Distance in Partial Metric Spaces and Common Fixed Point Theorems

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Abstract:
In this paper, we have proved some results of fixed point on partial metric spaces.

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1. Introduction

In 1992, Matthews [1] introduced a concept, and basic properties of partial metric (pmetric) functions. The partial metric space is a generalization of the usual metric space in which the self-distance is no longer necessarily zero. The failure of a metric function in computer studies was the primary motivation behind the introductory of the partial metrics [2]. After introducing the partial metric functions, Matthews [2] also proved the partial metric version of the Banach fixed point theorem; this makes the partial metric function relevant in fixed point theory. The topological properties of partial metric space studied by [10]. In 1999, Heckmann [7] established some results using a generalization of the partial metric function called a weak partial metric function. In 2004, Oltra and Valero [8] also generalized the Matthews's fixed point theorem in a complete partial metric space, in the sense of O'Neill. In 2013, Shukla et al.[6] introduced the notion of asymptotically regular mappings in a partial metric space, and established some fixed point results. Recently, Onsod et al. [3] established some fixed point results in a complete partial metric space endowed with a graph. Very recently, Batsari and Kumam [9] established the existence, and uniqueness of globally stable fixed points of an asymptotically contractive mappings. Also Dhanorkar proved some results [4, 5] using some of the properties of a partial metric function. In order to understand and develop the theory of partial metric space better, we shall draw our attention to certain fixed point theorems in this paper.

2. Preliminary Notes

Huang and Zhang [2] defined following cone metric space

Definition 2.1 [2] Let X be a non-empty set. Suppose the mapping $p: X \times X \to E \to [0, \infty)$ is said to be a partial metric on X if for any $x, y, z \in X$ the following conditions hold:

(p1) $p(x, y) = p(y, x)$ (symmetry),

(p2) If $p(x,x) = p(x,y) = p(y,y)$ then $x = y$ (equality),

(p3) $p(x,x) \leq p(x,y)$ (small self distances),

(p4) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$ (traingularity) Then (X,p) is called a partial metric space .

Notice that For a given partial metric p on X, the function $d_p: X \times X \to E \to [0,\infty)$ given by

$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$

is a metric on X. Observe that each partial metric p on X generates a T_0 topology

T_p on X with a base of the family of open p-balls $\{B_p(x,\epsilon)/x \in X, \epsilon > 0\}$,

where $B_p(x,\epsilon) = \{y \in X/p(x,y) < p(x,x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$. similarly, closed p-ball is defined as $B_p(x,\epsilon) = \{y \in X/p(x,y) \leq p(x,x) + \epsilon\}$

Definition 2.2 [2] (i) A sequence $\{x_n\}$ in a partial metric space (X,p) converge to $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$

(ii) A sequence $\{x_n\}$ in a partial metric space (X,p) is called Cauchy if and only if $x \in X$ if and only if $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$ is finite,

(iii) A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to T_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$

(iv) A mapping $f: X \to X$ is said to be continuous at $x_0 \in X$ if for every $\epsilon > 0$, there exist $\delta > 0$ such that $f(B((x_0),\delta)) \subset B(fx_0,\epsilon)$

Definition 2.3 [2] (i) A sequence $\{x_n\}$ is Cauchy in a partial metric space (X,p) if and only if $\{x_n\}$ is Cauchy in a metric space (X,d_p),

(ii) A partial metric space (X,p) is complete if and only if a metric space (X,d_p) is complete.

Moreover

$\lim_{n\to\infty} d_p(x,x_n) = 0 \Rightarrow p(x,x) = \lim_{n\to\infty} d_p(x,x_n) = \lim_{n,m\to\infty} d_p(x_n,x_m)$

3. Main results

In this section, a common fixed point theorem is proved for a pair of self mapping defined on a cone metric space under a plane contractive condition.

Theorem 3.1 Let (X,d) be a partial metric space. Suppose the mappings $f,g: X \to X$ satisfy

$$d(fx,fy) \leq kd(gx,gy), \text{for all } x,y \in X \tag{3.1}$$

where $k \in [0,1/3]$ is a constant. if the $f(X) \subset g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X.

