Sp Semi-Linear Parabolic Spatial Des Using Robust Numerical Method

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Abstract

In this research, a resilient numerical approach for a coupled system of singly perturbed parabolic delay partial differential equations is designed and analyses. On the regular and layer components of the solution and their derivatives, design a priori bounds are derived. Shishkin and generalized Shishkin type appropriate layer adaptive meshes are defined in the spatial direction based on these a priori constraints. The problem is then discretized using the central difference scheme on layer-adapted Shishkin and generalized Shishkin type meshes in the spatial direction and an implicit Euler scheme on a uniform mesh in the temporal direction. A numerical approach to the issue is proposed, taking into account both generalized Shishkin meshes and relevant layer adapted Shishkin meshes.

Keywords:partial differential equations, Parabolic Spatial

1. Introduction

We arrive at the numerical solution to the 1D parabolic CDPs with singly perturbed device and overlapping boundary layers. The proposed numerical approach includes both the implicit-Euler procedure for temporal spinoff and an upwind finite distinction scheme for spatial derivatives. To do so, build up the consistent convergence including the perturbation parameters, we learn about the plan for a piecewise uniform mesh shishkin. The steadiness assessment and Estimation of parameter-uniform error are produced for the cautious technique. The Richardson extrapolation method is used to raise the order of convergence from nearly first-order to essentially 2nd-order. Numerical experiments are done using the provided methods to support the theoretical conclusions.

Think about the following: 1D parabolic system perturbed singly CD$Q := \Omega_x \times (0, T], \Omega_x = (0,1)$:

$$\begin{cases} \frac{\partial \vec{u}}{\partial t} + L_{x,\vec{\in}}\vec{u} = \vec{f}, (y,w) \in Q \\ \vec{u}(Y,0) = \overrightarrow{u_0}(x), x \in \overline{\Omega_x}, \\ \vec{u}(0,w) = \vec{0}, \vec{u}(1,w) = \vec{0}\,, w \in [0,W], \end{cases} \qquad (1)$$

As an example of the spatial differential opeator$L_{x,\vec{\varepsilon}}$ is given by

$$L_{y,\vec{\in}} \equiv -\varepsilon\frac{\partial^2}{\partial y^2} - A(y)\frac{\partial}{\partial y} + B(y),$$

With $\in = \mathrm{diag}(\in_1, \in_2)$, $A(y) = diag\big(a_1(y), a_2(y)\big), B(y) = \{b_{lm}(y)\}_{l,m=1}^2$

We assume that $\in_1, \in_2$ fulfill $0 < \varepsilon_1 \leq \varepsilon_2 \ll 1$, and the matrix of convection coefficients A must meet the following positive criteria:

$$a_1(y) \geq \alpha > 0,\ a_2(y) \geq \alpha > 0. \qquad (2)$$

Additionally, let us suppose that B is an L_0 matrix with

$$\min{}_{y \in \overline{\Omega_x}}\{b_{11}(y) + b_{12}(y), b_{21}(y) + b_{22}(y)\} \geq \beta > 0 \qquad (3)$$

If the model problem's data (1) are sufficiently smooth functions and meet adequate compatibility constraints, then the model problem (1) has a unique solution$\vec{u}(y,w) \in (C_\lambda^4(Q))^2$. Examples of typical assumptions for the source term and beginning condition are provided by

$$\vec{f} \in (C_\lambda^2(\bar{Q}))^2 \ and\ \overrightarrow{u_0} \in (C_0^4(\overline{\Omega_y}))^2 \qquad (4)$$

Problem (3.1) compatibility criteria are as follows.

$$\begin{cases} \overrightarrow{u_0}(y) = \vec{0}, y \in \{0,1\}, \\ \vec{f}(y,0) - L_{y,\vec{\varepsilon}}\overrightarrow{u_0}(y) = \vec{0},\ y \in \{0,1\}, \\ \overrightarrow{f_w}(y,0) + (L_{y,\vec{\varepsilon}})^2\overrightarrow{u_0}(y) - L_{y,\vec{\varepsilon}}\vec{f}(y,0) = \vec{0},\ y \in \{0,1\}. \end{cases} \qquad (5)$$

