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Detour Global Domination for k-Regular Graphs with Girth 3 Where k is Even

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Abstract

In this paper, we introduced the new concept detour global domination number for k-regular graph with girth 3, where k is even. First we recollect the concept of k-regular graphs and we produce some results based on the detour global domination number for k-regular graph with girth 3, where k is even. A set S is called a *detour global dominating set* of G if S is both detour and global dominating set of G. The *detour global domination number* is the minimum cardinality of a detour global dominating set in G.

Keywords: Detour set, Dominating set, Detour Domination, Global Domination, Detour Global Domination, regular graphs.

Subject Classification Number: 05C12,05C69

1 Introduction

By a graph $G=(V,E)$ we mean a finite, connected, undirected graph with neither loops nor multiple edges. The *order* $|V|$ and *size* $|E|$ of G are denoted by p and q respectively. For graph theoretic terminology we refer to West[9]. For vertices x and y in a connected graph G, the *detour distance* $D(x,y)$ is the length of a longest $x-y$ path in G[1]. An $x-y$ path of length $D(x,y)$ is called an $x-y$ detour. The closed interval $I_D[x,y]$ consists of all vertices lying on some $x-y$ detour of G. For $S\subseteq V, I_D[S]=\cup_{x,y\in S} I_D[x,y]$. A set S of vertices is a detour set if $I_D[S]=V$, and the minimum cardinality of a detour set is the detour number $dn(G)$. A detour set of cardinality $dn(G)$ is called a *minimum detour set* [2].

A set $S\subseteq V(G)$ in a graph G is a *dominating set* of G if for every vertex v in V-S, there exists a vertex $u\in S$ such that v is adjacent to u. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G[3]. The complement $\overline{G}$ of a graph G also has $V(G)$ as its point set, but two points are adjacent in $\overline{G}$ if and only if they are not adjacent in G. A set $S\subseteq V(G)$ is called a *global dominating set* of G if it is a dominating set of both G and $\overline{G}$[7].

A graph is *k-regular* if every vertex has degree k. The *girth* of a graph is the length of its shortest cycle.[5]

Definition 1.1 *Let $G = (V, E)$ be a connected graph with atleast two vertices. A set $S \subseteq V(G)$ is said to be a detour global dominating set of G if S is both detour and global dominating set of G. The detour global domination number, denoted by $\bar{\gamma}_d(G)$ is the minimum cardinality of a detour global dominating set of G and the detour global dominating set with cardinality $\bar{\gamma}_d(G)$ is called the $\bar{\gamma}_d$-set of G or $\bar{\gamma}_d(G)$-set.[4]*

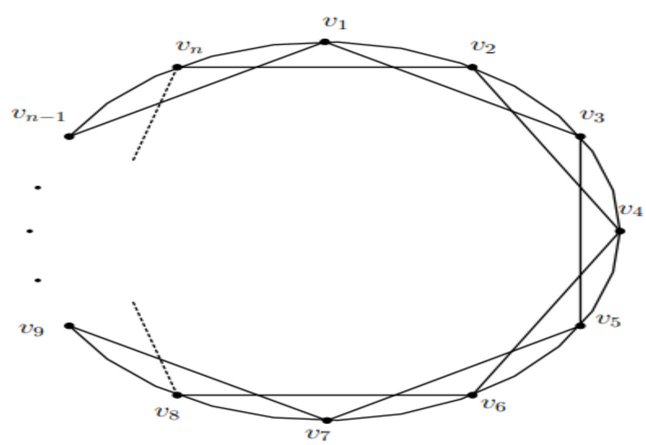
In 2015, N. Mohanapriya, et. Al. [6] investigated the domination number and its parameters for four regular graphs $G(n)$ on n vertices with girth 3. In 2019, Primo Potocnik and Jano Vidali[8] studied girth regular graphs. In 2020, C. Jayasekaran, S. Delbin Prema and S.V. Ashwin Prakash[5] studied irredundance and domination number for six regular graph with girth 3. This motivated us to determine detour global domination number for k-regular graph with girth 3 where k is even.

In Section 2 we deal with the structure for 4 regular graph with girth 3 and 6 regular graph with girth 3.

