A Review on Applications of Fractional Differential Equations in Engineering Domain

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Article Info

Page Number: 7147 - 7166

Publication Issue:

Vol 71 No. 4 (2022)

Abstract

The application of fractional calculus could be found all around the world. The concepts of fractions have been utilized in a wide variety of physical processes across many different scientific disciplines, including engineering, physics, and chemistry, amongst others. There are a lot of various methods that can be employed, such as the fractional analysis method, the Caputo fractional differential operator, and the Laplace transform, etc. The applications of Fractional Differential Equations (FDEs) in engineering in many different areas, such as in fractional cross product, electronic circuits, control engineering, electronic system designing, and modelling of speech. An innovative method that is now known as Reference model tuning was established. A comparison of the Z-N, Integral Square Time Error (ISTE), and Reference model tuning methods is carried out to ascertain which one is the most effective. The suggested method gives reasonable overshoots and shorter settling times and gives fewer oscillations.

Article History

***Article Received**: 25 March 2022*

***Revised**: 30 April 2022*

***Accepted**: 15 June 2022*

***Publication**: 19 August 2022*

Keywords: - Fractional complex transform, Laplace transform, Fractional Cross product, Control engineering, Modelling of speech, electronic system designing.

1. Introduction

A physical phenomenon that occurs on a global scale could be described using the notion of derivatives and fractional-order integrals (FOI). Consequently, fractional notions have been seen as a tool for expressing physical processes in domains such as engineering, physics, and chemistry. Consequently, a big number of powerful methods, like Galerkin finite element, variational iteration, homotopy analysis, Sumudu transform, iteration, modified homotopyperturbation, Adomian decomposition, homotopy perturbation, generalized trigonometry functions,fractional linear multistep methods, or developed trial equation method, have been presented in the text [1-8]. Additionally, several authors have examined the features of fractional concepts in a variety of methods [9].

A non-exponential relaxation agomalicious diffusion and pattern could alter the dynamics of transport in complex systems, among other things, are studied using FDEs. Derivative fractional derivatives could be used to describe a broad range of phenomena more correctly in several fields of research, including mathematics and science; mechanical engineering; economics, and bioengineering; among others. This discovery has implications for a wide range of phenomena in many fields including engineering and bioengineering, science and mathematics, and economics. Using fractional derivatives (FD), scientists and mathematicians may better understand a broad variety of phenomena across a wide range of scientific and mathematical disciplines. At present, FDEs are becoming increasingly popular as a new powerful tool for modelling a wide range of complex systems, including those with overlapping microscopic and macroscopic dimensions, as well as those with wide-range temporal memory and interactions in space, among other characteristics [10][11].Differential equations (DEs) can be used to study complex systems that exhibit non-exponential relaxation patterns and anomalous diffusion [12]. There has been a resurgence of interest in evolving numerical approaches for finding the solutions of FDEs, and there is a substantial amount of studies accessible on the topic. The two most challenging aspects of having to cope with FDEs are as follows:

i. FDs are operators that are not local.

ii. FDs entail solitary kernel/weight functions, and FDE solutions are often singular at their borders.

1.1 Fractional Complex Transformation

Classical DEs are diverse as signal processing and control theory, finance, mechanical engineering, and fractal dynamics among others.As a result, advanced calculus analytical techniques may be simply applied to FC via the use of FCT, which is suggested.The following fractional DE serves as an example of how the FCT approach may be used [13].

$$\partial^{\alpha}, \mathrm{u}(\mathrm{x},\mathrm{t}) = \mathrm{k}\,\partial_{\mathrm{x}}^{\beta}\mathrm{u}(\mathrm{x},\mathrm{t}), \mathrm{t} \in \mathrm{R}^{+}, \mathrm{x} \in \mathrm{R} \qquad (1)$$

if the initial condition is met,

$$\mathrm{u}(\mathrm{x}, 0) = \mathrm{u}_0(\mathrm{x}) = 2\mathrm{x}, \qquad (2)$$

k = positive coefficient, $0 < \alpha < 1,\ 0 < \beta < 1,\ u(x,t)$is the real-valued variable, ∂^{α} and $\partial_{\mathrm{x}}^{\beta}$ are changed Riemann-Liouville fractional derivatives (RLFD).

The FCT is an uncomplicated operation. Adding a complex variable ξ that is specified as:

$$\xi = \frac{\mathrm{p}\mathrm{x}^{\beta}}{\Gamma(1+\beta)} + \frac{\mathrm{q}\mathrm{t}^{\alpha}}{\Gamma(1+\beta)} \quad (3)$$

There are two unknown constants, p, and q, that need to be found out more about.