Compatibility criteria for the scalar case can be found here.

a 1D parabolic CD IBVP system with SP diffusion coefficients $\varepsilon_{1,}\varepsilon_2$ associated with each equation. To describe this case, boundary layers overlap along y = 0 on the left side of the spatial domain, hence a non-uniform mesh is used for the spatial variable while a uniform mesh is used for the temporal variable. Time semi discretization using the implicit-Euler scheme is combined with spatial discretization using the upwind DS in the numerical approach. Estimates of magnitude of error $O(N^{-1}In\ N + \Delta w)N$ is the spatial discretization parameter and t is the time step in the numerical solution. The estimated numerical solution is then refined using the Richardson extrapolation approach.

The following is how this section is organized: several analytical properties of the continuous problem are established. We also go through the exact solution's derivative bounds for the temporal semi discrete scheme.

2. The Solution's Limits And Derivatives

An exact solution for a continuous issue (1) has analytical properties, for example its highest fundamental and its restrictions on solution derivatives.

Lemma 1 Let $(\frac{\partial}{\partial x} + L_{x,\vec{\varepsilon}})$ Let t be the Differential operator described in (1), and we consider that the matrices A and B satisfy the conditions of (2) and (3)After that, $\vec{z} \geq \vec{0}\ on\ \partial Q\ and\ \left(\frac{\partial}{\partial x} + L_{x,\vec{\varepsilon}}\right)\vec{z} \geq \vec{0}\ and\ Q, we\ have\ \vec{z} \geq \vec{0}\,, for\ all\ (x,t) \in \bar{Q}.$

Proof. There is no other way to prove this lemma. Suppose that a point exists.

$(x_0, t_0) \in Q$such that.

$min\{z_1(x_0,t_0), z_2(x_0,t_0)\} = \min\{\min_{(x,t)\in\bar{Q}} z_1(x,t), \min_{(x,t)\in\bar{Q}} z_2(x,t)\} < 0.$

To keep things as broad as possible, we assume that $z_1(x_0,t_0) \le z_2(x_0,t_0)$Then, the first part of the structure$(\frac{\partial}{\partial \mathrm{x}} + L_{x,\vec{\varepsilon}})\vec{z}$ satisfies

$$\frac{\partial z_1}{\partial t} + L_{x,\overrightarrow{\varepsilon_1}}\vec{z}(x_0,t_0) \le b_{11}(x_0)z_1(x_0,t_0) + b_{12}(x_0)z_2(x_0,t_0) < 0,$$

Because this lemma's hypothesis is false, it follows that

$$\vec{z} \ge 0, for\ all\ (x,t) \in \bar{Q}.$$

The following lemma will assist us in this endeavor, as will the bound for the precise answer$\vec{u}$.

Lemma 2 The solution is as follows: $\vec{u}$The following estimate applies to the solution of the problem (1).

$$|\vec{u}(x,t) - \vec{u}(x,0)| \le \vec{C}t, (x,t) \in \bar{Q},$$

In which C is not dependent of $\varepsilon_1, \varepsilon_2$

Proof. We will only make an estimate for the first component u_1, but the same procedure may be used to demonstrate the outcome for the second component u_2.

Set

$$\vec{\Phi}(x,t) = \vec{\mathrm{u}}(x,t) - \overrightarrow{u_0}(x), where\ \vec{\mathrm{u}}(x,0) = \overrightarrow{u_0}(x)$$

Then $\vec{\Phi}$ fulfils the requirements of the following problem

$$\begin{cases} \dfrac{\partial \phi_1}{\partial t} + L_{x,\epsilon_1}\vec{\phi}(x,t) = f_1(x,t) - L_{x,\epsilon_1}\overrightarrow{u_0}(x), \\ \phi_1(x,0) = 0\ for\ 0 < x < 1, \\ \phi_1(x,t) = 0\ and\ \phi_1(1,t) = 0\ for\ 0 \le t \le T. \end{cases}$$

