Semi Centralizing Pair of Automorphisms of Prime rings

H.HabeebRani*

V.Thiripurasundari†

G.Gopala Krishnamoorthy‡

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Abstract
The concept of semi centralizing automorphism of rings is generalized as Semi-centralizing pair of automorphisms of rings and more general results are obtained. In this paper we generalize the concept of semi-centralizing automorphism of a ring as semi – centralizing pair of automorphism of a ring and more general results are proved.

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1. Introduction

Let T be an automorphism of ring. It is called commuting automorphism of R if xT(x)=T(x)x,∀x∈R and it is called a semi-commuting automorphism if either xT(x)=T(x)x(or)xT(x)=-T(x)x,foreachx∈R.In[1]L.O.ChungandJ.Luhhave proved that a prime ring R of characteristic ≠ 2,3 possessing a non – trivial semi – commuting automorphism is necessarily a commutative integral domain. In[5] A.Kaya and C. Koc have defined semi – centralizing automorphism of a ring and proved that every semi – centralizing automorphism of a prime ring is commuting.

In this paper we generalize the concept of semi-centralizing automorphism of a ring as semi – centralizing pair of automorphism of a ring and more general results are proved.

2. Preliminary

In this section we shall recollect some known definitions and results for easy reference.

Definition2.1 let T be an automorphism of a ring R.T is called

a) a commuting automorphism if xT(x)=T(x)x,∀x∈R and

*AssistantProfessorofMathematics,H.K.R.H.College,Uthamapalayam,Theni district,Tamilnadu.Email habeebrani23@gmail.com
†AssistantProfessorofMathematics, Sri S.R.N.M.College, Sattur,Virudhunagardistrict,Tamilnadu, Email Id:.thiripurasund@gmail.com
‡Advisor, Sri Krishnasamy Arts and Science College, Mettamalai, Sattur, Tamilnadu.
Email Id:. ggrmoorthy@gmail.com
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b) an anti-commuting automorphism if xT(x)=-T(x)x,∀x∈R.

c) a semicommuting automorphism if either xT(x)=T(x)x(or)xT(x) =-T(x)x

i.∀x∈R.

ii.a centralizing automorphism if xT(x)–T(x)x∈Z,∀x∈R.

iii.an anti-centralizing automorphism if xT(x)+T(x)x∈Z,∀x∈R.

iv.a semi–centralizing automorphism if either xT(x)–T(x)x∈Z(or)xT(x)+T(x)x∈Z,∀x∈R.

Definition2.2 [5] let Tbe an automorphism of R, we definer+ = {X∈R/XT(X)–T(X)X∈Z}and R-={X∈R/XT(X) + T(X)X∈Z}

Lemma2.3 [5] Let R be any ring and T be a semi centralizing automorphism of R.

Ifx,y∈R+ {respinR-}thenx+y∈R +(respinR -)ifandonlyifx–y∈R+(respinR -).

Lemma2.4 [5] Let R be a primering and if $y^n=0$, forall y∉R+ where n>1 is a fixed integer, then$y^{n-1}=0$.

Lemma2.5 [4] Let R be prime ring with Non-Trivial centralizing Pair of automorphisms S and T such that S≠T. Then R is a Commutative Integral Domain.

3. MainResults

Definition3.1 Let S and T be two non–trivial automorphisms of a ring. They are called,

a) a commuting pair of automorphism if S(x)T(x)=T(x)S(x),∀x∈R.

b) an anti– commuting pair of automorphisms if S(x)T(x)=-T(x)S(x),∀x∈R.

c) a semi–commuting pair of automorphism if either S(x) T(x)=T(x)S(x)(or)

S(x)T(x)=-T(x)S(x),∀x∈R.

d) a centralizing pair of automorphism if S(x)T(x)–T(x)S(x)∈Z,∀x∈R.

e) an anti– centralizing pair of automorphisms if S(x)T(x) +T(x)S(x)∈Z,∀x∈R.

f) a semi–centralizing pair of automorphism if either S(x)T(x)–T(x)S(x)∈Z(or)

S(x) T(x)+T(x)S(x)∈Zfor allx∈R.

Definition3.2 Let r be any ring and S and T be two maps from R into R.We define, R+= {X∈R /S(X)T(X)–T(X)S(X)∈Z}AND, R-= {X∈R/S(X)T(X)+T(X)S(X)∈Z}.