Substituting Eq. (3) into Eq. (1), It has obtained

$$\mathrm{k}\,(\mathrm{p} - \mathrm{q})\mathrm{u}_{\xi} = 0 \quad (4)$$

Case 1: Kp - q $\neq 0$,

In such a circumstance, the appropriate answer is,

$$\mathrm{u}(\xi) = \mathrm{c} \qquad (5)$$

where c denotes constant. Without satisfying the initial condition, it is a simple solution, Eq. (2).

Case 2: Kp – q = 0,

Assuming the initial condition, it choose

$$\mathrm{u}(\xi) = 2\,(\frac{\Gamma(1+\beta)}{\mathrm{p}})^{\frac{1}{\beta}}\xi^{\frac{1}{\beta}} \qquad (6)$$

the first condition is met in this case.

When Eq. (3) is substituted into Eq. (6), an exact solution that meets the starting condition is obtained:

$$\mathrm{u}(\mathrm{x},\mathrm{t}) = 2\left(\frac{\Gamma(1+\beta)}{\mathrm{p}}\right)^{\frac{1}{\beta}}\left(\frac{\mathrm{p}\mathrm{x}^{\beta}}{\Gamma(1+\beta)} + \frac{\mathrm{q}\mathrm{t}^{\alpha}}{\Gamma(1+\alpha)}\right)^{\frac{1}{\beta}} \quad (7)$$

1.2 Some special functions of Fractional Calculus

Some fundamental ordinary DEs are solved, partial DEs are solved using the separation of variables approach, and special functions are produced. Special functions were employed in more applications because of the rising number of special functions available and the consistency of the procedures leading to them [14].

1.2.1 The Mittag-Leffler Function

It was initiated by Mittag-Laffer in 1903 [15] and defined as:

$$E_a(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+1)}, \quad (\alpha \epsilon C, \mathrm{Re}(\alpha) > 0) \quad (8)$$

Wiman gave a generalization of the Mittag-Leffler Function in 1905 [16] and defined it as:

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+\beta)}, \quad (\alpha, \beta \in C, \mathrm{Re}(\alpha) > 0, Re(\beta) > 0) \ (9)$$

Generalization of Eq. (9) and defined it as:

$$E_{\alpha,\beta}^{\gamma}(x) = \sum_{k=0}^{\infty} \frac{(\gamma)_k x^k}{\Gamma(\alpha k+\beta)k!}, \quad (\alpha, \beta, \gamma \in C, \mathrm{Re}(\alpha) > 0), \mathrm{Re}(\beta) > 0 \ (10)$$

1.2.2 Caputo Fractional derivative

The Caputo FD of order $\alpha > 0$ is given by Caputo in the form. If$m - 1 < \alpha \leq m, Re(\alpha) > 0, m \in N$): ${}_a^cD_t^{\alpha} f(t) = I^{m-\alpha} D^m f(t)$

$${}_a^cD_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^m(\tau) d\tau, t > 0$$

$$= \frac{d^m f(t)}{dt^m}, \text{if } \alpha = m \qquad (11)$$

Where, $\frac{d^m f(t)}{dt^m}$ is m-th derivative of order m of the function f(t) concerning t.According to this definition, for a constant Caputo's FD of a constant is zero.

$${}_a^cD_t^{\alpha} = 0 \quad f(t) = a = \text{constant} \qquad (12)$$

2. Preliminaries

There are many definitions, and theorems that would help for getting the specific outcome like Laplace transform (LT), Riemann Liouville (RL) definition of integral, Caputo's FD, Cauchy's theorem etc.

The following provides a condensed overview of several of the conventional definitions for, as well as properties associated with, fractional differ-integrals. A well-known fractional integral definition is the RL integral, which is written as

$$aD_x^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_a^x (x-u)^{-\alpha-1} f(u)du, \text{ for } \alpha < 0 \text{ } and \text{ } x > a. \text{ (13)}$$

Where aD_x^αrepresents the $(\text{fractional})_\alpha$ th order integration of the f(x)function, $\Gamma(.)$ is known as the Gamma function with a lower limit of integration. This idea is a broader use of the method that Cauchy described for repeated integration.The previousexplanation changed into the equation that is given belowfor $\alpha = -n$:

$$aD_x^\alpha f(x) = \frac{1}{(n-1)!} \int_a^x (x-u)^{n-1} f(u)du, \text{ for } a < x. \text{ (14)}$$

Following the repeated-integration formula developed by Cauchy, the earlier equation can be represented as,

$$aD_x^\alpha f(x) = \frac{1}{(n-1)!} \int_a^x (x-u)^{n-1} f(u)du = \int_a^x dx_{n-1} \int_a^{x_{n-1}} dx_{n-2} \dots \int_a^{x_1} f(x_0)dx_0. \text{ (15)}$$