In Section 3 we introduce detour global domination number for k-regular graph with girth 3 where k is even and condition for minimum number of detour global dominating set.

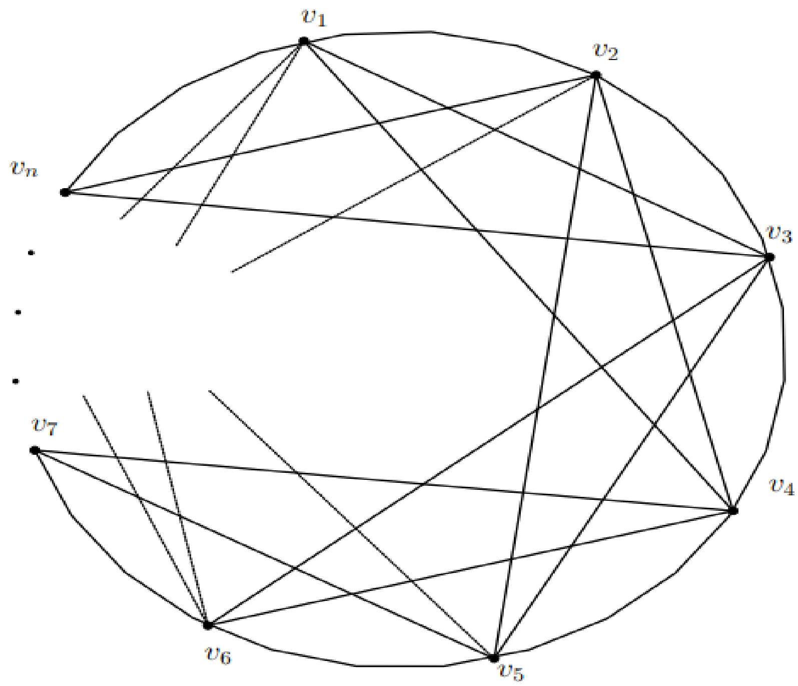
2 Basic Definitions

Definition 2.1 *If v_1 is adjacent with v_{n-1}, v_n, v_2, v_3; v_2 is adjacent with v_n, v_1, v_3, v_4; v_i is adjacent with $v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}$, where $i = 3$ to $n-2$, v_{n-1} is adjacent with $v_{n-3}, v_{n-2}, v_n, v_1$ and v_n is adjacent with $v_{n-2}, v_{n-1}, v_1, v_2$ such that $v_1 v_2 \cdots v_n$ forms a cycle, then clearly each vertex is of degree 4. Hence, the graph has $2n$ edges. Thus, from the construction, we have a 4-regular graph of girth 3 with n vertices and $2n$ edges. In Figure 1, a four regular graph on n vertices with girth 3 is shown.*



$Figure\ 1 : G(n, 4, 3)$

Definition 2.2 *If v_1 is adjacent with $v_{n-2}, v_{n-1}, v_n, v_2, v_3, v_4; v_2$ is adjacent with $v_{n-1}, v_n, v_1, v_3, v_4, v_5; v_i$ is adjacent with $v_{i-3}, v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}, v_{i+3}$, where $i = 3$ to $n-2$, v_{n-1} is adjacent with $v_{n-4}, v_{n-3}, v_{n-2}, v_n, v_1, v_2$ and v_n is adjacent with $v_{n-3}, v_{n-2}, v_{n-1}, v_1, v_2, v_3$ such that $v_1 v_2 \cdots v_n$ forms a cycle, then clearly each vertex is of degree 6. Hence, the graph has $3n$ edges. Thus, from the construction, we have a 6-regular graph of girth 3 with n vertices and $3n$ edges. In Figure 2, a six regular graph on n vertices with girth 3 is shown.*



