseparatereverseofProposition4.5.1neednotbetrue.

Example4.5.3

Let X= {1,2,3,4},τ= {ϕ,X,{1},{1,2},{1,3,4}},Y={1,2,3}and σ={ϕ,Y,{1}}.WehaveG̈αO(X)={ϕ,X,{1},{1,2},{1,3},{1,4},{1,2,3},{1,2, 4},{1, 3, 4}}andG̈αO(Y)={ϕ,Y,{1},{1,2},{1, 3}}.Ifthemapfis definedasf(a)=1,f(b)=3,f(c)=f(d)=2,thenfisg̈α-quotientbutnotquotient. Since for the g̈α-open {1, 2}in Y f^{-1}({1, 2}) = {1, 3, 4}is open in X but {1, 2} is not open in Y.

Example4.5.4

LetX={1,2,3,4},τ={ϕ,X,{1},{2},{1,2}},Y={1,2,3}andσ={ϕ,Y,{1,2}}.WehaveG̈αO(X)={ϕ,X,{1},{2},{1,2},{1,2,3},{1,2,4}}and G̈αO(Y)={ϕ,Y,{1},{2},{1,2}}.Ifthemapfisdefinedasf(a)=1,f(b)=2,

f(c)=f(d)=3,thenfisg̈α-quotientbutnotα-quotientbecausef^{-1}({1})={1} isopeninXbut{1}isnotα-openinY.

Theorem 4.5.5

Eachstronglyg̈α-quotientmapisg̈α-quotientbutnotreversed.

Proof.Let P be an open in Y. Since f is strongly g̈α-quotient, f^{-1}(P) is g̈α- open in X. Thus f is g̈α-continuous.Let f^{-1}(P) be open in X. Then f^{-1}(P) isg̈α-openinX.Sincefisstronglyg̈α-quotient,PisopeninY.ItimpliesthatPis g̈α-openinY.Thisshowsthatfisang̈α-quotientmap.

Example4.5.6

Let X= {1,2,3,4},τ= {ϕ,X,{1},{1,2},{1,3,4}},Y={1,2,3}and σ= {ϕ,Y,{1}}.WehaveG̈αO(X)={ϕ,X,{1},{1,2},{1,3},{1,4},{1,2,3},{1,2, 4},{1, 3, 4}}andG̈αO(Y)={ϕ,Y,{1},{1,2},{1, 3}}.Ifthemapfis definedasf(a)=1,f(b)=3,f(c)=f(d)=2,thenfisg̈α-quotientbutnotstrongly g̈α-quotient because f^{-1}({1, 2}) = {1, 3, 4} is g̈α-open in X but {1, 2} is not open in Y.

Proposition4.5.7

Eachα*-quotientmapisg̈α*-quotientmap.

Proof.Let f be an α*-quotient map.Then f is surjective, α-irresolute and f^{-1}(K)isanα-openinXimpliesKisanopeninY.ThenKisg̈α-openin Y.Sinceeachα-irresolutemapisg̈α-irresoluteandeachα-irresolutemapisα-continuous,f^{-1}(K)isanα-openwhichisang̈α-openinX.Sincefisα*-quotient map, K is open in Y. Hence f is an g̈α*-quotient map.

Remark4.5.8

ThereverseofProposition4.5.7neednotbetrue.

Example4.5.9

LetX={1,2,3},τ={ϕ,X,{2,3},{b,c,d},{1,2,3}},Y={1,2,3}andσ ={ϕ,Y,{2},{3},{2,3}}.WehaveG̈αO(X)={ϕ,X,{2},{3},{2,3},{1,2, 3},{b,c,d}}andG̈αO(Y)={ϕ,Y,{2},{3},{2,3}}.Ifthemapfisdefined as f(a) = 1 = f(d), f(b) = 2, f(c) = 3, then f is g̈α*-quotient but not α*-quotient because f^{-1}({2}) = {2}, f is not α-irresolute.

Proposition4.5.10

Eachg̈α*-quotientmapisstronglyg̈α-quotient.

Proof. LetPbeanopeninY.Thenitis$\ddot{g}_\alpha$-openinY.Since,byProposition 4.4.11[46],fis$\ddot{g}_\alpha$-irresolute,f^{-1}(P)isan$\ddot{g}_\alpha$-openinX.ThusPisopensetinY impliesf^{-1}(P)isan$\ddot{g}_\alpha$-openinX.Conversely,iff^{-1}(P)isan$\ddot{g}_\alpha$-openinX,since f is an $\ddot{g}_\alpha$*-quotient map, P is an open in Y. Hence f is a strongly $\ddot{g}_\alpha$-quotient map.