It is still possible to use the RL fractional integration definition for fractional derivatives with a>0 if the following additional step is followed:

$$aD_x^\alpha f(x) = \frac{d^m}{dx^m{}_a} D_x^{\alpha-m} f(x), \text{for } 0 < a \text{(16)}$$

where m is selected therefore$0 > (\alpha - m)$, and thus the RL integration is a useful tool that can be used for $aD_x^{\alpha-m} f(x)$. Then, $\frac{d^m}{dx^m{}_a}$ is the differential operator of ordinary mth order. A different way to compute FDs is available. Caputo's FD can be found here. M. Caputo first presented it in 1967. When using Caputo's formulation to solve DEs, unlike when using the RLFD technique, it is not important to explain the initial conditions in fractions order. This contrasts with the RL method. Following is an example of Caputo's definition of action [16].

$$aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^n(u)}{(x-a)^{\alpha+1-n}} f(u)du \text{(17)}$$

2.1 Background Study

FC has been elevated to the rank of a new mathematical tool for the infusion of many difficulties in engineering and science fields because of its capacity to deal with issues in domains such as signal processing, physics, fluid dynamics, control engineering, and other fields. This has enabled it to play a great role in the advancement of these fields. The mathematical machinery of fractional integrals and FDs, as well as the mathematical machinery itself, is being studied for engineering applications and physical implications. A brief review of some of the most often used definitions is followed by a discussion of the properties of fractional differ integrals and a discussion of the various practical uses for FC in engineering. Application areas such as fractional calculus in an electronic system, electronic circuit analysis, antenna radiation engineering, and control engineering design are discussed [17].

3. Problem Formulation

FDEs are becoming more popular on a global scale. A component of calculus, FDEs are now widely employed at a high degree of sophistication. In the Research work, FDEs is shown that is specially used in engineering. Before the development of FC, many problems were faced by engineers like in the development of machinery, computational work, etc. After this, efforts were undertaken in FC to investigate the physical consequences of the results as well as their prospective applications in mathematical machinery. Several strategies may be used for this, including the Caputo FD method, the LT, the Fractional Adam-Bashforth-Moulton method, and others. In the research, the main focus is on the PID controller and its step point, disturbance response and its oscillations.

4. Research Methodology

FDE is used in many places, engineering is one of them. In this section, some, applications of FDEs in engineering are FC in Radiation Engineering, FC in electronic system Designing, FC in Control Engineering, FC in Modelling of Speech signals, and FC in the tuning of Proportional Integral Derivative(PID) controller discussed.

5Application of Fractional Calculus in Engineering

FC is used broadly in the engineering domain. But FDEs plays a vital role in the development of engineering fields (electronic circuit analysis, control engineering, electronic system designing etc.). Here some of the applications of FC are discussed briefly.

- **Analysis of Electronic circuit and Fractional calculus in Engineering**

Wireless communication relies on electromagnetic, and both are concerned with integer-calculus orders. When it comes to electrostatic fractional image methods for conducting spheres, the invention of a novel fractional multiple notion and a mathematical relationship between dielectric sphere image methods are among the most important achievements of this research.

Canonical solutions for the scalar Helmholtz equation are determined as cylindrical, plane, and spherical waves, respectively, for one-dimensional (D), 2D, and 3D. Dirac delta functions in one, two, and three dimensions are the appropriate sources. There was an intermediate wave predicted by the author to occur between two classic instances. To put it another way, don't think of the transition as discrete. This sort of answer is not predicted by standard calculus, which is predicated on positive integers. It has been demonstrated that FD and integration, two mathematical methods explored in the topic of FC, may be used to locate intermediate sources. Electromagnetic and wireless researchers may profit from waves that obey the standard scalar Helmholtz equation. For determining the intermediate values, a fractional parameter denoted by the letter v is used. This parameter takes on fractional values that lie between unity and zero. The consequences are such that, when v = 0, the cylindrical wave propagation case is represented, and when v = 1, the plane wave propagation case is represented.

The factor was a term used by the author.It is utilized to generate a big circular loop antenna, which is then emulated, by using a basic RLC type input tuning network that incorporates fractional-order capacitive and inductive components as well as a resistor. Instead of integer components, fractional-order components increase the antenna's radiation pattern[18].

In this instance, a capacitor and inductor are used as part of an RLC circuit, which is linked in series with a voltage source and a resistor. Theresistor R, capacitance C, and inductance L are assumed to be positive constants and $\psi(t)$ is the ramp function earlier [19], consider the $\psi(t)$ is the Heaviside function

The constitutive equations for an RLC electrical circuit consisting of three parts are as follows:

The voltage dropacross an inductor

$$U_L(t) = L\frac{d}{dx}I(T), \qquad (18)$$

The voltage dropacross a resistor

$$U_R(t) = RI(t),(19)$$

The voltage dropacross a capacitor

$$U_c(t) = \frac{1}{C}\int_0^t I(\xi)d\,\xi, \qquad (20)$$

where I(t) is the current.

