Study of Some Results on the Factor Group $K(C_n \times S_3)$

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Abstract
The main goal of this paper is to calculate the cyclic decomposition of the finite commutative factor group $(C_n \times S_3)$, where $n = q_1^{\eta 1} \cdot q_2^{\eta 2} \dots q_m^{\eta m}$, q_I are distinct primes for all $i = 1, 2, 3, \dots, m$ and $\eta_1, \eta_2, \cdots, \eta_m$ are positive integers then:

$$K(C_n \times S_3) = \bigoplus_{i=1}^{3} K(C_n) \bigoplus_{i=1}^{(\eta_1+1)(\eta_2+1)(\eta_3+1)\cdots(\eta_m+1)} C_6 .$$

We found the general table of irreducible characters for the group $(C_n \times S_3)$.

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Introduction:

The commutative G group of all $Z-$ valued characters of a finite G group constant of the $\Gamma -$ classes forms a finitly generated a commutative group $cf(G, Z)$of a rank equal to the number of Γ–classes . Intersection of $cf(G, Z)$ with the group of all generalized characters of G ,is a normal subgroup of $cf(G, Z)$ denoted by $\overline{R}(G)$,then $cf(G, Z)/\overline{R}(G)$ is a finite commutative factor group that is set to be $K(G)$.The matrix form $\overline{R}(G)$ consists of terms of the $cf(G, Z)$basis is $\equiv^* (G)$.We use the theory of invariant factors to obtain the direct sum of the cyclic $Z-$module of orders the distinct invariant factors of $\equiv^* (G)$ to find the cyclic decomposition of $K(G)$."M. S.Kirdar [11] studied the of $K(C_n)$ in 1982". "The factor group $cf(G, Z)/\overline{R}(G)$for the special linear group SL(2,P)", was studied by N.S.Jasim [13] in 2005. AL-Harere.M.N and AL-

Heety.F.A [1] "had studied the primary decomposition of the factor group $K(Z_p^n)$" in 2011. "The some combinatorial results on the factor group $K(G)$" , had been studied by M.N.Yaqoob and A.A.Ali [10] in 2016 . Finally ,we would like to form the reader of this paper that we have found the $\equiv^*(C_n \times S_3)$, in addition to that we calculated the cyclic decomposition of the group $K(C_n \times S_3)$.

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Definition(1.1): [3]

Suppose that the group $GL(\mathbf{n}, F)$ is a multiplicative group of all non–singular $\mathbf{n} \times \mathbf{n}$ matrices over the field F, the group $GL(\mathbf{n}, F)$ general linear group is called .

Definition(1.2): [4]

A homomorphism of G into $GL(n, F)$, be a matrix representation of a group G ,where n is known as a degree of matrix representation T. In particular case , T is a unit representation (principal) if $T(g) = 1$, for all $G \ni g$.

Example (1.3) :

Assume the symmetric group S_3, then we determine the matrix representation of the group.

$\beta_1: S_3 \to GL(1, \mathbb{C})$ for all $g \in S_3$ …..(trivial representation)

$\beta_2: S_3 \to GL(1, \mathbb{C}) \Rightarrow \rho_2(g) = \begin{cases} 1 & \text{if g is even} \\ -1 & \text{if g is odd} \end{cases}$

for all $g \in S\_3$ …..(alternating representation)

$\beta_3: S_3 \to GL(3, \mathbb{C})$ for all $g \in S_3$……(linear representation)

$$\beta_3((I)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \beta_3((12)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \beta_3((13)) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\beta_3((23)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \beta_3((123)) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \beta_3((132)) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Note that the actions are on column's represent reducible representation because there exist invertible matrix

$$T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \text{ such that}$$

$$T.\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}.T^{-1}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=(1)\oplus\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$T.\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}.T^{-1}=\begin{pmatrix}1&0&0\\0&-1&0\\0&-1&1\end{pmatrix}=(1)\oplus\begin{pmatrix}-1&0\\-1&1\end{pmatrix}$$

$$T.\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}.T^{-1}=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}=(1)\oplus\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$$T.\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}.T^{-1}=\begin{pmatrix}1&0&0\\0&1&-1\\0&0&-1\end{pmatrix}=(1)\oplus\begin{pmatrix}1&-1\\0&-1\end{pmatrix}$$

$$T.\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}.T^{-1}=\begin{pmatrix}1&0&0\\0&-1&1\\0&-1&0\end{pmatrix}=(1)\oplus\begin{pmatrix}-1&1\\-1&0\end{pmatrix}$$

$$T.\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}.T^{-1}=\begin{pmatrix}1&0&0\\0&0&-1\\0&1&-1\end{pmatrix}=(1)\oplus[\quad]\begin{pmatrix}0&-1\\1&-1\end{pmatrix}.$$