Set $\vec{\varphi}(x,t) = \vec{C}t$ when the positive constant is sufficiently large $\vec{C} = (C,C)^T$, as a result, it is simple to show that

$$\begin{cases} \dfrac{\partial \psi_1}{\partial t} + L_{x,\epsilon_1}\vec{\psi}(x,t) = C + C(b_{11}+b_{12})t, \\ \psi_1(x,0) = 0\ for\ 0 < x < 1, \\ \psi_1(x,t) = \ \psi_1(1,t) = Ct\ for\ 0 \le t \le T. \end{cases}$$

We can achieve this result by employing the maximal principle stated in Lemma2

$$|\phi_1(x,t)| = |u_1(x,t) - u_1(x,0)| \le Ct.$$

Similarly, we can get $|\phi_2(x,t)| = |u_2(x,t) - u_2(x,0)| \le Ct$. This brings the proof to a close.We examine the qualitative behavior graphically before getting into the thorough study of derivative bounds for the solution$\vec{u}(x,t)$of (1).

Example 1: Consider the following problem (1), in which the values of A and B are provided as

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \ B = \begin{pmatrix} 2+x & -1 \\ -1 & 2+2x \end{pmatrix}$$

as well as the source phrase$\vec{f} = (1,1)^T$without any baseline or limit conditions.

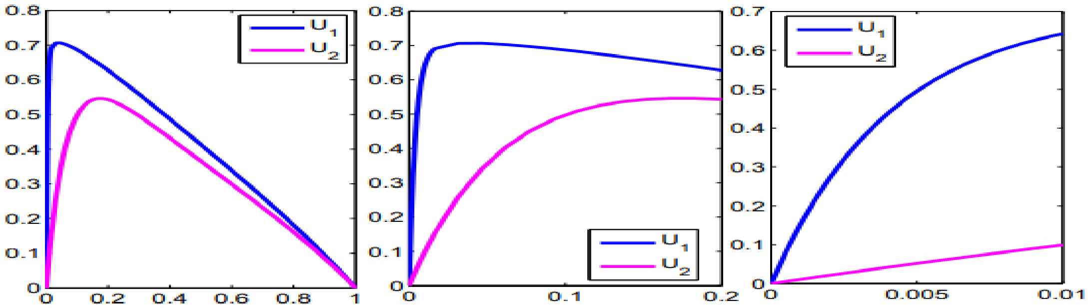

Fig. 1: Example 3.1 numerical solution for ε_1= 2-[8], ε_2= 2-[4] and N = 64 at time *t = 1*

Figure 1 depicts the overlapping boundary layers along the side x = 0. While both components have width boundary layers $O\ (\varepsilon_2 In\ \varepsilon_2), only\ u_1(x,t)$ has an extra width sublayer $O\ (\varepsilon_1 In\ \varepsilon_1)$, this phenomenon is depicted in Fig. 1. The differences in behavior between the two curves in the rightmost figure are very obvious.

Theorem 1. For any non-zero integersk; k_0, satisfying $0 \le k + k_0 \le 2$the exact solution's derivatives $\vec{u} = (u_1, u_2)^T$The following estimates are satisfied by the IBVP (1):

$$\left|\frac{\partial x^{k+k_0} u_t}{\partial x^k \partial t^{k_0}}\right| \le \begin{cases} C, for\ k = 0 \\ C\left(1 + \epsilon_l^{-1}\ B_{\epsilon_l}^0(x)\right), for\ k = 1, \\ C\left(1 + \epsilon_l^{-1}\ (\epsilon_1^{-1}\ B_{\epsilon_1}^0(x) + \epsilon_2^{-1}\ B_{\epsilon_2}^0(x))\right), for\ k = 2, \end{cases}$$

for all (x; t) $\epsilon\ \bar{Q}$ and l = 1, 2.