This equation can be written utilizing Kirchhoff's voltage law and the constitutive equations linked to the three sections of non-homogeneous second-order ordinary DEs as

$$RC\frac{d^2}{dt^2}U_c(t) + \frac{d}{dt}U_c(t) + \frac{R}{L}U_c(t) = \frac{d}{dt}\psi(t) \qquad (21)$$

where $U_c(t)$is the voltage on the capacitor, Figure 1 shows a parallel connection between the inductor and capacitor.

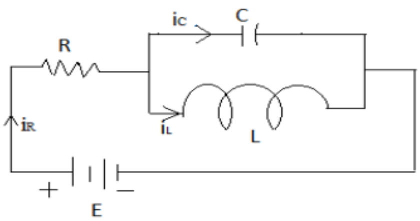

Figure 1: Three elements electrical LCR circuit

The inductor's current can be found in other non-homogeneous second-order ordinary DEs as

$$RLC\frac{d^2}{dt^2}i_L(t) + L\frac{d}{dt}i_L(t) + Ri_L(t) = \psi(t) \qquad (22)$$

These two non-homogeneous ordinary DEs of the second order can be brought to comparable integrodifferential equations by utilising the constitutive equation for the inductor once more,

$$R\frac{d}{dt}i_c(t) + \frac{1}{C}i_c(t) + \frac{R}{LC}\int_o^t i_c(\xi)d\xi\,, = \frac{d}{dt}\psi(t) \qquad (23)$$

And

$$RC\frac{d}{dt}U_L(t) + U_L(t) + \frac{R}{L}\int_0^t U_L\,(\xi)d\xi, = \psi(t) \qquad (24)$$

respectively. It has been brought to our attention that integrodifferential equations consist of different forms. Only the first of these will be considered here. The LT is the conventional approach that is utilised while talking about this integrodifferential equation. To achieve this goal, it is considered the initial condition iC (0) = 0, and the answer can be expressed as an exponential function [20].

- **Fractional Calculus in Control Engineering**

It is customary in FC to utilise an operator that combines differentiation and integration, known as the different integral operator (${}_aD_t^q$). In a single equation, it is a representation for calculating the FD and the fractional integral and expressed as [21]

$$ {}_aD_t^q = \begin{cases} \frac{d^q}{dt^q} & q > 0 \\ 1 & q = 0 \\ \int_a^t (d\tau)^{-q} & q < 0 \end{cases} \quad (25)$$

Here, q indicates the fractional-order and could be a complex number, with the operation's limits denoted by a and t. There are a few different ways that fractional derivatives can be defined. The Grunwald–Letnikov, RL, and Caputo definitions are the ones that are most frequently put into practice (Podlubny, 1999b). The Grunwald–Letnikov definition can be derived from the following:

$$ {}_aD_t^q f(t) = \frac{d^q f(t)}{d(t-a)^q} = \lim_{N\to\infty} \left[\frac{t-a}{N}\right]^{-q} \sum_{j=0}^{N-1} (-1)^j \binom{q}{j} f(t - j\left[\frac{t-a}{N}\right]) \quad (26)$$

One can easily apply the RL definition because it is the most basic and straightforward. It is given by,

$$ {}_aD_t^q f(t) = \frac{d^q f(t)}{d(t-a)^q} = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^1 (t-\tau)^{n-q-1} f(\tau) d\tau \quad (27)$$

where the first integer is n which is more than q i.e., $n - 1 \leq q < n$ and Γ is the Gamma function.

$$ \Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt \quad (28)$$

For functions f(t) having n continuous derivatives for 0< t where $n - 1 \leq q < n$, the Grunwald – Lernikov and the RL definitions are equivalent. Following are the LT of the RL fractional derivative and integral:

$$ L\{{}_0D_t^q f(t)\} = s^q F(s) - \sum_{k=0}^{n-1} s^k\, {}_0D_t^{q-k-1} f(0) n - 1 < q \leq n \epsilon N \quad (29)$$

LT appears unsuitable for the RL fractional derivative because it demands information on the non-integer-order functions at t=0. The Caputo definition, which is commonly referred to as a smooth FD in literature, does not have this issue. A derivative is defined by this definition

$$ {}_aD_t^q f(t) = \begin{cases} \frac{1}{\Gamma(m-q)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{q+1-m}} d\tau & m - 1 < q < m \\ \frac{d^m}{dt^m} f(t) & q = m \end{cases} \quad (30)$$

the first integer is m which is greater than q. Due to homogeneous beginning conditions, the RL operators and Caputo operators are equivalent. Using the LT, the Caputo FD can be calculated as

$$L\{{}_0D_t^q f(t)\} = s^q F(s) - \sum_{k=0}^{n-1} s^{q-k-1} f^{(k)}(0) \qquad n-1 < q \leq n\epsilon N \text{ (31)}$$

Unlike the LT of the RLFD, the LT of the Caputo FD only includes integer-order derivatives of function f Equation (32) is reduced to

$$L\{{}_0D_t^q f(t)\} = s^q F(s) \text{(32)}$$

In the next part, the notation of D^q, indicates the Caputo FD [21].