Remark3.3 If SandTare SemiCentralizingAutomorphismsofR.thenR = R+∪R-**.**

Lemma3.4 Let RbeanyRing and Sand Tbenon–trivialAutomorphisms ofR.

Ifx,y∈R-{respinR+}thenx+y∈R-(respinR+)ifandonlyifx–y∈R-

(respinR+).

Proof:

Letx,y∈R-.

ThenS(x)T(x)+T(x)S(x)∈Z → 1

S(y)T(y)+T(y)S(y)∈Z → 2

x+y∈R-iffS (x+y)T (x+y)+T (x+y)S (x+y)∈Ziff(S(x)+S(y))(T(x)+T(y))+(T(x)+T(y))(S(x)+S(y))∈Z

iffS(x)T(x)+S(x) T(y)+S(y)T(x)+S(y)T(y)

T(x)S(x) +T(x)S(y)+T(y) S(x)+T(y) S(y) ∈Z

Using(1)and (2),weget

x+y∈R-iffS(x)T(y) + S(y) T(x) +T(x) S(y)+T(y) S(x)∈Z.

iff-S(x)T(y) - S(y)T(x)-T(x)S(y) -T(y)S(x)∈Z.

iffS(x) T(x) –S(x)T(y) –S(y) T(x)+S(y)T(y) +T(x)S(x)–T(x)S(y)- T(y)S(x)

+T(y) S(y) ∈ Z.iffS(x) (T(x)–T(y))–S(y)(T(x)–T(y)) +T(x)(S(x)–S(y))–T(y) (S(x)–S(y))∈Z

iff(S(x)–S(y)) (T(x)–T(y)) +(T(x)–T(y))(S(x)-S(y)) ∈Z,∀x,y∈R.

iffS(x-y)T(x-y)+T(x– y)S(x–y)∈Z,∀ x,y∈R.iffx-y∈R -.

Repeatingtheabove argument,itcanbeprovedthatx+y∈R+iffx–y∈R+.

Remark3.5 TakingS=I,TheIdentityAutomorphismOfR,WeGet Lemma 1[5]

Lemma3.6 LetRBeAPrimeRingAnd SAndTBeSemiCentralizing PairOfAutomorphismsOfR.

Ify∉R+,thenS $(y^2)T(y^2)=0$.

Proof

<u>Casei)</u>CharR≠2.

SinceSand Taresemicentralizingautomorphismsof R,wehaveR=R+ ∪R -.

Ify∉R +,theny∈R -.

So,S(y) T(y)+T(y)S(y)∈Z 1

Hence[S(y) T(y)+T(y) S(y), S(y)]=0

i.e,[S(y) T(y),S(y)] +[T(y)S(y),S(y)]=0

S(y)[T(y),S(y)]+[S(y),S(y)]T(y)+T(y)[S(y),S(y)]+[T(y),S(y)] S(y)=0 i.e, $S(y)[T(y),S(y)]+[T(y),S(y)]S(y)=0$

$S(y)(T(y)S(y)–S(y)T(y))+(T(y)S(y)–S(y)T(y))S(y)=0$

$S(y)|(T(y)S(y)–S(y^2)T(y)|+|T(y)S(y^2)–S(y)T(y)S(y))|=0 S(y^2)T(y)–T(y)S(y^2)= 0$

i.e, $[S(y^2), T(y)] = 0, \forall y \notin R+$ 2

Also $[S(y)T(y)+T(y)S(y),T(y)] =0$

i.e, $[S(y)T(y),T(y)]+[T(y)S(y)T(y)]=0$

$S(y)[T(y),T(y)]+[S(y),T(y)]T(y)+T(y)[S(y)T(y)]+[T(y),T(y)]S(y) =0$

$(S(y)T(y)–T(y)S(y))T(y)+T(y) (S(y)T(y)–T(y)S(y)) =0.$

i.e, $S(y)T(y^2)–T(y)S(y)T(y)+T(y)S(y)T(y)–T(y^2)S(y)=0$. i.e, $T(y^2)S(y)–S(y)T(y^2)=0$.