Proof. Let x_0 be any arbitrary point in X. Choose a point x_1 in X such that $f(x_0) = g(x_1)$. this holds, since the range of g contains the range of f. Continuing this process, having choose x_n in X, we obtain x_{n+1} in X such that $f(x_n) = g(x_{n+1})$, then

$$d(gx_{n+1}, gx_n) = d(fx_n, fx_{n-1}) \leq kd(gx_n, gx_{n-1})$$

$$\leq k[d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n-1}) - d(gx_{n+1}, gx_{n+1})]$$

$$\leq k[d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n-1})]$$

$$\leq k[d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_n) + d(gx_n, gx_{n-1}) - d(gx_n, gx_n)]$$

$$\leq k[2d(gx_{n+1}, gx_n) + d(gx_n, gx_{n-1})]$$

$$(1-2k)d(gx_{n+1}, gx_n) \leq kd(gx_n, gx_{n-1})$$

$$d(gx_{n+1}, gx_n) \leq \frac{k}{(1-2k)} d(gx_n, gx_{n-1}), \tag{3.2}$$

where $\lambda = \frac{k}{(1-2k)} < 1$ with $0 < k < 1/3$. Hence we can write

$$\therefore d(gx_{n+1}, gx_n) \leq \lambda d(gx_n, gx_{n-1})$$

$$\leq \lambda^2 d(gx_{n-1}, gx_{n-2})$$

$$\leq \lambda^3 d(gx_{n-2}, gx_{n-3})$$

$$\vdots$$

$$\leq \lambda^n d(gx_1, gx_0) \tag{3.3}$$

So for $n > m$,

$$d(gx_n, gx_m) \leq d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_m) - d(gx_{n+1}, gx_{n+1})$$

$$\leq d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_m)$$

$$\vdots$$

$$\leq d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + \ldots + d(gx_{m-1}, gx_m)$$

$$\leq (\lambda^n + \lambda^{n+1} + \ldots + \lambda^{m-1}) d(gx_1, gx_0)$$

$$d(gx_n, gx_m) = \frac{\lambda^n}{(1-\lambda)} d(gx_0, gx_1) \tag{3.4}$$

gives $d(gx_n, gx_m) \to 0$ as $n \to \infty$. We get $\{g(x_n)\}$ is Cauchy sequence in complete partial metric space $g(X)$, there exists a point $y \in g(X)$ such that $g(x_n) \to y$ as $n \to \infty$. and $\lim_{n\to\infty} d(gx_n, y) = d(y, y) = \lim_{n\to\infty} d(gx_n, gx_n) = 0$.

Now

$$d(gy, y) \leq d(gy, gx_n) + d(gx_n, y) - d(gx_n, gx_n)$$

$$\leq d(gy, gx_n) + d(gx_n, gx_{n+1}) + d(gx_{n+1}, y) - d(gx_{n+1}, gx_{n+1}) - d(gx_n, gx_n)$$

$$\leq d(gy, gx_n) + \frac{\lambda^n}{1-\lambda} d(gx_0, gx_1)$$

$$< kd(gy, y) \text{ as } n \to \infty$$

which is contradiction. Hence $gy = y$. Now

$$d(gx_n, fy) = d(fx_{n-1}, fy) \tag{3.5}$$

$$\leq kd(gx_{n-1}, gy) \tag{3.6}$$

Consequently we can write $d(gx_n, fy) \to 0$ as $n \to \infty$. Also $d(gx_n, gy) \to 0$ as $n \to \infty$. gives $fy = gy$.

Uniqueness: Suppose there exist point $x \in X$ with $x \neq y$ such that $fx = gx$.

We get

$$d(gy, gx) = d(fx, fy) \leq kd(gx, gy) \tag{3.7}$$

We can write $d(gx, gy) = 0$ i.e. $gx = gy$

Hence f and g having unique common fixed point.

Theorem 3.2 Let (X, d) be a partial metric space. Let a_1, a_2, a_3, a_4 are nonzero real numbers with $a_1 + 2a_2 + a_3 + a_4 < 1$ and Suppose the mappings $f, g: X \to X$ satisfy

$$d(fx, fy) \leq a_1 d(gx, gy) + a_2 d(gx, fy) + a_3 d(gx, fx) + a_4 d(gy, fy) \tag{3.8}$$

for all $x, y \in X$ Suppose that the $f(X) \subset g(X)$ and $g(X)$ is a complete subspace of X. If f and g satisfy

$$\inf\{d(fx, y) + d(gx, y) + d(gx, fx): x \in X\} > 0 \tag{3.9}$$

or all $y \in X$ with $y \neq fy$ or $y \neq gy$, then f and g have a common fixed point in X.

