Reliability Estimation Of N-Component Parallel System by Bayesian Approach Using Inverse Rayleigh Distribution

K. Sandhya Rani [1], M. Tirumala Devi [2] and T. Sumathi Uma Maheswari [3]

[1, 2, 3] Department of Mathematics, Kakatiya University, Warangal, Telangana State, India.

Email: sandhya.bommera@gmail.com

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Abstract

In this paper reliability estimation subjected to common stress using inverse Rayleigh distribution if $X_1, X_2,\ldots,X_n$ are strengths of n- components under one stress is derived. The estimation is performed by using Bayesian estimation under Jeffery's prior and exponential prior distribution. Numerical calculations are done by matlab.

Keywords: Inverse Rayleigh distribution (IRD), Maximum likelihood estimator, Bayesian estimation.

1. INTRODUCTION:

IRD was introduced by Trayer (1964) in reliability and survival analysis. A system with n-components is known as an n-unit parallel system if and only if the successful functioning of any one of the components leads to the system success. Gaver, D.P., [1] discussed time to failure and availability of paralleled systems with repair. Voda R. GH [2] studied on Inverse Rayleigh variable. Hirose, H., [3] obtained estimation of threshold stress in accelerated life testing. Sanku Dey [4] studied on bayesian estimators of the parameter and reliability function of an inverse Rayleigh distribution. Abdel-Monem (2005) obtained bayesian estimators of the parameter of the inverse Rayleigh distribution under four loss functions. In (2013) Tabassum and others studied the Bayes estimation of the parameters of the Inverse Rayleigh distribution for left censored data under Symmetric and asymmetric loss functions. In (2015) Guobing discussed Bayes estimation for Inverse Rayleigh model under different loss functions represented by squared error loss, linex loss and entropy loss functions. Karam and Ramak H.G. [5] described estimation of the reliability for a component exposed to two or three independent stresses based on weibull and inverse Lindley distribution.

2. Statistical model:

Let X be the strength and Y be the stress ,f(x) and g(y) is the probability density functions ,F(x) and G(y) is the cumulative distribution functions of X and Y random variables of inverse Rayleigh distribution respectively.

$$f(x,\delta)=\frac{2\delta}{x^3}e^{\frac{-\delta}{x^2}};x>0,\delta>0.$$

$$F(x,\delta)=e^{\frac{-\delta}{x^2}};x>0,\delta>0.$$

$$g(y,\beta)=\frac{2\beta}{y^3}e^{\frac{-\beta}{y^2}};y>0,\beta>0.$$

Two components system reliability under common stress:

If (X_1,X_2) are strength of two components of random stress Y.

Reliability $R_{12}=p[Y<\max(X_1,X_2)]$.

$$R_{12}=\int_0^\infty \overline{H}_Z(y)f_y(y)dy$$

Where $Z=Max(x_1,x_2)$

$$H_Z=P(x_1<z).P(x_2<z)$$

$$=F_{x_1}(z)F_{x_2}(z)$$

$$H_Z=\left(e^{\frac{-\delta_1}{z^2}}\right)\left(e^{\frac{-\delta_2}{z^2}}\right)$$

$$\overline{H}_Z=1-e^{-\frac{(\delta_1+\delta_2)}{z^2}}$$

$$R_{12}=\int_0^\infty\left[1-e^{\frac{-(\delta_1+\delta_2)}{y^2}}\right]\frac{2\beta}{y^3}e^{\frac{-\beta}{y^2}}dy$$

$$R_{13}=\frac{\delta_1+\delta_2}{\delta_1+\delta_2+\beta}$$

Three components system reliability under common stress:

If (X_1,X_2,X_3) are strength of three components of random stress Y.

Reliability $R_{13}=p[Y<\max(X_1,X_2,X_3)]$.

$$H_Z=F_{x_1}(z)F_{x_2}(z)F_{x_3}(z)$$

$$H_Z=\left(e^{\frac{-\delta_1}{z^2}}\right)\left(e^{\frac{-\delta_2}{z^2}}\right)\left(e^{\frac{-\delta_3}{z^3}}\right)$$

$$\overline{H}_Z=1-e^{-\frac{(\delta_1+\delta_2+\delta_3)}{z^2}}$$

$$R_{13} = \frac{\delta_1 + \delta_2 + \delta_3}{\delta_1 + \delta_2 + \delta_3 + \beta}$$

$$R_{13} = \frac{\sum_{i=1}^{3} \delta_i}{\sum_{i=1}^{3} \delta_i + \beta}$$

n- components system reliability under common stress:

If $(X_1, X_2, X_{3,\dots,} X_n)$ are strength of n-components of random stress Y.