i.e, $[T(y^2),S(y)]=0, \forall y \notin R+$ 3

Now, $[S(y^2+y),T(y^2+y)]=[S(y^2)+S(y),T(y^2)+T(y)]$

$=[S(y^2),T(y^2)]+[S(y^2),T(y)]+[S(y),T(y^2)]+[S(y),T(y)]$

$=S(y)[S(y),T(y^2)]+[S(y),T(y^2)]S(y)+[S(y),T(y)][S(y^2+y),T(y^2+y)]=[S(y),T(y)]$4

Similarly,

$[S(y^2-y), T(y^2-y)]=[S(y),T(y)]$ 5

Using(4)and(5), gives

$[S(y^2+y),T(y^2+y)]=[S(y^2-y),T(y^2-y)]$

$=[S(y),T(y)], \forall y \notin R+$ 6

Now $y \notin R+ => [S(y),T(y)] \notin Z$.

Hence, $[S(y^2+y),T(y^2+y)]=[S(y^2–y), T(y^2–2)] \notin Z$.

So,neither y^2+y nor $y^2–y$ belongsto R+.So $y^2+y \in R–$ and $y^2-y \in R–$.

Since $y \in R\_$,itisclearthat $ny \in R–$ forallpossibleinteger n.Now, $(y^2+y)+(y^2–y)=2y \in R–$,

So,bylemma3.4

$(y^2+y)+(y^2–y)=2y^2 \in R–$.

Hence $y^2 \in R–$.

Hence $S(y^2)T(y^2)+T(y^2)S(y^2) \in Z$ 7

Now, $S(y^2)T(y^2)=S(y)(S(y)T(y^2))$

$=S(y)(T(y^2)S(y))$(Using(3))

$=(S(y)(T(y^2))S(y)$

$=(T(y^2)S(y))S(y)$(Using(3))

$S(y^2)T(y^2)=T(y^2)S(y^2)$ 8

From(7)and(8)we get,

$2S(y^2)T(y^2)=2T(y^2)S(y^2)\in Z$

Hence$S(y^2)T(y^2)=T(y^2)S(y^2)\in Z,\forall y\notin R+$ 9

Since$y^2+y\in R+$using(2)weget,$[S(y^2+y)^2,T(y^2+y)]=0$

$[S(y^4+2y^3+y^2),T(y^2+y)]=0$

$[S(y^4),T(y^2)]+[S(y^4),T(y)]+2[S(y^3),T(y^2)]+2[S(y^3),T(y)]+[S(y^2),T(y^2)]$

$+[S(y^2),T(y)]=0.$

Since$y^2\notin R+$,by(2)$[S(y^4),T(y^2)]=0$.

Now,$[S(y^4),T(y)]=S(y^2)[S(y^2),T(y)]+[S(y^2),T(y)]S(y^2)=0$.

Also, $[S(y^2),T(y^2)]=S(y^2)[S(y),T(y^2)]+[S(y^2),T(y^2)]S(y)$.

$=\{S(y)[S(y),T(y^2)]+[S(y),T(y^2)]S(y)\}S(y)$ (using (3))

$=0.$

Also,$[S(y^2),T(y^2)]=S(y)[S(y),T(y^2)]+[S(y),T(y)]S(y)=0$

Hence,$2[S(y^3),T(y)]=0$.

Since,Char$R\neq 2$,weget$[S(y^3),T(y)]=0$

i.e,$S(y^2)[S(y),T(y)]+[S(y^2),T(y)]S(y)=0$i.e,$S(y^2)[S(y),T(y)]=0,\forall y\notin R+$.

$\therefore T(y^2)S(y^2)[S(y),T(y)]=0,\forall y\notin R+$.

By(9),$T(y^2)S(y^2)\in Z$,SinceRisprimeand$[S(y),T(y)]\neq 0$,wehave$T(y^2)S(y^2)=0$.i.e,$S(y^2)T(y^2)=T(y^2)S(y^2)=0,\forall y\notin R+$.

Case(ii):

Char(R)=2.Then$x=-x\forall x\in R$.

So,everysemicentralizing pairofautomorphismsarecentralizing pair ofautomorphisms.So,byTheorem2.5[4] theyarecommutingpairofautomorphisms.