$Figure\ 2 : G(n, 6, 3)$

3 Detour global domination number for *k*-regular graph with girth 3

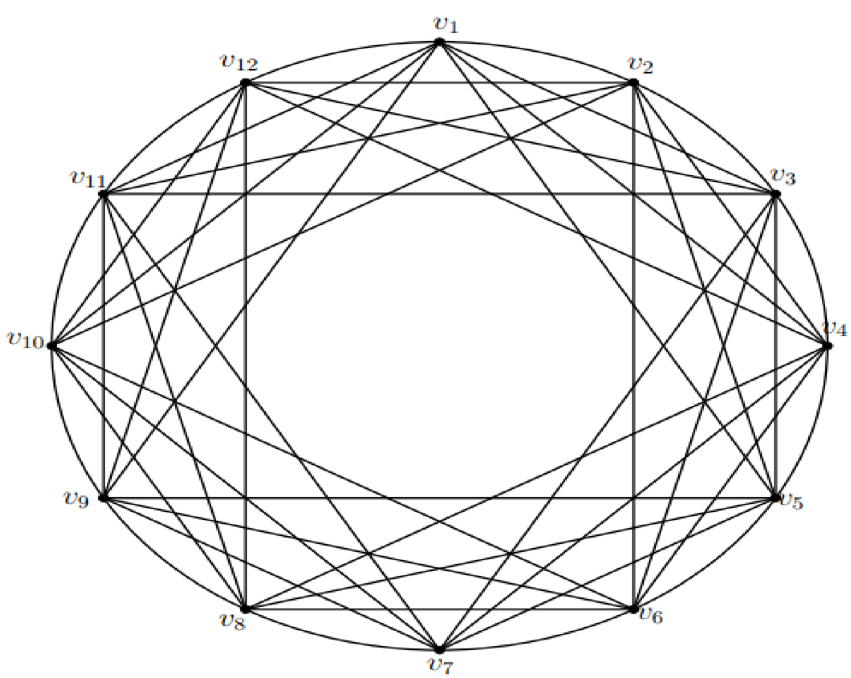
Definition 3.1 *If v_1 is adjacent with $v_{n-(\frac{k}{2}-1)}, \cdots, v_{n-2}, v_{n-1}, v_n, v_2, v_3, \cdots, v_{\frac{k}{2}+1}; v_2$ is adjacent with $v_{n-(\frac{k}{2}-2)}, \cdots, v_n, v_1, v_3, v_4, \cdots, v_{\frac{k}{2}+2}; v_i$ is adjacent with $v_{i-\frac{k}{2}}, \cdots, v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}, \cdots, v_{i+\frac{k}{2}}$, where $i = 3$ to $n-2$, v_{n-1} is adjacent with $v_{n-(\frac{k}{2}+1)}, \cdots, v_{n-3}, v_{n-2}, v_n, v_1, v_2, \cdots, v_{\frac{k}{2}-1}$ and v_n is adjacent with $v_{n-\frac{k}{2}}, \cdots, v_{n-2}, v_{n-1}, v_1, v_2, \cdots, v_{\frac{k}{2}}$ such that $v_1 v_2 \cdots v_n$ forms a cycle, then clearly each vertex is of degree k. Hence, the graph has $\frac{nk}{2}$ edges. Thus, from the construction, we have a k-regular graph of girth 3 with n vertices and $\frac{nk}{2}$ edges where k is even. In this section we denote the k regular graph on n vertices with girth 3 as $G(n, k, 3)$.*

Theorem 3.2 *For any integer* $n \geq k+1, \bar{\gamma}_d(G(n,k,3)) = \begin{cases} n & if \quad n = k+1 \\ \left\lceil \frac{n}{k+1} \right\rceil & for \quad n \geq k+2 \end{cases}$

Proof. Let $v_1, v_2, v_3, \cdots, v_n$ be the vertices of $G(n,k,3)$ such that $v_1 v_2 v_3 \cdots v_n v_1$ forms a cycle. Now consider for $n = k+1$, $G(k+1,k,3)$ is isomorphic to K_{k+1}. We know that all the vertices are isolated vertices in the complement graph of $G(k+1,k,3)$. Therefore, the detour global dominating set must contain all the vertices of $G(k+1,k,3)$ and so, for $n = k+1$, $\bar{\gamma}_d(G(k+1,k,3)) = k+1$.