Proof. We shall explore several situations to verify the boundaries of the derivative of the exact solution $\vec{u}$ of (1).

Case 1: We will explore the situation k_0 =0. In this section, we will explore the situation for from lemma 2 we have

$$\|\vec{u}\|_\infty \le \frac{1}{1-\gamma}\left(\frac{1}{\gamma_\beta}\|\vec{f}\|_\infty + \|\vec{u}\|_\infty\right)$$

Case 2: Let's have a look at it. *k*=0 and k_0 =1. Since $\vec{u}(0,t) = \vec{u}(1,t) = \vec{0}\ for\ t \in [0,1]$because of which it follows $\vec{u}_t = \vec{0}$.In addition, by employing the regularity condition (4), we obtain$|\vec{u}_t(x,0)| \le \vec{C}$ for all$x \in [0,1]$.When we differentiate (1) regarding *t*, we get

$$\left(\frac{\partial}{\partial t} + L_{x,\vec{\varepsilon}}\right)\overrightarrow{u_t}(x,t) = \overrightarrow{u_{tt}} - \varepsilon\overrightarrow{u_{txx}} - A\overrightarrow{u_{tx}} + B\overrightarrow{u_t} = \overrightarrow{f_t}. \qquad (6)$$

Since$\vec{f}$is a reasonably smooth function, thus by applying the highest principle on it Q, We can then assume that$|\vec{u}_t| \le \vec{C}$

Case 3lets look at the situation. In which *k = 1* and $k_0 = 0$*:* We get

$$\frac{\partial u_1}{\partial t} - \varepsilon_1\frac{\partial^2 u_1}{\partial t^2} - a_1\frac{\partial u_1}{\partial x} + b_{11}u_1 + b_{12}u_2 = f_1, on\ \bar{Q}.$$

The preceding equation can be rewritten as follows:

$$\varepsilon_1\frac{\partial^2 u_1}{\partial x^2} = \frac{\partial u_1}{\partial t} - a_1\frac{\partial u_1}{\partial x} + b_{11}u_1 + b_{12}u_2 - f_1, (7)$$

This suggests that $\epsilon_1\left|\frac{\partial u_1}{\partial x}(\theta,t)\right| \le 2\|\vec{u}\|_\infty$. We get by integrating (7) regarding *x* and then integrating by parts.

$$\in_1 \left|\frac{\partial u_1}{\partial x}(\theta,t)-\frac{\partial u_1}{\partial x}(0,t)\right| = \int_0^\theta \frac{\partial u_1}{\partial t}(s,t)ds-[a_1(s)u_1(s,t)]_0^\theta+\int_0^\theta \frac{\partial a_1}{\partial s}(s)u_1(s,t)ds + \int_0^\theta \big(b_{11}(s)u_1(s,t)+b_{12}(s)u_2(s,t)\big)ds-\int_0^\theta f_1(s,t)ds.$$

As a result, we have obtained

$$\in_1 \left|\frac{\partial u_1}{\partial x}(0,t)\right| \le \|f_1\|_\infty + \left\|\frac{\partial u_1}{\partial t}\right\|_\infty + C(\|u_1\|_\infty+\|u_2\|_\infty)$$

By making use of the bound of $|\vec{u}|$ and $\vec{u}_t$one is able to obtain

$$\left|\frac{\partial u_1}{\partial x}(0,t)\right| \le C\varepsilon_1^{(-1)}$$

The following is an expression for the equation (7):

$$\varepsilon_1\frac{\partial^2 u_1}{\partial \mathrm{t}^2}+a_1\frac{\partial u_1}{\partial \mathrm{x}}=\frac{\partial u_1}{\partial \mathrm{t}}+b_{11}u_1+b_{12}u_2-f_1\equiv A_1(x,t) \qquad (8)$$

We get the following result by integrating (8) with regard to *x*:

$$\frac{\partial u_1}{\partial x}(x,t)=\frac{\partial u_1}{\partial x}(0,t)\exp\left(\frac{-\big(\eta_1(x)-\eta_1(0)\big)}{\varepsilon_2}\right)-\varepsilon_1^{(-1)}\int_0^x \Lambda_1(s,t)\exp\left(\frac{-\big(\eta_2(x)-\eta_1(0)\big)}{\varepsilon_1}\right)ds$$