An important goal for researchers is to create new and effective time-domain analytic methods for fractional-order dynamic systems to address control theory issues [22]. PID controllers are a modern extension of the conventional PID controller $PI^{\lambda}D^{\mu}$ controller evolved by researchers. The idea of $PI^{\lambda}D^{\mu}$- controlleris a more effective way of controlling fractional-order dynamical systems which are involving a fractional-order integrator and fractional-order differentiator[23].

PID controllers have utilized widely control algorithms in the industrial environment. The Ziegler-Nicholas(Z-N) technique is one of the numerous PID controller methods that exist, and it is widely used for calculating the PID controller settings. The Z-N principles only apply to plants that react monotonically[24].For the fractional order control system shown in Figure 2, L(s) is the open-loop transfer function.

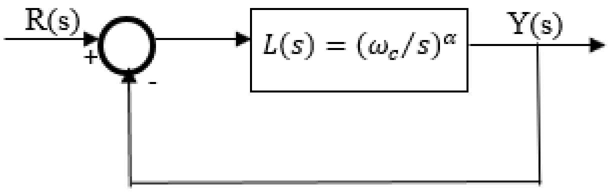

Figure 2: Open-loop transfer function L(s) of fractional-order control system

The fractional-order control system that will be utilised to fine-tune the PID controller is shown in Figure 2. The following is the expression for the open-loop transfer function L(s):

$$L(s) = (\frac{\omega_c}{s})^{\alpha} \quad (33)$$

Where, gain cross over frequency is ω_c, i.e., $|L(j\omega_c)| = 1$, α is the magnitude curve's slope, on a log scale, and may presume integer as well as non-integer values.

The construction of a PID controller would involve determining the PID set gains (T_i, K, T_d) that are optimal for the given system that minimizes J, the integral of the square error (ISE) which has defined as:

$$J = \int_0^{\infty} [y(t) - y_d(t)]^2 dt \qquad (34)$$

Where, $y_d(t)$ are the desired step response of the fractional-order transfer function and the closed − loop system with the PID controller in Figure 3 has a step response of $y(t)$.

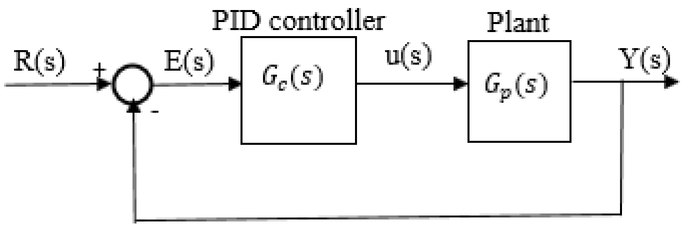

Figure 3: PID controller-based closed-loop control system

A system framework which can accept mathematical peculiarities provided by the different integral operators must be designed to examine the importance of the different integral in systems theory. This is done using the different integral operator, which stands for differentiable integral. This could be achieved by using a typical linear systems model and loosening some of the restrictions on integer order that are often imposed. A consequence of this research would be utilized to investigate the impact of different integrals within the context of traditional control theory.

Recall the integrator's or the derivative operator's LT:

$$L\left\{\frac{d^q f}{dx^q}\right\} = s^q L\{f\} - \sum_{k=0}^{q-1} s^k \frac{d^{q-1-k} f}{dx^{q-1-k}}(0) \qquad (35)$$

$$q = 0, \pm 1, \pm 2 \ldots$$

$$L\left\{\frac{d^q f}{dx^q}\right\} = s^q L\{f\} - \sum_{k=0}^{n-1} s^k \frac{d^{q-1-k} f}{dx^{q-1-k}}(0)(36)$$

This is a very helpful conclusion with fascinating mathematical implications for engineers working with control systems.

Eq. (36) demonstrates how to differ integrals are related to S- domain and time domaindifferintegral. Although this discovery is pleasant on the surface, it decreases the amount of time required to do the research dramatically. It is possible to carry out complex gamma function administration by transforming (Laplace) the differintegral operator into the S-domain, $\Gamma(x)$ can be lessened to simple algebraic handling of the s^q operator. Many well-known control system ideas and processes are only expressed in the s-domain, therefore concerns about this domain are mostly the focus of the next inspection.