Now consider for $n \geq k+2$, starting with the vertex v_i for $1 \leq i \leq n$ $N[v_i] = \{v_{i-\frac{k}{2}}, \cdots, v_{i-2}, v_{i-1}, v_i, v_{i+1}, v_{i+2}, \cdots, v_{i+\frac{k}{2}}\}$ where the suffices modulo n and $|N[v_i]| = k+1$. Now, we choose the next vertex to be v_{i+k+1} where, $N[v_{i+k+1}] = \{v_{i+\frac{k}{2}+1}, \cdots, v_{i+k-1}, v_{i+k}, v_{i+k+1}, v_{i+k+2}, v_{i+k+3}, \cdots, v_{i+\frac{3k}{2}+1}\}$. Clearly, $N[v_i] \neq N[v_{i+k+1}]$. Proceeding like this we obtain a set $S = \{v_i, v_{i+k+1}, v_{i+2(k+1)}, \cdots, v_{i+(\lceil\frac{n}{k+1}\rceil-1)(k+1)}\}$ which dominates every vertices in $G(n,k,3)$. Also, $v_i - v_{i+k+1}$ detour path covers all the vertices of $G(n,k,3)$. As a result, S is a minimum detour dominating set. We now show that S is a global dominating set of $G(n,k,3)$. In $\overline{G(n,k,3)}$, $N[v_i] \cup N[v_{i+k+1}] = V(G(n,k,3))$. Since $v_i, v_{i+k+1} \in S$, S is a dominating set of $\overline{G(n,k,3)}$. Therefore, $S = \{v_i, v_{i+k+1}, v_{i+2(k+1)}, \cdots, v_{i+(\lceil\frac{n}{k+1}\rceil-1)(k+1)}\}$ is a minimum detour global dominating set for $1 \leq i \leq n$ and the suffices modulo n and hence, $\bar{\gamma}_d(G(n,k,3)) = |S| = \left\lceil\frac{n}{k+1}\right\rceil$ for $n \geq k+2$.

Example 3.3 *Consider the graph G(12,8,3) given in Figure 3. for which the minimum detour global dominating sets are* $\{v_1, v_{10}\}, \{v_2, v_{11}\}, \{v_3, v_{12}\}, \{v_4, v_1\}, \{v_5, v_2\}, \{v_6, v_3\}, \{v_7, v_4\}, \{v_8, v_5\}, \{v_9, v_6\}, \{v_{10}, v_7\}, \{v_{11}, v_8\}$ *and* $\{v_{12}, v_9\}$ *and hence by Theorem 3.2,* $\bar{\gamma}_d(G(12,8,3)) = \left\lceil\frac{12}{9}\right\rceil = 2.$



$Figure\ 3 : G(12, 8, 3)$

Theorem 3.4 *In $G(n,k,3)$, for $n > k+1$ and if n is a multiple of $k+1$, then $G(n,k,3)$ contains only $k+1$ minimum detour global dominating set.*