Whereη_1(x) is a definite integral of the indefinite a_1 (x) (x; t), we are able to obtain that By making use of the bounds of $\frac{\partial u_1}{\partial x}(0,t)$ and$\Lambda_1(x,t)$ We are able to obtain that

$$\left|\frac{\partial u_1}{\partial x}\right| \le C\left(1+\varepsilon_1^{(-1)}B_{\varepsilon_1}^0(x)\right).$$

We can derive that in a similar method as well.

$$\left|\frac{\partial u_2}{\partial x}\right| \le C\left(1+\varepsilon_2^{(-1)}B_{\varepsilon_2}^0(x)\right).$$

Case 4 consider k=0 and k_0=2. From the $\vec{u}(0,t)=\vec{u}(1,t)=\vec{0}\, for\ t\in[0,1]$because$\vec{u}_t=\vec{u}_{tt}=\vec{0}$ Using the regularity criteria and the estimate from Case 2**,** we have $|\vec{u}_{tt}(x,0)|\le\vec{C}$ for all x ∈ [0, 1]. Following that, distinguishing (6) in relation to *t*, we obtain

$$\left(\frac{\partial}{\partial \mathrm{x}}+L_{x,\vec{\varepsilon}}\right)\overrightarrow{u_{tt}}=\overrightarrow{u_{ttt}}-\varepsilon\overrightarrow{u_{ttxx}}-A\overrightarrow{u_{ttx}}+B\overrightarrow{u_{tt}}=\overrightarrow{f_{tt}}$$

Since$\vec{f}_{tt}$ maximal concept is used in Lemma 3.4 to obtain bounded function $|\vec{u}_{tt}|\le\vec{C}\ on\ \bar{Q}$

Case 5 In this section, we'll look at the situation. k = 1 and k_0 = 1: To begin, we'll talk about derivative boundaries. u_1, For a fixed $t\in[0,T]$,, there exist$\theta\in(0,1)$ such that

$$\frac{\partial^2 u_1}{\partial \mathrm{x}\,\partial \mathrm{t}}(\theta,t)=\frac{1}{\varepsilon_1}\left(\frac{\partial u_1}{\partial t}(\varepsilon_1,t)-\frac{\partial u_1}{\partial t}(0,t)\right),$$

This suggests that $\varepsilon_1\left|\frac{\partial^2 u_1}{\partial \mathrm{x}\,\partial \mathrm{t}}(\theta,t)\right|\le 2\left|\frac{\partial u_1}{\partial t}\right|_\infty$,we have (9) by differentiating (7) regarding t and rearranging the terms.

$$\varepsilon_1\frac{\partial^3 u_1}{\partial \mathrm{x}^2\,\partial \mathrm{t}}=\frac{\partial^2 u_1}{\partial t^2}-a_1\frac{\partial^2 u_1}{\partial \mathrm{x}\,\partial \mathrm{t}}+b_{11}\frac{\partial u_1}{\partial t}+b_{12}\frac{\partial u_2}{\partial t}-\frac{\partial f_1}{\partial t}. \qquad (9)$$

We obtain by integrating (9) with corresponding to x and using the Case 3 technique.

$$\varepsilon_1\left|\frac{\partial^3 u_1}{\partial \mathrm{x}^2\,\partial \mathrm{t}}(0,t)\right|\le\left\|\frac{\partial f_1}{\partial t}\right\|_\infty+\left\|\frac{\partial^2 u_1}{\partial \mathrm{t}^2}\right\|_\infty+C\left\|\frac{\partial u_1}{\partial t}\right\|_\infty+\left\|\frac{\partial u_2}{\partial t}\right\|_\infty$$