So,$[S(y^2),T(y^2)]=0$

i.e,$S(y^2)T(y^2)–T(y^2)S(y^2)=0$

i.e,$S(y^2)T(y^2)=T(y^2)S(y^2)=-T(y^2)S(y^2)$

Now,$2S(y^2)T(y^2)=S(y^2)T(y^2)+S(y^2)T(y^2)$.

$=S(y^2)T(y^2)–T(y^2)S(y^2)$.

$=0$.

SinceCharR=2,weget$S(y^2)T(y^2)=0,\forall y\notin R+$.

Hencetheproof.

Remark3.8 TakingS=R,theIdentityautomorphismofR,wegetLemma2[5].

Theorem3.9 LetRbeaprimeringand SandTbeasemicentralizing PairofautomorphismsofR.thenSandTarecommutingpair ofautomorphisms**.**

Proof

Let$x\in R$and$y\in R+$. Consider theelementxy^2+y, sinceS andTaresemi

centralizingpairofautomorphismsofR.Wehave

$c=S(xy^2+y^2)+T (x y^2+y^2)\pm T (xy^2+y^2)S(x y^2+y^2)\in Z$.

i.e,$c=S(xy^2)T(x y^2)+S(xy^2)T(y^2)+S(y^2)T(xy^2)+S(y^2)T(y^2)\pm T(x y^2)S(xy^2)$

$+T(xy^2)S(y^2)+T(y^2)S(xy^2)+T(y^2)S(y^2)\in Z$.

Usinglemma3.6weget

$c=S(xy^2)T(xy^2)+S(x)S(y^2)T(y^2)+S(y^2)T(x)T(y^2)\pm T(xy^2)S(xy^2)+T(x)T(y^2)S(y^2)$

$+T(y^2)S(x)S(y^2)\in Z$.

Again,usinglemma3.6,weget

$c=S(x)S(y^2)T(x)T(y^2)+S(y^2)T(x)T(y^2)\pm T(x)T(y^2)S(x)S(y^2)+T(y^2)S(x)S(y^2)\in Z$.

$cT(y^2)=S(x)S(y^2)T(x)T(y^2)$ T $(y^2)+S(y^2)T(x)T(y^2)T(y^2)\pm T(x)T(y^2)S(x)S(y^2)T(y^2)+T(y^2)S(x)S(y^2)$ $T(y^2)\in Z$.

i.e,$cT(y^2)=S(x)S(y^2)T(x)T(y^4)+S(y^2)T(x)T(y^4)\in Z$. 1

Similarly consideringtheelement$xy^2–y^2$,weget

$dT(y^2)=S(x)S(y^2)T(x)T(y^4)–S(y^2)T(x)T(y^4)\in Z$,forsome$d\in Z$. 2

(1)–(2)gives,

$(c–d)T(y^2)=2S (y^2)T(x)T (y^4)$.

$\therefore T(y^2)(c\text{-}d)T(y^2) = 2T(y^2)S(y^2)T(x)T(y^4) \in Z$. 3

Usinglemma3.6,weget, $(c–d)T(y^4)=0$(using$c–d\in Z$)

If$c\neq d$,$T(y^4)=0$andso$y^4=0$ 4

Ifc =d,thenfrom(4)and(2),weget

$S(x)S(y^2)T(x)T(y^4) + S(y^2)T(x)T(y^4) = S(x)S(y^2)T(x)T(y^4) – S(y^2)T(x)T(y^4)$.i.e,$2S(y^2)T(x)T(y^4)=0$.

Since R is prime either $S(y^2) = 0$ (or) $T(y^4) = 0$.Ineithercasewegety$y^4=0$.

Thus,if$y\notin R+$, then$y^4=0$ 5

Thenusinglemma2.4repeatedly,wegety=0,HenceR=R+andsoSandTarecommutingpairofautomorphismofR.

Corollary3.10 Prime ring R possessing a non–trivial semi centralizing pair of automorphism is a Commutative integral Domain**.**

Corollary3.11 A prime ring R possessing a non–trivial semi commuting pair of automorphism is a commutative integral domain.

Remark3.12 Taking S=R,the identityautomorphismofR,we getTheorem[5]

Conclusions

In this papers we investigated we generalize the concept of semi-centralizing automorphism of a ring as semi – centralizing pair of automorphism of a ring and more general results are proved.

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