Remark4.5.11

ThereverseofProposition4.5.10neednotbetrue.

Example4.5.12

LetX={1,2,3,4},τ={ϕ,X,{1},{1,2}},Y={1,2,3}andσ={ϕ,Y,{1}}.
WehaveG̈αO(X)={ϕ,X,{1},{1,2},{1,3},{1,4},{1,2,3},{1,2,4},{1, 3,4}}andG̈αO(Y)={ϕ,Y,{1},{1,2},{1,3}}.Ifthemapfisdefinedasf(a) =1,f(b)=3=f(c),f(d)=2,thenfisstrongly$\ddot{g}_\alpha$-quotientbutnot$\ddot{g}_\alpha$*-quotient because f^{-1}({1, 3}) = {1, 2, 3}is $\ddot{g}_\alpha$-open in X but {1, 3}is not open in Y.

Remark4.5.13

Quotientmapsandstrongly$\ddot{g}_\alpha$-quotientmapsareindependentofanyother.

Example4.5.14

LetX={1,2,3},τ={ϕ,X,{1},{2},{1,2}},Y={1,2,3}andσ={ϕ, Y,{1,2}}.WehaveG̈αO(X)={ϕ,X,{1},{2},{1,2}}andG̈αO(Y)={ϕ, Y,{1},{2},{1,2}}.Ifthemapfisdefinedasidentitymap,thenfisstrongly $\ddot{g}_\alpha$-quotientbutnotquotientbecausef^{-1}({1})={1}isopeninXbut{1}isnot open in Y.

Example4.5.15

LetX={1,2,3},τ={ϕ,X,{1},{1,2},{1,3}},Y={1,2,3}andσ={ϕ, Y,{1}}.WehaveG̈αO(X)={ϕ,X,{1},{1,2},{1,3}} andG̈αO(Y)={ϕ,Y, {1},{1,2},{1,3}}.Ifthemapfisdefinedasidentitymap,thenfisquotient butnotstrongly$\ddot{g}_\alpha$-quotientbecausef^{-1}({1,3})={1,3}is$\ddot{g}_\alpha$-openinXbut {1,3}isnotopeninY.

Theorem 4.5.16

Each$\ddot{g}_\alpha$*-quotientmapis$\ddot{g}_\alpha$-quotient.

Proof. Let f be an $\ddot{g}_\alpha$*-quotient map.Then f is $\ddot{g}_\alpha$-irresolute.By Remark 4.2.9, f is $\ddot{g}_\alpha$-continuous.Let f^{-1}(P) be an open in X. Then f^{-1}(P) is an $\ddot{g}_\alpha$- open inX. Sincef is $\ddot{g}_\alpha$*-quotient, P is openin Y.It means Pis $\ddot{g}_\alpha$-open inY. Therefore f is $\ddot{g}_\alpha$-quotient map.

Remark4.5.17

ThereverseofTheorem4.5.16neednotbetrue.

Example4.5.18

LetX={1,2,3,4},τ={ϕ,X,{1},{1,2},{1,3,4}},Y={1,2,3}andσ={ϕ, Y,{1}}.Wehave$\ddot{G}_{\alpha}O(X)$={ϕ,X,{1},{1,2},{1,3},{1,4},{1,2,3},{1,2,4},{1,3,4}}and$\ddot{G}_{\alpha}O(Y)$={ϕ,Y,{1},{1,2},{1,3}}. Ifthemapfisdefined asf(a)=1,f(b)=3,f(c)=f(d)=2,thenfis$\ddot{g}_{\alpha}$-quotientbutnot$\ddot{g}_{\alpha}$*-quotient becausef^{-1}({1,2})={1,3,4} is$\ddot{g}_{\alpha}$-openinXbut{1,2} isnotopeninY.

Theorem4.5.19[46]

α-quotientmapsand$\ddot{g}$-quotientmapsareindependentofanyother.

Theorem 4.5.20

Each$\ddot{g}_{\alpha}$-quotientmapis$\ddot{g}_{\alpha}$-quotient.