Setting up a model that fits the specific needs of your s-domain Look at the system below in Figure 4, which shows a basic control with unity gain feedback of S – the domain model:

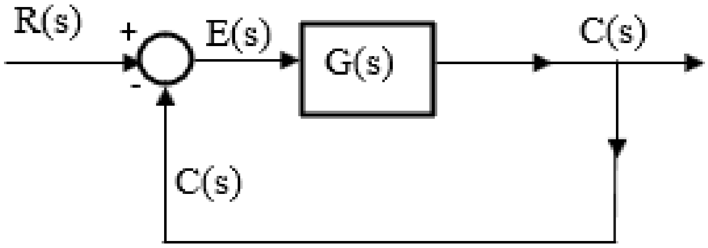

Figure 4: S - Domain model

The following is the definition of the open-loop transfer function for this system:

$$G(S)H(S) = \frac{K \prod_{h=1}^{w}(S - Z_h)}{S^m \prod_{c=1}^{4}(S - p_c)} \tag{37}$$

The kind of system, which has a distinct effect on the dynamics of the system, is determined using the m value in Eq. (37). Functionally, s^m represents m cascaded time-domain integrators in an s-domain presentation. It is possible to rewrite Equation (37) as by letting m take the non-integer value q:

$$G(S)H(S) = \frac{K \prod_{h=1}^{w}(S - Z_h)}{S^q \prod_{c=1}^{4}(S - p_c)} \tag{38}$$

The obtained Eq. (38) system is referred to as a Fractional type of system. To classify a system's transfer functions according to some specific performance evaluation, a standard categorization is used (e.g., steady-state error). It is possible to detect intermediate types that bridge what are thought of as distinct classes if a fractional type is first established in the proper context, and then the intermediate kinds are searched for using the appropriate keywords. As a result of the differ integral's fractional order, and because of the capacity to continuously alter q in Eq. (38).

The following is the optimal PID controller time-domain equation:

$$u(t) = K\left(e(t) + \frac{1}{T_i}\int_0^1 e(\tau)d\tau + T_d \frac{d}{dt} e(t)\right) \tag{39}$$

A PID controller's integral time constant, derivative time constant, and proportional gain are all correlatively represented by $T_i, T_d,$ and K. The error signal (e(t)) and control signal (u(t)) are also represented in equation 39. It's important to note that the $\mathrm{K}, \mathrm{T_d}, \mathrm{T_i}$variables are the parameters that would be adjusted, in above equation.

$$G_c(s) = \frac{U(s)}{E(s)} = K\left(1 + \frac{1}{T_i s} + T_d s\right) \qquad (40)$$

The derived term ${}_s\mathrm{T_d}$ is typically employed by the band-limited differentiator ${}_s\mathrm{T_d}/(1 + s\mathrm{T_d}/n)$, where $3 \leq N \leq 20$ This is done to minimize any measurement of high frequency noise or control effort.It is reasonable to suppose that the filter constant N is one of the parameters to be determined. To recognize a realistic PID controller, the transfer function must be right:

$$G_c(s) = \frac{U(s)}{E(s)} = K\left(1 + \frac{1}{T_i s} + \frac{sT_d}{1+sT_d/N}\right) \qquad (41)$$

$$G_{pc}(s) = \frac{1.2e^{-10s}}{(5s+1)(2.5s+1)} \qquad (42)$$

Systematic testing and evaluation of various parameters combinations, for all implementations of controller, was used to tune the various controllers.In order to achieve a settlementamong the instantaneous minimization of E_{av} and $\varepsilon_{xy}H$, the$G_{c1}(s)$parameters. Furthermore, it is expected that the joint actuators will be high-performance ones with a maximum actuator torque of $\tau_{ijMax} = 400\ Nm$, and that the required angle between the foot and the ground (which is supposed to be horizontal) will be set at$\theta_{i3hd} = -15°$.

Let's assume that joint 3 of the leg is mechanically activated, whereas joints 1 and 2 are motorized. It is necessary to fine-tune the PD^{α} joint controllers for fractions of the order α_j, in this example $\alpha_j = \{-0.9, -0.8, \ldots, +1.4, +1.5\}$ for various values of the fractional order $\Delta\alpha_j = 0.1$.

- **Fractional Calculus in Designing of Electronic System**

A fractional-order system can be reacted to in three different ways by employing analogical circuits with fractional order behaviour [25].

i. Component by component implementation: A recursive circuit is used to approximate the transfer function. Using the Laplace Transform, the gain between V_0 and V_1 is continued fraction approximation to the original system.

$$\frac{V_0}{V_1} = 1 + \frac{w_n}{s + \frac{w_{n-1}}{1 + \frac{w_{n-2}}{s + \frac{w_{n-3}}{\cdots}}}} \qquad (43)$$

Where,

$$w_{n-2j} = \frac{1}{R_j C_j} \text{ and } w_{n-2j+1} = \frac{1}{R_{j+1}+C_j} \quad (44)$$

The approach given has two major drawbacks: the first one is that it has a restricted frequency range of work, and the second is that it approximates the true frequency range. Therefore, based on the quantity of accuracy required by the designer, a huge number of low tolerance components are required.

ii. Field Programmable Analog Array (FPAA): The circuit is built constituent by component into an FPAA, allowing the fractional-order system's dynamical behaviour to be adjusted with a few simple adjustments. Each component has a certain liberality.

iii. Fractional Order Impedance Component: It is a modified capacitor that demonstrates fractional-order behaviour. For example, one of the plates in a fractal dimensioned sequence of parallel plates has a fractal dimension. Each branch of a low-pass capacitor/resistor (CR) circuit filter may be described as linked to the main branch.