Reliability $R_{1n} = p[Y < \max(X_1, X_2, X_3, ..., X_n)]$.

$$R_{1n} = \frac{\sum_{i=1}^{n} \delta_i}{\sum_{i=1}^{n} \delta_i + \beta}$$

3. Maximum likelihood estimation:

$$L(x_1, x_2, ..., x_n, \delta_i) = 2^n \delta_i^{\ n} \prod_{i=1}^{n} \frac{1}{x_i^3} \exp\left[-\delta_i \sum_{i=1}^{n} \frac{1}{x_i^2} \right]$$

$$\frac{\partial LnL(x, \delta)}{\partial \delta_i} = \frac{n}{\delta_i} - \sum_{i=1}^{n} \frac{1}{x_i^2} = 0$$

$$\hat{\delta}_i = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i^2}} = \frac{n}{T}, where T = \sum_{i=1}^{n} \frac{1}{x_i^2}$$

Similarly

$$\hat{\beta} = \frac{n}{\sum_{i=1}^{n} \frac{1}{y_i^2}} = \frac{n}{T}, where T = \sum_{i=1}^{n} \frac{1}{y_i^2}$$

4. Bayes estimator under Jeffrey's distribution:

Assume that δ has non informative prior density, Jeffery's prior information g(δ)

$$g_1(\delta) \propto \sqrt{I(\delta)}$$

Where I(δ) represented Fisher information:

$$I(\delta) = -nE\left[\frac{\partial^2 Lnf(x,\delta)}{\partial \delta^2}\right]$$

Hence

$$g_1(\delta) = b\sqrt{-nE\left[\frac{\partial^2 Lnf(x,\delta)}{\partial \delta^2}\right]} \text{..........(1)}$$

$$\frac{\partial Lnf(x,\delta)}{\partial \delta} = \frac{1}{\delta} - \frac{1}{x^2} = 0$$

$$\frac{\partial^2 Lnf(x,\delta)}{\partial \delta^2} = \frac{1}{\delta^2}$$

Hence

$$E\left[\frac{\partial^2 Lnf(x,\delta)}{\partial \delta^2}\right] = \frac{-1}{\delta^2}$$

Substituting in (1) we get,

$$g_1(\delta) = b\sqrt{-n\left[\frac{-1}{\delta^2}\right]} = \frac{1}{\delta}\sqrt{n}, \delta > 0$$

$$h_1(\delta / x_1, x_2, ..., x_n) = \frac{g_1(\delta) L(\delta; x_1, x_2, ..., x_n)}{\int_0^\infty g_1(\delta) L(\delta; x_1, x_2, ..., x_n) d\delta}$$

$$= \frac{\frac{1}{\delta}\delta^n e^{-\delta T}}{\int_0^\infty \frac{1}{\delta}\delta^n e^{-\delta T} d\delta}$$

$$= \frac{\delta^{n-1} e^{-\delta T}}{\int_0^\infty \delta^{n-1} e^{-\delta T} d\delta}$$

Hence

$$h_1(\delta / x_1, x_2, ..., x_n) = \frac{T^n \delta^{n-1} e^{-\delta T}}{\overline{)n}}$$

$\delta \sim Gamma(n, T)$

$$E(\delta)=\frac{n}{T},Var(\delta)=\frac{n}{T^{2}}$$

$$E(\delta^{m})=\int_{\forall\delta}\delta^{m}h_{1}(\delta/X)d\delta$$

$$=\frac{T^{n}}{\Gamma n}\int_{\forall\delta}\delta^{(m+n)-1}e^{-\delta T}d\delta$$

$$E(\delta^{m}/x)=\frac{\Gamma n+m}{\Gamma nT^{m}}$$

Substituting under generalized square error loss function

$$\hat{\delta}=\frac{a_{0}E(\delta/x)+a_{1}E(\delta^{2}/x)+....+a_{k}E(\delta^{k+1}/x)}{a_{0}+a_{1}E(\delta/x)+......+a_{k}E(\delta^{k}/x)}$$

We get

$$\hat{\delta}_{J}=\frac{a_{0}\frac{\Gamma n+1}{\Gamma nT}+a_{1}\frac{\Gamma n+2}{\Gamma nT^{2}}+.......+a_{k}\frac{\Gamma n+k+1}{\Gamma nT^{k+1}}}{a_{0}+a_{1}\frac{\Gamma n+1}{\Gamma nT}+a_{2}\frac{\Gamma n+2}{\Gamma nT^{2}}+......+a_{k}\frac{\Gamma n+k}{\Gamma nT^{k}}}$$

$$\hat{\delta}_{Ji}=\frac{a_{0}\frac{n}{T}+a_{1}\frac{(n+1)n}{T^{2}}+.......+a_{k}\frac{(n+k)(n+k-1).....(n+1)n}{T^{k+1}}}{a_{0}+a_{1}\frac{n}{T}+......+a_{k}\frac{(n+k-1)(n+k-2).....(n+1)n}{T^{k}}},i=1,2,....,n$$