Proof. Letfbe$\ddot{g}_{\alpha}$-quotient map. Then by definition, f is $\ddot{g}_{\alpha}$-continuousand hence, by Remark 4.2.6, f is $\ddot{g}_{\alpha}$-continuous.Let f^{-1}(P) be an open in X. By definitionof$\ddot{g}_{\alpha}$-quotientmap,Pis$\ddot{g}_{\alpha}$-openinY.ByRemark4.2.5,Vis$\ddot{g}_{\alpha}$-open in Y. Therefore f is $\ddot{g}_{\alpha}$-quotient map.

Remark4.5.21

ThereverseofTheorem4.5.20neednotbetrue.

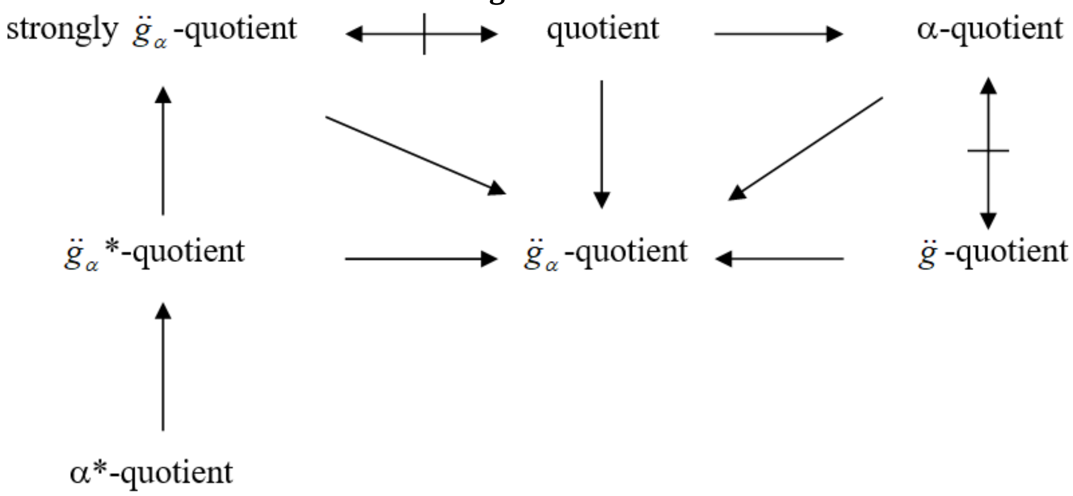
Example4.5.22

LetX={1,2,3,4},τ={ϕ,X,{1},{1,2},{1,3,4}},Y={1,2,3}andσ={ϕ,Y,{1}}.Wehave$\ddot{G}_{\alpha}O(X)$={ϕ,X,{1},{1,2},{1,3},{1,4},{1,2,3},{1,2,4},{1,3,4}}and$\ddot{G}_{\alpha}O(Y)$={ϕ,Y,{1},{1,2},{1,3}}. Ifthemapfisdefined asf(a)=1,f(b)=3,f(c)=f(d)=2,thenfis$\ddot{g}_{\alpha}$-quotientbutnot$\ddot{g}_{\alpha}$-quotient becausef^{-1}({1,2})={1,3,4} is$\ddot{g}_{\alpha}$-openinXbut{1,2} isnotopeninY.

Remark4.5.23

From the previous Theorems, Propositions, Examples and Remarks, we obtain thefollowingdiagram,whereA→ B(resp. A↔ B)representsAimpliesBbut not reversed (resp. A and B are independent of each other).

Diagram



e. Application

Proposition4.6.1
Letf:X→Ybeanopensurjective$\ddot{g}_\alpha$-irresolute map and g:Y→Zbea $\ddot{g}_\alpha$-quotient map. Then their composition g ○f: X→Zisa$\ddot{g}_\alpha$-quotient map.

Proof. Let P be any open in Z. Then $g^{-1}(P)$ is a $\ddot{g}_\alpha$-open in Y since g is a $\ddot{g}_\alpha$-continuous map. Since f is $\ddot{g}_\alpha$-irresolute, $f^{-1}(g^{-1}(P)) = (g \circ f)^{-1}(P)$ is a $\ddot{g}_\alpha$-open in X. This implies $(g\circ f)^{-1}(P)$isa$\ddot{g}_\alpha$-open in X. This shows that g ○f isa$\ddot{g}_\alpha$-continuous map. Also, assumethat$(g\circ f)^{-1}(P)$isopeninXforP⊆ Z, thatis,$(f^{-1}(g^{-1}(P)))$isopeninX.Sincefisopen$f(f^{-1}(g^{-1}(P)))$isopeninY.It

followsthat$g^{-1}(P)$isopeninY,becausefissurjective. Sincegisa$\ddot{g}_\alpha$-quotient map, P is a $\ddot{g}_\alpha$-open in Z. Thus g ○ f : X → Z is a $\ddot{g}_\alpha$-quotient map.