The following table includes the irreducible representation for each elements of S_3:

S_3	(1)(2)(3)	(123)	(132)	(12)(3)	(13)(1)	(23)(1)
ρ_1	[1]	[1]	[1]	[1]	[1]	[1]
ρ_1	[1]	[1]	[1]	[−1]	[−1]	[−1]
ρ_1	$\begin{bmatrix}1&0\\0&1\end{bmatrix}$	$\begin{bmatrix}-1&0\\-1&1\end{bmatrix}$	$\begin{bmatrix}0&-1\\1&-1\end{bmatrix}$	$\begin{bmatrix}-1&0\\-1&1\end{bmatrix}$	$\begin{bmatrix}0&1\\1&0\end{bmatrix}$	$\begin{bmatrix}1&-1\\0&-1\end{bmatrix}$

Table(1,1)

Definition (1.4): [4]

Let A is a matrix of the size $n \times n$ the sum of the main diagonal elements is said to be trace and denoted by $\mathrm{tr}(A)$.

Definition(1.5): [4]

Let G be a finite group over the field F, T be a matrix representation of degree n of the group G. The function $\partial : G \to F$ defined by $\partial(g) = \mathrm{tr}(T(g))$ for all $g \in G$, ∂ is a character of degree n of T . In particular, the character of the principal representation ($\partial(g) = 1$, for all $g \in G$) is called the principal character.

Definition(1.6): [7]

Γ –conjugate consists of two elements in group G ,if the cyclic subgroups of generate are conjugate in G, so we can define it as an equivalence relation on G. Its classes are called Γ–classes.

Definition(1.7): [9]

A irreducible characters of The G′s irreducible characters which is denoted by ϑ has integer values which is called character ,such that $\vartheta(g) \in Z$, $\forall g \in G$.

Proposition (1.8):[11]

The number of Γ–classes on G equals to the number of all distinct irreducible characters of a finite group G.

Theorem (1.9): [2]

Let S_n be a symmetric group so it has a k is a subgroup , and the function $\zeta: G \to \mathbb{C}$ defined by the set:

$$\zeta_{(g)} = \mathrm{fix}(g) = \{u: gu = u, \forall g \in S_n\}$$

Then $\partial_{\zeta(g)} = |\mathrm{fix}(g)| - 1$ is an irreducible character of k.

Example (1.10):

Consider $S_3 \leq S_n$ and the elements of S_3 are known from [theorem (1.9)] Then:

$\zeta((I)) = |\text{fix}(I')| - 1 = 3 - 1 = 2.$

$\zeta\big((12)(3)\big) = |\text{fix}((12)(3))| - 1 = 1 - 1 = 0$ the same for (13)(2) and (23)(1).

$\zeta((123)) = |\text{fix}((123))| - 1 = 0 - 1 = -1$ the same for (132).

Then ∂_ζ=(2,0,-1) is irreducible character of S_3.

$\langle \partial_\zeta . \partial_\zeta \rangle = \frac{1}{6}[(1)(1)(1) + (1)(1)(3) + (1)(1)(2)] = 1.$

Example (1.11):

From example (1.3) we can calculate the irreducible characters and characters table for symmetric group S_3,

$\partial'_{\beta_1} = (1,1,1,1,1,1), \partial'_{\beta_2} = (1,1,1,-1,-1,-1),$

$\partial'_{\beta_1} = (2,-1,-1,0,0,0)$ .We construct the characters table for S_3.