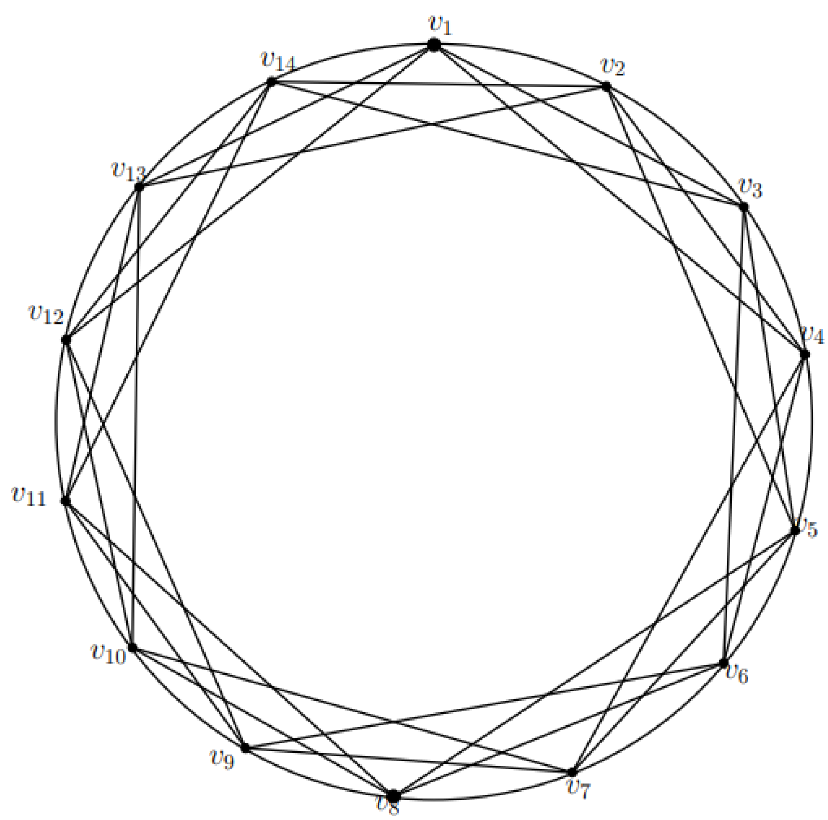
Proof. Let $G(n,k,3)$ be a k-regular graph with girth 3 where k is even and n be a multiple of $k+1$. Then $n = (k+1)m$ where $m \geq 1$. If $m = 1$, then $G(n,k,3) = G(k+1,k,3) \cong K_{k+1}$ which contains every vertices of $G(k+1,k,3)$. Hence, for $n = k+1$ there exists only one minimum detour global dominating set. Now consider for $n = (k+1)m, m \geq 2$ where the minimum detour global dominating sets is $S = \{v_i, v_{i+k+1}, v_{i+2(k+1)}, \cdots, v_{i+(\lceil\frac{n}{k+1}\rceil-1)(k+1)}\} = \{v_i, v_{i+k+1}, v_{i+2(k+1)}, \cdots, v_{i+(m-1)(k+1)}\} = \{v_i, v_{i+k+1}, v_{i+2(k+1)}, \cdots, v_{i+n-(k+1)}\}$ for $1 \leq i \leq n, m \geq 2$ and the suffices modulo n.

Now we consider the following two cases for $m = 2$ and $m > 2$.

Case 1. $m = 2$

Here, $G(n,k,3) = G(2(k+1),k,3)$. Then The minimum detour global dominating sets are $\{v_1, v_{k+2}\}, \{v_2, v_{k+3}\}, \{v_3, v_{k+4}\}, \cdots, \{v_k, v_{2k+1}\}$ and $\{v_{k+1}, v_{2k+2}\}$. Thus, $G(n,k,3)$ contains only $k+1$ minimum detour global dominating sets.

For example consider the graph $G(n,6,3) = G(14,4,3)$, given in Figure 4. The minimum detour global dominating sets are $\{v_1, v_8\}, \{v_2, v_9\}, \{v_3, v_{10}\}, \{v_4, v_{11}\}, \{v_5, v_{12}\}$ and $\{v_6, v_{13}\}$. Thus, $G(n,6,3)$ contains only 7 minimum detour global dominating sets.