Proposition4.6.2
If h :X →Y is a $\ddot{g}_\alpha$-quotient map and g :X →Z is a continuous map that is constantonanyset$h^{-1}(y)$,fory∈Y,thenginducesa$\ddot{g}_\alpha$-continuousmapf:Y →Zsuchthatf○h=g.

Proof. Sincegisconstanton$h^{-1}(y)$,foranyy∈ Y,thesetg$(h^{-1}(y))$isaone pointinZ.Iff(y)denotethispoint, thenitisclearthatfiswelldefinedandfor any x ∈ X, f(h(x)) = g(x).We claim that f is $\ddot{g}$-continuous.Let P be any open inZ,then$g^{-1}(P)$isanopeninXasgiscontinuous.But$g^{-1}(P)=h^{-1}(f^{-1}(P))$ is open in X. Since h is $\ddot{g}_\alpha$-quotient map, $f^{-1}(P)$ is a $\ddot{g}$-open in Y. Hence f is $\ddot{g}_\alpha$-continuous.

Proposition4.6.3
Letf:X→Ybeanstrongly$\ddot{g}_\alpha$-opensurjectiveand$\ddot{g}_\alpha$-irresolutemapandg:Y → Zbeastrongly$\ddot{g}_\alpha$-quotientmapthengof: X→ Zisastrongly$\ddot{g}_\alpha$-quotient map.

Proof.Let P be any open in Z. Then $g^{-1}(P)$ is a $\ddot{g}_\alpha$-open in Y (since g is strongly $\ddot{g}_\alpha$-quotient).Since f is $\ddot{g}_\alpha$-irresolute, $f^{-1}(g^{-1}(P))$ is a $\ddot{g}_\alpha$-open in X. Conversely, assume that $(g \circ f)^{-1}(P)$ is a $\ddot{g}_\alpha$-open in X for P ⊆ Z. Then $f^{-1}(g^{-1}(V))$ is a $\ddot{g}_\alpha$-open in X. Since f is strongly $\ddot{g}_\alpha$-open, $f(f^{-1}(g^{-1}(P)))$ is a $\ddot{g}_\alpha$- openinY.Itfollowsthat$g^{-1}(P)$isa$\ddot{g}_\alpha$-openinY.ThisgivesthatPisanopen inZ(sincegisstrongly$\ddot{g}_\alpha$-quotient). Thusg○fisastrongly$\ddot{g}_\alpha$-quotientmap.

Definition 4.6.4
AspaceXiscalledaT$\ddot{g}_\alpha$-spaceifeach$\ddot{g}_\alpha$-cldinitiscld.

Theorem 4.6.5
Letp:X→Ybea$\ddot{g}_\alpha$-quotient map where X and Y are T $\ddot{g}_\alpha$-spaces. Then f: Y →Z is a strongly $\ddot{g}_\alpha$-continuous if f the composite map f○p:X→Z is strongly $\ddot{g}_\alpha$-continuous.

Proof. Let f be strongly $\ddot{g}_\alpha$-continuous and K be each $\ddot{g}_\alpha$-open in Z. Then $f^{-1}(K)$ is openin Y.Then$(f\circ p)^{-1}(K)=p^{-1}(f^{-1}(K))$ is $\ddot{g}_\alpha$-open X. Since X is a T $\ddot{g}_\alpha$-space, $p^{-1}(f^{-1}(K))$ is open in X. Thus the composite map is strongly $\ddot{g}_\alpha$- continuous. Conversely let the composite map f○p be strongly $\ddot{g}_\alpha$-continuous. Then for any $\ddot{g}_\alpha$-open Kin Z, $p^{-1}(f^{-1}(K))$ is open in X. Since p is a$\ddot{g}_\alpha$-quotient map, it implies that $f^{-1}(K)$ is $\ddot{g}_\alpha$-open in Y. Since Y is a T$\ddot{g}_\alpha$-space, $f^{-1}(K)$ is open in Y. Hence f is strongly $\ddot{g}_\alpha$-continuous.