CL_α	$[L_1]$	$[L_2]$	$[L_3]$
$\lvert CL_\alpha \rvert$	1	2	3
$\lvert C_G(CL_\alpha) \rvert$	6	3	2
∂'_1	1	1	1
∂'_1	1	1	-1
∂'_1	2	-1	0

Table (1,2)

Where $[L_1] = \{I'\}$, $[L_2] = \{(123)\}$, $[L_3] = \{(12)(3)\}$,

Character table of finite commutative group(1.12): [4]

Let C_n be a cyclic group with order n, which are generated by u . Then the Character table of C_n is given :

$\equiv (\boldsymbol{C_n}) =$

CL_α	I	u	u^2	$\cdots$	u^{n-1}
$\|CL_\alpha\|$	1	1	1	$\cdots$	1
$\|C_G(CL_\alpha)\|$	n	n	n	$\cdots$	n
γ_1	1	1	1	$\cdots$	1
γ_2	1	φ	φ^2	$\cdots$	φ^{n-1}
γ_3	1	φ^2	φ^4	$\cdots$	φ^{n-2}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
γ_n	1	φ^{n-1}	φ^{n-2}	$\cdots$	φ

Table (1.3)

where $\boldsymbol{\varphi = e^{2\pi i/n}}$

Theorem (1.10):[5]

Let G_1 *and* G_2 are two group .Suppose $T^1: G_1 \to GL(n_1, F)$ and $T^2: G_2 \to GL(n_2, F)$ are two irreducible representations of the groups G_1 and G_2 with characters ∂_1 *and* ∂_2 respectively , then $T^1 \otimes T^2$ is irreducible representation of the group $G_1 \times G_2$ with the character $\partial_1.\partial_2$.

2. The Factor Group AC(G):

We devote our work to study the group of $Z-$ valued class function of a group G ,with its factor group on $\overline{R}(G)$in this section ,also we includes the irreducible characters tables of C_n and S_3 and the factor group $K(C_n)$ and $K(S_3)$.

Definition(2.1): [8]

A K− minor of T is the determinat of K × K. where T is a matrix entries in a principle with domain $\mathfrak{R}$.

Definition(2.2): [8]

The greatest common divisor $(g.\mathrm{c}.d)$ of all $K - minor\ is$ a $K - th$ determinant divisor of T, denoted by $DK(T)$.

Theorem (2.3): [8]

Suppose $N\ and\ M$ are two matrices of degree $s\ and\ v$ respectively, then $\det(N \otimes M) = (\det(N))^s . (\det(M))^v$.

Theorem (2.4): [9]

Let N and M be non-singular matrices with rank α and m respectively ,on a principal domain
$\mathfrak{R}$ and let :

$Q_1 N J_1 = D(N) = diag\{d_1(N), d_2(N), \cdots, d_\propto(N)\}$ and

$Q_2 M J_2 = D(M) = diag\{d_1(M), d_2(M), \cdots, d_m(M)\}$ the invariant factor matrices of N and M then ,

$(Q_1 \otimes Q_2)(N \otimes M)(J_1 \otimes J_2) = D(N) \otimes D(M)$ and from this we get that the invariant factor matrices of $N \otimes M$ can bewritten.

Theorem(2.5): [4]

Let M be a matrix with entries in a principal domain $\mathfrak{R}$ then there is matrices Q, J, D such that
Q and J are invertible, $QMJ = D$, D is diagonal matrix and then , $D_k(QDJ) = D_k(M)$ module the group of unites A .

Remark(2.6):[11]

Let $cf(G, Z) = Z^l\ basis\ is\ \equiv^* (G)$.using theorem (2.5), we evaluate two matrices $Q\ and\ J$ in addition a determent ∓ 1 where $Q. \equiv^* (G). J = D\left(\equiv^* (G)\right) = diag\{d_1, d_2, \cdots, d_\propto\}, d_i = \mp D_i(\equiv^* (G)) / \mp D_{i-1}(\equiv^* (G))$.

The Z − module $K(\mathrm{G})$ represent the direct sum of the cyclic sbmodules and with annihilating ideals $< d_1 >, < d_2 >, \cdots, < d_l >$.

Theorem(2.7): [11]

$\left| K(\mathrm{G}) \right| = det\ (\equiv^* (G))$.

Proposition (2.8): [11]

The basis of $\overline{R}$(G) is formed by irreducible characters $\vartheta_i = \sum_{\sigma \in Gal\left(Q(\gamma_i)/_Q\right)} \sigma(\gamma_i) = \vartheta_i$ form, where γ_i are the irreducible characters of G and their numbers are equal to the number of all distinct $\Gamma -$ classes of G.