Making use of the bound of $|\vec{u}_t|$ and $|\vec{u}_{tt}|$ we obtain

$$\left|\frac{\partial^2 u_1}{\partial x \partial t}(0,t)\right| \leq C\varepsilon_1^{(-1)}$$

It is now possible to express (9) as

$$\varepsilon_1 \frac{\partial^3 u_1}{\partial \mathrm{x}^2\, \partial \mathrm{t}} + a_1 \frac{\partial^2 u_1}{\partial \mathrm{x}\, \partial \mathrm{t}} = A_2(x,t), where\ A_2(x,t) = \frac{\partial^2 u_1}{\partial \mathrm{t}^2} + b_{11}\frac{\partial u_1}{\partial t} + b_{12}\frac{\partial u_2}{\partial t} - \frac{\partial f_1}{\partial t}$$

Case 3 argument can lead us to this conclusion.

$$\left|\frac{\partial^2 u_1}{\partial x \partial t}\right| \leq C\left(1 + \varepsilon_2^{(-1)} B_{\varepsilon_2}^0(x)\right).$$

Similarly, the needed bound for component may be determined in the same way u_2.

Case 6. Take, for example, the situation in which k = 2 and k_0 = 0: We will offer you with an estimate for the component u_1To begin, consider the first component of (1) as shown in the following form

$$\varepsilon_1 \frac{\partial^2 u_1}{\partial \mathrm{x}^2} = \frac{\partial u_1}{\partial \mathrm{t}} - a_1 \frac{\partial u_1}{\partial \mathrm{x}} + b_{11}u_1 + b_{12}u_2 - f_1 \qquad (10)$$

Following the approach stated in Case 3, one could be

$$\left|\frac{\partial^2 u_1}{\partial x^2}(0,t)\right| \leq C\varepsilon_1^2$$

It is straightforward to calculate that from (10) and the preceding estimate.

$$\left|\frac{\partial^2 u_1}{\partial x^2}(0,t)\right| \leq C\varepsilon_1^2$$

Differentiating (10) in relation to another x, There is nothing we can do about it

$$\varepsilon_1 \frac{\partial^3 u_1}{\partial \mathrm{x}^3} + a_1 \frac{\partial^2 u_1}{\partial \mathrm{x}^2} = A_3(x,t).\, on\ \bar{Q} \qquad (11)$$

$$where\ A_3(x,t) = -\frac{\partial f_1}{\partial x} - \frac{\partial u_1}{\partial x}\frac{\partial u_1}{\partial x} + \frac{\partial(b_{11}u_1 + b_{12}u_2)}{\partial x} + \frac{\partial^2 u_1}{\partial x\, \partial \mathrm{t}}$$

Case 3 and Case 5 can be used to obtain the desired result.

$$|A_3(x,t)| \leq C\left(1 + \varepsilon_1^1 B_{\varepsilon_1}^0(x) \varepsilon_2^1 B_{\varepsilon_2}^0(x)\right).$$

Then, using Case 3's reasoning and the bound of A3(x, t), we may deduce that

$$\left|\frac{\partial^2 u_1}{\partial x^2}\right| \leq C\left(1 + \varepsilon_1^{-1}(\varepsilon_1^{-1} B_{\varepsilon_2}^0(x) + \varepsilon_2^{-1} B_{\varepsilon_2}^0(x)\right).$$

In a similar vein, one can get an estimate for u_2. This brings the proof to a close.

3. Conclusion

In this study, a coupled system of singularly perturbed parabolic PDEs with time delay has been taken into consideration (2). We discretized the issue using the finite difference operator, which consists of an implicit Euler scheme for time and a central difference scheme for space, on a rectangular mesh consisting of Shishkin mesh or generalised Shishkin mesh in the space direction and uniform mesh in the time direction. It has been demonstrated that the suggested numerical method for Shishkin and generalised Shishkin meshes uniformly converges in the maximum norm. It is demonstrated that, regardless of the perturbation parameters, the suggested numerical technique converges with first order in time and nearly second-order in space. The theoretical convergence conclusions were supported by numerical results.

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