Theorem 4.6.6

Letf:X→Ybeasurjectivestrongly$\ddot{g}_\alpha$-openand$\ddot{g}_\alpha$-irresolutemapandg:Y →Z be a$\ddot{g}_\alpha$*-quotient map then g ○f is $\ddot{g}_\alpha$*-quotient map.

Proof. LetPbe$\ddot{g}_\alpha$-openinZ.Theng^{-1}(P)isa$\ddot{g}_\alpha$-open in Y because g is a $\ddot{g}_\alpha$*-quotient map. Since f isg $\ddot{}_\alpha$-irresolute,$f^{-1}(g^{-1}(P))$isa$\ddot{g}_\alpha$-openin X. Then g ○f is a$\ddot{g}_\alpha$-irresolute. Suppose$(g\circ f)^{-1}(P)$isa$\ddot{g}_\alpha$-open in X for a subset P⊆Z. That is $(f^{-1}(g^{-1}(P)))$ is $\ddot{g}_\alpha$-open in X. Since f is strongly $\ddot{g}_\alpha$-open, $f(f^{-1}(g^{-1}(P)))$ is$\ddot{g}_\alpha$-openinY.Thus$g^{-1}(P)$is$\ddot{g}_\alpha$-openinY.Sincegisa$\ddot{g}_\alpha$*-quotientmap,P is an open in Z. Hence g ○ f is a $\ddot{g}_\alpha$*-quotient map.

Proposition4.6.7

Letf:X→Y bean strongly $\ddot{g}_\alpha$-quotient $\ddot{g}_\alpha$-irresolute map and g:Y→Zbea $\ddot{g}_\alpha$*-quotient map then g○ f is a$\ddot{g}_\alpha$*-quotient map.

Proof. LetPbeany$\ddot{g}_\alpha$-openinZ.Theng^{-1}(P)isa$\ddot{g}_\alpha$-openinY(Since g is $\ddot{g}_\alpha$*-quotient map).We have $f^{-1}(g^{-1}(P))$ is also $\ddot{g}_\alpha$-open in X (Since f is $\ddot{g}_\alpha$-irresolute). Thus,$(g\circ f)^{-1}(P)$is$\ddot{g}_\alpha$-openinX.Henceg○fis$\ddot{g}_\alpha$-irresolute. Let$(g\circ f)^{-1}(P)$bea$\ddot{g}_\alpha$-openinXforP⊆Z.i.e.,$f^{-1}(g^{-1}(P))$isa$\ddot{g}_\alpha$-openin
X. Then $g^{-1}(P)$ is an open in Y because f is a strongly $\ddot{g}_\alpha$-quotient map. This meansthat$g^{-1}(P)$isa$\ddot{g}_\alpha$-open in Y.Since g is$\ddot{g}_\alpha$*-quotient map, P is an open in Z. Thus g ○ f is a $\ddot{g}_\alpha$*-quotient map.

Theorem 4.6.8

Thecompositionoftwo$\ddot{g}_\alpha$*-quotientmapsis$\ddot{g}_\alpha$*-quotient.

Proof. Letf:X→Yandg:Y→Zbetwo$\ddot{g}_\alpha$*-quotientmap.LetP be a $\ddot{g}_\alpha$-open in Z. Since g is $\ddot{g}_\alpha$*-quotient, $g^{-1}(P)$ is $\ddot{g}_\alpha$-open in Y. Since f is $\ddot{g}_\alpha$*-quotient, $f^{-1}(g^{-1}(P))$ is $\ddot{g}_\alpha$-open in X. That is $(g \circ f)^{-1}(P)$ is $\ddot{g}_\alpha$-open in X. Hence g ○ f is $\ddot{g}_\alpha$-irresolute. Let $(g \circ f)^{-1}(P)$ be $\ddot{g}_\alpha$-open in X. Then $f^{-1}(g^{-1}(P))$ is$\ddot{g}_\alpha$-openinX.Sincefis$\ddot{g}_\alpha$*-quotient,$g^{-1}(V)$isanopeninY.Then$g^{-1}(P)$isa $\ddot{g}_\alpha$-openinY.Sincegis$\ddot{g}_\alpha$*-quotient,PisopeninZ.Thusg○fis$\ddot{g}_\alpha$*-quotient.

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