Proof. Let x_0 be any arbitrary point in X. Since $f(X) \subset g(X)$, there exists an $x_1 \in X$ such that $fx_0 = gx_1$. Consider sequence x_n such that

$$f(x_n) = g(x_{n+1}), n = 0,1,2, \cdots \tag{3.10}$$

We can write

$$d(gx_n, gx_{n+1}) = d(fx_{n-1}, fx_n)$$

$$\leq a_1 d(gx_{n-1}, gx_n) + a_2 d(gx_{n-1}, fx_n) + a_3 d(gx_{n-1}, fx_{n-1}) + a_4 d(gx_n, fx_n)$$

$$\leq a_1 d(fx_{n-1}, gx_n) + a_2 d(gx_{n-1}, gx_{n+1}) + a_3 d(gx_{n-1}, gx_n) + a_4 d(gx_n, gx_{n+1}$$

$$\leq (a_1 + a_3) d(gx_{n-1}, gx_n) + (a_2 + a_4) d(gx_{n-1}, gx_{n+1})$$

$$\leq (a_1 + a_3)d(gx_{n-1}, gx_n) + (a_2 + a_4)[d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1}) - d(gx_n, gx_n)]$$

$$\leq (a_1 + a_2 + a_3 + a_4)d(gx_{n-1}, gx_n) + (a_2 + a_4)[d(gx_n, gx_{n+1}) - d(gx_n, gx_n)]$$

$$\leq \frac{a_1+a_2+a_3+a_4}{1-a_2-a_4} d(gx_{n-1}, gx_n)$$

$$\leq hd(gx_{n-1}, gx_n)$$

$$\text{where } \frac{a_1+a_2+a_3+a_4}{1-a_2-a_4} < 1 \tag{3.11}$$

$$\Rightarrow a_1 + a_2 + a_3 + a_4 < 1 - a_2 - a_4 \tag{3.12}$$

$$\Rightarrow a_1 + 2a_2 + a_3 + 2a_4 < 1 \tag{3.13}$$

Now

$$d(gx_n, gx_{n+1}) \leq hd(gx_{n-1}, gx_n)$$

$$\leq h^2 d(gx_{n-2}, gx_{n-1})$$

$$\vdots$$

$$\leq h^n d(gx_0, gx_1) \tag{3.14}$$

Let m, n with $m < n$, We get

$$d(gx_n, gx_m) \leq d(gx_, gx_{n+1}) + d(gx_{n+1}, fx_m) - d(gx_{n+1}, gx_{n+1})$$

$$\leq d(gx_, gx_{n+1}) + d(gx_{n+1}, fx_m)$$

$$\vdots$$

$$\leq d(gx_, gx_{n+1}) + d(gx_{n+1}, fx_{n+2}) + \ldots + d(gx_{m-1}, fx_{nm})$$

$$\leq h^n d(gx_0, gx_1) + h^{n+1} d(gx_0, gx_1) + \ldots + h^{m-1} d(gx_0, gx_1)$$

$$\leq (h^n + h^{n+1} + \ldots + h^{m-1}) d(gx_0, gx_1)$$

$$\leq \frac{h^n}{1-h} d(gx_0, gx_1)$$

We get sequence $\{g(x_n)\}$ is Cauchy sequence in X. Since $g(X)$ is complete, there exists some point $y \in g(x)$ such that $gx_n \to y$ as $n \to \infty$.

Hence, $d(gx_n, y) \to 0$ as $n \to \infty$ Suppose that $y \neq gy$ or $y \neq fy$.

We get

$$0 \leq \inf\{\| d(fx, y) \| + \| d(gx, y) \| + \| d(gx, fx) \| : x \in X\}$$

$$0 \leq \inf\{\| d(fx_n, y) \| + \| d(gx_n, y) \| + \| d(gx_n, fx_m) \|\}$$

$$0 \leq \inf\{\| d(gx_{n+1}, y) \| + \| d(gx_n, y) \| + \| d(gx_n, gx_{n+1}) \|\}$$

$$0 \leq \inf\{\| d(gx_{n+1}, y) \| + \| d(gx_n, y) \| + \| d(gx_n, y) \| + \| d(y, gx_{n+1}) \| - \| d(y, y) \|\}$$

$$0 \leq -d(y, y)$$

This is contradiction. hence $y = gy = fy$ this completes the proof.

Example 3.3 A mapping $f: X \to X$ defined by $f(x) = \frac{x}{2}$ for $x \neq 2$ and $f(x) = 2$ for $x = 2$ and the mapping $g: X \to X$ by $gx = x$ for all $x \in X$.

Since $d(f(1), f(2)) = d(g(1), g(2))$, there is no $k \in [0,1)$ such that $d(fx, fy) \leq kd(fx, gy)$ for all $x \in X$, since $d(f(1), f(2)) = d(1/2,2) = \max\{1/2,2\} = 2$, $d(g(1), g(2)) = d(1,2) = \max\{1,2\} = 2$.

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