Theorem (2.9): [4]

The irreducible character table of the cyclic group C_{q^δ} of the rank $\delta + 1$ and where q is an prime number which is denoted by($\equiv^*$ (C_{p^δ})) given by:

$\Gamma -$ classes	[1]	$[r^{q^{\delta-1}}]$	$[r^{q^{\delta-2}}]$	$[r^{q^{\delta-3}}]$	$\cdots$	$[r^q]$	[r]
ϑ_1	$q^{\delta-1}(q-1)$	$-q^{\delta-1}$	0	0	$\cdots$	0	0
ϑ_2	$q^{\delta-2}(q-2)$	$q^{\delta-2}(q-1)$	$-q^{\delta-2}$	0	$\cdots$	0	0
ϑ_3	$q^{\delta-3}(q-3)$	$q^{\delta-3}(q-2)$	$q^{\delta-3}(q-1)$	$-q^{\delta-3}$	$\cdots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$\vdots$	$q(q-1)$	$q(q-1)$	$q(q-1)$	$q(q-1)$	$\cdots$	$-q$	0
ϑ_δ	$(q-1)$	$(q-1)$	$(q-1)$	$(q-1)$	$\cdots$	$(q-1)$	-1
$\vartheta_{\delta+1}$	1	1	1	1	$\cdots$	1	1

Table (2.1)

Example (2.10):

For finding the irreducible character table of a cyclic group C_{49} by using theorem above as follows:

$\equiv^*$ (C_{49}) =
$\equiv^*$ (C_{7^2})=

Γ-classes	[I]	$[r^7]$	[r]
∂_1	42	-7	0
∂_2	6	6	-1
∂_3	1	1	1

Table(2.2)

Proposition(2.14):

Let $n = \prod_{i=1}^{k} q_i^{\delta i}$,where q_i are distinct primes and δ is a positive integer then :

$K(C_n) = \oplus \sum_{i=1}^{k} (\oplus \sum \quad K(C_{q_i^{\delta i}}))[\prod_{\substack{j \neq i \\ j=1}}^{k} (\delta_j + 1)]\ time$.

The group$(C_n \times S_3)$ (2.15):

The tensor product group $(C_n \times S_3)$,where (C_n is a group of order n and cyclic generated by u) and S_3 is a group of order 6 and symmetric . The direct product group $(C_n \times S_3)$ = {(q , c): q ∈ C_n, c ∈ S_3} and

$|C_n \times S_3| = |C_n| \cdot |S_3|$=6n

3. The main results:

we devote our work to study irreducible character table of the group $(C_n \times S_3)$ and for finding the cyclic decomposition of the factor group $K(C_n \times S_3)$, in this section .

Proposition(2.11):[11]

If P is a prime number, then $\left(\equiv^* \left(C_{q^\delta}\right)\right) = \{q^\delta, q^{\delta-1}, \cdots, q \quad, 1\}$.

Remark (2.12) :

Hence forth $if\ n = q_1^{\eta 1} \cdot q_2^{\eta 2} \ldots \ldots \ldots . q_m^{\eta m}$where $q_1, q_2, \ldots \ldots ., q_m\ are$ distinct primes then:

$D(\equiv^* (C_n)) = D(\equiv^* \left(C_{q_1}^{\eta 1}\right)) \otimes D(\equiv^* \left(C_{q_2}^{\eta 2}\right)) \otimes \ldots \ldots . . D(\equiv^* \left(C_{q_m}^{\eta m}\right))$.

Theorem (2.13): [11]

Let δ is a positive integer and q be a prime number, then:

$$K\left(C_{q^\delta}\right) = \oplus \sum_{i=1}^{\delta} C_{q^i} .$$

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Proposition(3,1): The general form of the irreducible character table of the group$(C_n \times S_3)$ is given as follows:

$$\equiv^{*}(C_n \times S_3) =$$

Γ-classes	$[I, L_1]$	$[I, L_2]$	$[I, L_3]$	$[x^{q^{\delta-1}}, L_1]$	$[x^{q^{\delta-1}}, L_2]$	$[x^{q^{\delta-1}}, L_3]$	$\cdots$	$[x^q, L_3]$	$[x^q, L_3]$	$[x^q, L_3]$	$[x, L_3]$	$[x, L_3]$	$[x, L_3]$
$\partial_{(1,1)}$	$q^{\delta-1}(q-1)$	$q^{\delta-1}(q-1)$	$q^{\delta-1}(q-1)$	$-q^{\delta-1}$	$-q^{\delta-1}$	$-q^{\delta-1}$	$\cdots$	0	0	0	0	0	0
$\partial_{(1,2)}$	$q^{\delta-1}(q-1)$	$q^{\delta-1}(q-1)$	$-q^{\delta-1}(q-1)$	$-q^{\delta-1}$	$-q^{\delta-1}$	$q^{\delta-1}$	$\cdots$	0	0	0	0	0	0
$\partial_{(1,3)}$	$2q^{\delta-1}(q-1)$	$-q^{\delta-1}(q-1)$	0	$-2q^{\delta-1}$	$q^{\delta-1}$	0	$\cdots$	0	0	0	0	0	0
$\partial_{(2,1)}$	$q^{\delta-1}(q-1)$	$q^{\delta-1}(q-1)$	$q^{\delta-1}(q-1)$	$q^{\delta-2}(q-1)$	$q^{\delta-2}(q-1)$	$q^{\delta-2}(q-1)$	$\cdots$	0	0	0	0	0	0
$\partial_{(2,2)}$	$q^{\delta-1}(q-1)$	$q^{\delta-1}(q-1)$	$-q^{\delta-1}(q-1)$	$q^{\delta-2}(q-1)$	$q^{\delta-2}(q-1)$	$-q^{\delta-2}(q-1)$	$\cdots$	0	0	0	0	0	0
$\partial_{(2,3)}$	$2q^{\delta-1}(q-1)$	$-q^{\delta-1}(q-1)$	***0***	$2q^{\delta-2}(q-1)$	$-q^{\delta-2}(q-1)$	0	$\cdots$	0	0	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\partial_{(\delta,1)}$	$(q-1)$	$(q-1)$	$(q-1)$	$(q-1)$	$(q-1)$	$(q-1)$	$\cdots$	$(q-1)$	$(q-1)$	$(q-1)$	-1	-1	-1
$\partial_{(\delta,2)}$	$(q-1)$	$(q-1)$	$-(q-1)$	$(q-1)$	$(q-1)$	$-(q-1)$	$\cdots$	$(q-1)$	$(q-1)$	$-(q-1)$	-1	-1	1
$\partial_{(\delta,3)}$	$2(q-1)$	$-(q-1)$	***0***	$2(q-1))$	$-((q-1)$	***0***	$\cdots$	$2(q-1(q-1)$	$(q-1)$	0	***-2***	***1***	0
$\partial_{(\delta+1,1)}$	1	1	1	1	1	1	$\cdots$	1	1	1	1	1	1
$\partial_{(\delta+1,2)}$	1	1	-1	1	1	-1	$\cdots$	1	1	-1	1	1	-1
$\boldsymbol{\partial_{(\delta+1,3)}}$	2	$\mathbf{-1}$	0	2	$\mathbf{-1}$	0	$\cdots$	2	$\mathbf{-1}$	0	2	-1	0

Table(3,1)

Theorem (3.2):

The irreducible character table of the group $C_{q^\delta} \times S_3$ when q is an prime number and δ is a positive integer , given as follows:

$\equiv^{*}(C_{q^\delta} \times S_3) = \equiv^{*}(C_{q^\delta}) \otimes \equiv^{*}(S_3)$.

Proof:

Since $S_3 = \{(1)(2)(3)$,(12)(3),(13)(2),(23)(1),(123),(132)$\}$ and the character table of S_3:

CL_α	$[L_1]$	$[L_2]$	$[L_3]$
$\lvert CL_\alpha \rvert$	1	2	3
$\lvert C_G(CL_\alpha) \rvert$	6	3	2
∂'_1	1	1	1

$\equiv \quad (S_3) =$

∂'_1	1	1	-1
∂'_1	2	-1	0

Where $[L_1] = \{(I')\}$, $[L_2] = \{(123)\}$, $[L_3] = \{(12),(3)\}$ and the irreducible valued character of S_3:

$\equiv^* S_3 =$

Γ-classes	$[L_1]$	$[L_2]$	$[L_3]$
$\lvert CL_\alpha \rvert$	1	2	3
$\lvert C_G(CL_\alpha) \rvert$	6	3	2
ϑ'_1	1	1	1
ϑ'_1	1	1	-1
ϑ'_1	2	-1	0