$Figure\ 4 : G(14,6,3)$

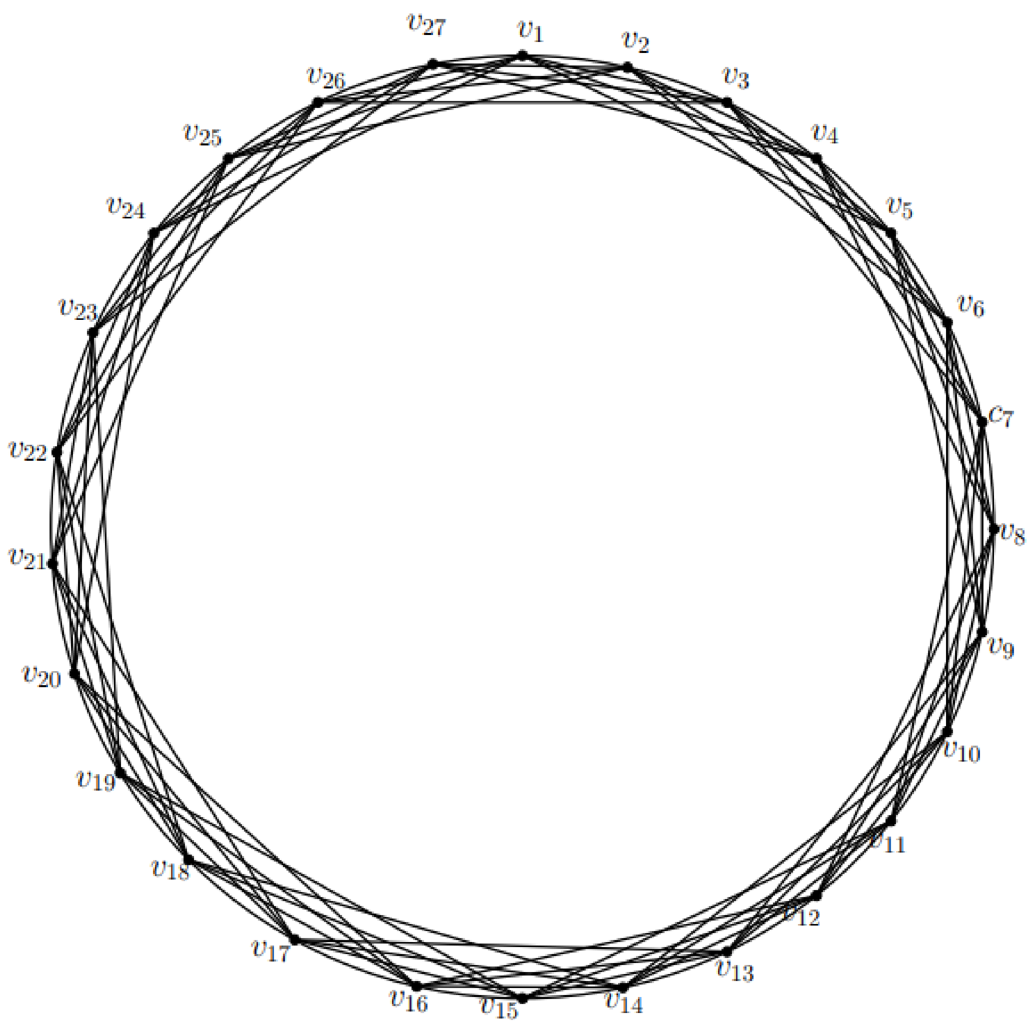
Case 2. $m > 2$

The minimum detour global dominating sets are $S_i = \{v_i, v_{i+k+1}, \cdots, v_{i+n-(k+1)}\}$ where

$1 \leq i \leq n$ and the suffices modulo n. Here, $S_1 = \{v_1, v_{k+2}, \cdots, v_{n-k}\}, S_2 = \{v_2, v_{k+3}, \cdots, v_{n-(k-1)}\}, \cdots, S_k = \{v_k, v_{2k+1}, \cdots, v_{n-1}\}$, and $S_{k+1} = \{v_{k+1}, v_{2(k+1)}, \cdots, v_n\}$. Proceeding like this we get $S_p = \{v_p, v_{p+k+1}, \cdots, v_{p+n-(k+1)}\}$ for $k+2 \leq p \leq n$. Since the suffices are modulo n, S_p is either S_1 or S_2 or $\cdots S_k$ or S_{k+1} according as $p \equiv 1(mod\ (k+1))$ or $p \equiv 2(mod\ (k+1))$ or $\cdots$ $p \equiv k(mod\ (k+1))$ or $p \equiv 0(mod\ (k+1))$, respectively. Thus, $G(n,k,3)$ contains only $k+1$ minimum detour global dominating sets.

The theorem follows from cases 1 and 2.

Example 3.5 *Consider the graph $G(27,8,3)$ given in Figure 5 where the minimum detour global dominating sets are $\{v_1, v_{10}, v_{19}\}, \{v_2, v_{11}, v_{20}\}, \{v_3, v_{12}, v_{21}\}, \{v_4, v_{13}, v_{22}\}, \{v_5, v_{14}, v_{23}\}, \{v_6, v_{15}, v_{24}\}, \{v_7, v_{16}, v_{25}\}, \{v_8, v_{17}, v_{26}\}$ and $\{v_9, v_{18}, v_{27}\}$. Thus, by Theorem 3.4, there is only 9 minimum detour global dominating sets.*



$Figure\ 5 : G(27, 8, 3)$

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