- **Fractional Calculus in Modelling of Speech**

The Linear Predictive Coding (LPC) technique is utilized for speech processing based on integer-order models. Speech signals may be properly modelled via simulation by employing a small number of integrals with fractional orders as basic functions [26].

i. Fractional order Speech Modelling: The demonstration of the effectiveness of newly established fractional-order modelling approaches in a variety of applications, it is recommended that similar techniques be used to model speech signals as well. The more typical LP-based strategy, which depends on integer-order DEs approximated by difference equations for accuracy and compactness, may not be the best choice when modelling speech signals because FDEs may be a superior option.

The FD is defined using an example of the RL definition, commonly used definitions, of order α of $x(t)$function as given below:

$$\frac{d^\alpha x(t)}{dt^\alpha} = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_{0(t-\tau)}^{t} \frac{x(\tau)}{\alpha-m+1} d\tau \quad (45)$$

Where$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$ is the Euler's gamma functionand'm' is an integer such that $m < \alpha \leq m+1$.

Fractional integrals and derivatives are often simulated using numerical methods depending on the RL definition. Numerical simulations employing the backward difference approach were done depending on the 'Grünwald-Letnikov' estimate of the FD:

$$D^{\alpha}x = \frac{d^{\alpha}x}{dt^{\alpha}} = h^{-\alpha}\sum_{j=0}^{n}(-1)^{j}C_j^{\alpha}x(n-j) \qquad (46)$$

Where, h = step size of integration and $C_j^{\alpha} = \binom{\alpha}{j} = \alpha(\alpha-1)\ldots{}^{\alpha-j+1}/_{j!}$.

The discrete form can be used to describe a voice signal in the context of a linear combination of its FDs, as seen below:

$$\hat{x}(n) = \sum_{k=1}^{Q}\mu_k D^{\alpha}x \qquad (47)$$

It is important to emphasize that a fractional integral of order αcorresponded by negative value of α, denoted as $I^{\alpha}x$. As a result, to improve numerical stability and noise immunity, it is feasible to rewrite equation (29) as follows:

$$\hat{x}(n) = \sum_{k=1}^{Q}\gamma_k I^{\beta_k}x = \sum_{k=1}^{Q}\gamma_k\,\varphi_k(n) \qquad (48)$$

Where $\{\gamma_k\}$ are the fractional model's sought prediction parameters, hereafter known as Fractional Linear Prediction (FLP) coefficients.

In vector-matrix representation the arrangement equivalent to the fractional integral $\varphi_k(n)$ is re as N× 1 column vectors ρ_k. As a result, the prediction coefficients, $\{\gamma_k\}$ presented by Q× 1vector, by a least-squares solution 'g' can be determined such as:

$$g = (\wedge^{T}\wedge)^{-1}\wedge^{T}x \qquad (49)$$

Where, $\wedge = \rho_1, \rho_2, \ldots\ldots \rho_Q$.

6. Results

According to the findings, it gives relatively high overshoots, which is something that the system probably doesn't want to happenwhen the Z-N setting is configured. An overshoot that falls within acceptable limits and a settling time that is greatly decreased are hallmarks of both the proposed methodology and the ISTE method. This is true for both inputs of setpoint as well as a disturbance. However, it is vital to note that this technique results in lower fluctuations of the setpoint and disturbance inputs.

In Table 2, there are different values given for PID control parameters (T_i, K, T_d) and transient time specification $(T_s, \%OS, T_r)$ for different methods like ZN, ISTE, and Reference model tuning. It is noticeable that there is a continuous reduction in their values that will lessen the oscillations.

Table 2. PID control parameters (T_i, T_d, K) and transient time specification $(\%OS, T_s T_r)$ for $G_{pc}(s)$

Method	K	T_i	T_d	T_r	T_s	%OS
ZN	0.772	15.110	4.076	16.52	91.31	10.59
ISTE	0.650	10.280	4.210	17.71	55.21	8.01
Reference model tuning	0.6113	9.4546	4.3275	17.94	55.46	6.35

Figure 5 shows the disturbance and step setpoint response with the PID controller of the closed-loop system tuned according to Z-N heuristics, optimal ISTE, and the suggested techniquefor $G_{pc}(s)$ is shown. In which the proposed model i.e., The reference model tuning found the smallest integrated square error (ISE) between thedesiredstep response and the step response with the PID controller that is provided by a fractional-order transfer. In the figure, α shows the ISE values of Z-N, ISTE and proposed method.