$$\hat{\delta}_{Ji}=\frac{\sum_{j=0}^{k}a_{j}\frac{\Gamma n+1+j}{T^{j+1}\Gamma n}}{\sum_{j=0}^{k}a_{j}\frac{\Gamma n+j}{T^{j}\Gamma n}},i=1,2,....,n.$$

$$where T=\sum_{i=1}^{n}\frac{1}{x_{i}^{2}}$$

Similarly we get $\hat{\beta}$

The first and second polynomials as follows:

$$\hat{\delta}_{Ji_1} = \frac{a_0 \frac{n}{T} + a_1 \frac{(n+1)n}{T^2}}{a_0 + a_1 \frac{n}{T}}, i = 1,2,...,n$$

$$\hat{\delta}_{Ji_2} = \frac{a_0 \frac{n}{T} + a_1 \frac{(n+1)n}{T^2} + a_2 \frac{(n+2)(n+1)n}{T^3}}{a_0 + a_1 \frac{n}{T} + a_2 \frac{n(n+1)}{T^2}}, i = 1,2,..,n$$

The estimation of system reliability

$$\hat{R}_{1n} = \frac{\sum_{i=1}^{n} \hat{\delta}_{J\,i}}{\sum_{i=1}^{n} \hat{\delta}_{J\,i} + \beta}$$

Bayes estimator under exponential prior distribution:

$$g_2(\delta) = \frac{1}{\lambda} e^{-\frac{\delta}{\lambda}}; \delta > 0, \lambda > 0$$

$$h_2(\delta / X) = \frac{L(x_1, x_2, ..., x_n) g_2(\delta)}{\int_0^\infty L(x_1, x_2, ..., x_n / \delta) g_2(\delta) d\delta} = \frac{\delta^n e^{-\delta\left(T + \frac{1}{\lambda}\right)}}{\int_0^\infty \delta^n e^{-\delta\left(T + \frac{1}{\lambda}\right)} d\delta}$$

$$= \frac{\left(T + \frac{1}{\lambda}\right)^{n+1} \delta^n e^{-\delta(T + \frac{1}{\lambda})}}{\Gamma(n+1)}$$

Notice that $\delta \sim (\Gamma(n+1), p)$ where $p = T + \frac{1}{\lambda} = \sum_{i=1}^{n} \frac{1}{x_i^2} + \frac{1}{\lambda}$

$$E(\delta^m) = \int_{\forall \delta} \delta^m h_2(\delta / X) d\delta = \frac{\Gamma(n+1+m)}{(\Gamma(n+1)) p^m}$$

Substituting

$$\hat{\delta}_{Ei} = \frac{a_0 \frac{\Gamma(n+2)}{\Gamma(n+1) p} + a_1 \frac{\Gamma(n+3)}{\Gamma(n+1) p^2} + ... + a_k \frac{\Gamma(n+k+2)}{\Gamma(n+1) p^{k+1}}}{a_0 + a_1 \frac{\Gamma(n+2)}{\Gamma(n+1) p} + ... + a_k \frac{\Gamma(n+k+1)}{\Gamma(n+1) p^k}}, i = 1,2,..,n$$

$$\hat{\delta}_{Ei} = \frac{\sum_{j=0}^{k} a_j \frac{\Gamma(n+2+j)}{p^{j+1}\Gamma(n+1)}}{\sum_{j=0}^{k} a_j \frac{\Gamma(n+1+j)}{p^{j}\Gamma(n+1)}}, i = 1,2,.....,n \text{ Where } p = T + \frac{1}{\lambda} = \sum_{i=1}^{n} \frac{1}{x_i^2} + \frac{1}{\lambda}$$

$$\hat{\delta}_{Ei} = \frac{a_0 \frac{n+1}{p} + a_1 \frac{(n+2)(n+1)}{p^2} + ... + a_k \frac{(n+k+1)(n+k).....(n+1)}{p^{k+1}}}{a_0 + a_1 \frac{n+1}{p} + a_2 \frac{(n+2)(n+1).....(n+1)n}{p^2} + ... + a_k \frac{(n+k)....(n+1)}{p^k}}, i = 1,2,...,n$$

Similarly we get $\hat{\beta}$

The first and second polynomials

$$\hat{\delta}_{Ei_1} = \frac{a_0 \frac{n+1}{p} + a_1 \frac{(n+2)(n+1)}{p^2}}{a_0 + a_1 \frac{n+1}{p}}, i = 1,2,..,n$$

$$\hat{\delta}_{Ei_2} = \frac{a_0 \frac{n+1}{p} + a_1 \frac{(n+2)(n+1)}{p^2} + a_2 \frac{(n+3)(n+2)(n+1)}{p^3}}{a_0 + a_1 \frac{n+1}{p} + a_2 \frac{(n+2)(n+1)}{p^2}}, i = 1,2,...,n$$

The estimation of system reliability

$$\hat{R}_{1n} = \frac{\sum_{i=1}^{n} \hat{\delta}_{Ei}}{\sum_{i=1}^{n} \hat{\delta}_{Ei} + \beta}$$

5. NUMERICAL CALCULATIONS:

Table 1: Expected values for a_0=5, a_1=4, a_2=2, n=5.