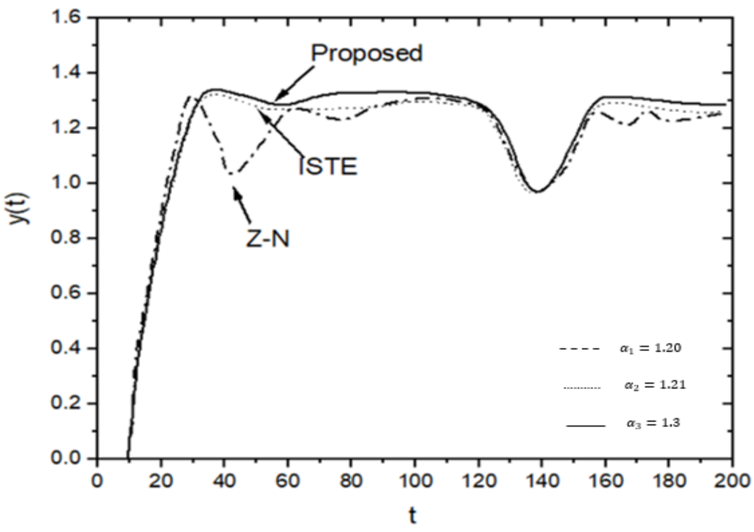

Figure 5: PID controller with optimum ISTE, Z-N heuristics, and Reference model tuning approach for stepping setpoint and disturbance responses for $Gpc(s)$.

In Figure 6, disturbance responses and closed-loop system step setpoint with PID controller tuned are shown as per the optimum ISTE, Z-N heuristics, and the reference model tuning (proposed method). The proposed one showed the maximum deduction in oscillations of PID control parameters, that's why it is better than others. The ISE of PID controller can be identified using the value of α.

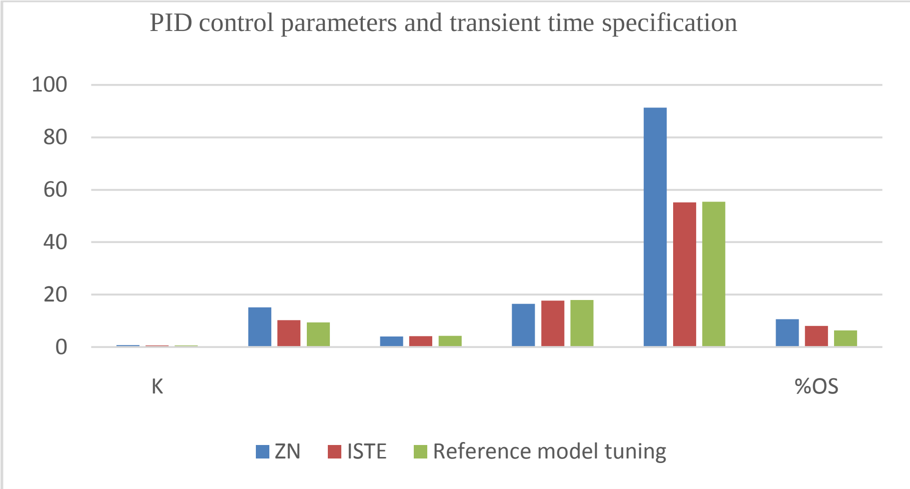

Figure 6: Closed loop system's Disturbance responses and step setpoint.

7. Conclusion and Future Scope

A new way of tuning the PID controller is developed and named areference model for tuning or reference model tuning. It is based on minimizing the Integrated Square Error (ISE) among the desired step response of the system and the step response with the PID controller that is produced by a fractional-order transfer. This is done to achieve the best possible results. The reference model consists of a closed-loop system that is perfect in every way. Consequently, there needs to be a compromise between the different tuning parameters (γ, ω_c). Now, the values that were shown by the Reference model tuning showed the maximum deduction. The value for different PID control parameters and Transient time specifications of the model are given as K=0.6113, T_i= 9.4546, T_q= 4.3275, T_r= 17.94, T_s= 55.46, and %OS= 6.35. These values are lessened concerning other methods (Z-N and ISTE). The value of %OS for Z-N is 10.59 and for ISTE is 8.01 i.e., greater than the applied one which gives only 6.35. Now, it is seen that it showed the maximum deduction and gives reasonable overshoots and shorter settling time and gives a less oscillatory system.The application of idea of fractional calculus to robotics in the future provided exciting possibilities for the progression of future advances. The procedure that was utilized can be used for multiple plants, all of which produce positive outcomes and demonstrate the strategy's applicability.

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