R	Jefferys prior for first polynomial			Jefferys prior for second polynomial		
	$\hat{\delta}$	$\hat{\beta}$	$\widehat{R_1}$	$\hat{\delta}$	$\hat{\beta}$	$\widehat{R_2}$
0.6760	3.6801	1.9329	0.6720	3.7589	1.9540	0.6755
0.7429	3.8147	1.9176	0.7365	3.8169	1.9244	0.7403
0.7623	3.8271	1.9081	0.7558	3.8381	1.9130	0.7586

0.7839	3.8534	1.8971	0.7686	3.9147	1.9009	0.7765
0.8436	3.9237	1.8394	0.8299	3.9280	1.8431	0.8336
0.8602	4.3315	1.7757	0.8469	3.9325	1.7834	0.8590
0.8784	4.3415	1.7543	0.8696	3.9574	1.7636	0.8725
0.9284	4.4812	1.6732	0.9226	3.9643	1.6934	0.9198
0.9430	4.5677	1.5162	0.9363	3.9848	1.5332	0.9387
0.9578	4.7643	1.4342	0.9458	3.9987	1.4485	0.9528

Table 2: Expected values for $a_0=5$, $a_1=4$, $a_2=2$, $n=5$, $\lambda=0.8$.

R	Exponential prior for first polynomial			Exponential prior for second polynomial		
	$\hat{\delta}$	$\hat{\beta}$	$\widehat{R_1}$	$\hat{\delta}$	$\hat{\beta}$	$\widehat{R_2}$
0.5467	2.4256	1.4029	0.5320	2.3576	1.3867	0.5131
0.6548	2.5271	1.3832	0.6267	2.4469	1.3643	0.6076
0.6721	2.7690	1.3668	0.6426	2.6798	1.3476	0.6232
0.6954	2.8798	1.3421	0.6744	2.7809	1.3212	0.6527
0.7134	2.9856	1.3276	0.7087	2.9575	1.3098	0.6921
0.7564	3.4532	1.3147	0.7217	3.2187	1.2865	0.7126
0.7760	3.5643	1.2954	0.7521	3.5464	1.2662	0.7432
0.8054	3.6987	1.2664	0.7678	3.6789	1.2473	0.7565
0.8432	3.7964	1.2265	0.8276	3.7243	1.2176	0.8125
0.8765	3.9976	1.1984	0.8543	3.9612	1.1785	0.8334

Table 3: Expected values for $a_0=5$, $a_1=4$, $a_2=2$, $n=5$, $\lambda=1.8$.

R	Exponential prior for first polynomial			Exponential prior for second polynomial		
	$\hat{\delta}$	$\hat{\beta}$	$\widehat{R_1}$	$\hat{\delta}$	$\hat{\beta}$	$\widehat{R_2}$
0.5238	2.1219	1.3845	0.5056	2.2431	1.3934	0.5161
0.5567	2.1549	1.3721	0.5323	2.2629	1.3823	0.5383
0.6721	2.3421	1.3543	0.6439	2.2839	1.3648	0.6535
0.7046	2.4932	1.3454	0.6847	2.5216	1.3534	0.6932
0.7256	2.6754	1.3174	0.7098	2.7439	1.3295	0.7128
0.7487	2.9854	1.2976	0.7230	3.1876	1.3184	0.7287
0.7639	3.4876	1.2765	0.7429	3.4321	1.2987	0.7532
0.7987	3.6578	1.2454	0.7765	3.5875	1.2638	0.7854
0.8176	3.7642	1.2176	0.7928	3.6538	1.2458	0.8012
0.8587	3.8723	1.1870	0.8342	3.7638	1.2065	0.8423

6. CONCLUSION:

In this paper reliability estimation subjected to common stress using inverse Rayleigh distribution is derived. The estimation is done by Bayesian estimation. It is observed that the Bayesian estimation under Jeffrey's prior information is the best estimator than with Exponential prior. The performance of Bayes estimator under Exponential prior when λ=0.8 with first polynomial is the best estimator compared to the second polynomial. The performance of Bayes estimator under Exponential prior when λ=1.8 with second polynomial is the best estimator compared to the first polynomial.

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