Then $\partial'_1(L_1) = \partial'_1(L_2) = \partial'_1(L_3) = \vartheta'_1(L_1) = \vartheta'_1(L_2) = \vartheta'_1(L_3)1$.
$\partial'_2(L_1) = \partial'_2(L_2) = \vartheta'_2(L_1) = \vartheta'_2(L_2) = 1$,
$\partial'_2(L_3) = \vartheta'_2(L_3) = -1$.
$\partial'_3(L_1) = \vartheta'_3(L_1) = 2$, $\partial'_3(L_2) = \vartheta'_3(L_2) = -1$, $\partial'_3(L_3) = \vartheta'_3(L_3) = 0$.
From the definition of $C_{q^\delta} \times S_3$,theorem(1.10)
$\equiv \quad (C_{q^\delta} \times S_3) = \equiv \quad (C_{q^\delta}) \otimes \equiv \quad (S_3)$.
Each element in $C_{q^s} \times S_3$.
$L_{ng} = J_n.L_g$, $\forall J_n \in C_{q^\delta}, L_g \in S_3$, $n = 1,2,3,\cdots,\delta+1$ and any irreducible character of $C_{q^\delta} \times S_3$ is $\partial(i,j) = \partial_i.\partial'_j$ where ∂_i represent an irreducible character of C_{q^δ} and ∂'_{j} is an irreducible character S_3 ;then ,

$$\partial_{(i,j)}(Lng) = \begin{cases} \partial_i(Ln) & if\ j = 1\ and\ g \in S_3 \\ \partial_i(Ln) & if\ j = 2\ and\ g \in \{I', (123), (132)\} \\ -\partial_i(Ln) & if\ j = 2\ and\ g \in \{(12)(3), (13)(2), (32)(1)\} \\ 2\partial_i(Ln) & if\ \ j = 3\ \ and\ \ g \in \{I'\} \\ -\partial_i(Ln) & if\ \ j = 3\ \ and\ g \in \{(123), (132)\} \\ 0 & if\ \ j = 3\ and\ g \in \{(12)(3), (13)(2), (23)(1)\} \end{cases}$$

From proposition(2.8)
$\vartheta_{(i,j)} = \sum_{\sigma \in Gal(^{Q\partial_{(\mathrm{i,j})}}/\ )} \sigma(\partial_{(\mathrm{i,j})})$ such that $\vartheta_{(\mathrm{i,j})}$ is an irreducible character of $C_{q^s} \times S_3$.

Then, $\vartheta_{(\mathrm{i,j})}(h_{ng}) = \sum_{\sigma \in Gal(^{Q\partial_{(\mathrm{i,j})}(h_{ng})}/_Q)} \sigma(\partial_{(i,j)}(h_{ng}))$.

1- if $= 1\ and\ g \in S_3$.
$\vartheta_{(i,j)}(h_{ng}) = \sum_{\sigma \in Gal(^{Q\partial_i(L_n)}/_Q)} \sigma(\partial_i(h_n)) = \vartheta_i(h_n).1 = \vartheta_i(h_n).\vartheta'_j(L_g)$ where ϑ_i is an

irreducible character of C_{q^s}.

2- (a) $j = 2 and\ g \in \{I', (123), (132)\}$.

$$\vartheta_{(i,j)}(h_{ng}) = \sum_{\sigma \in Gal\left(^{Q\partial_i(h_n)}/_Q\right)} \sigma(\partial_i\,(h_n)) = \vartheta_i(h_n).1 = \vartheta_i(h_n).\vartheta'_j\,(L_g)$$

(b) $j = 2\ and\ g \in \{(12)(3), (13)(2), (23)(1)\}$.

$$\vartheta_{(i,j)}(h_{ng})$$
$$= \sum_{\sigma \in Gal\left(^{Q\partial_i(h_n)}/_Q\right)} \sigma(-\partial_i\,(h_n))$$
$$= -\sum_{\sigma \in Gal\left(^{Q\partial_i(h_n)}/_Q\right)} \sigma(\partial_i\,(h_n))$$
$$= \sum_{\sigma \in Gal\left(^{Q\partial_i(h_n)}/_Q\right)} \sigma(\partial_i\,(h_n)).-1 = \vartheta_i(h_n)\,\vartheta'_j(L_g)$$

(3) (a) $j = 3\ and\ g \in \{I'\}$.

$$\vartheta_{(i,j)}(h_{ng})$$
$$= \sum_{\sigma \in Gal\left(^{Q\partial_i(h_n)}/_Q\right)} \sigma(2\partial_i\,(h_n))$$
$$= 2\sum_{\sigma \in Gal\left(^{Q\partial_i(h_n)}/_Q\right)} \sigma(\partial_i\,(h_n))$$
$$= \sum_{\sigma \in Gal\left(^{Q\partial_i(h_n)}/_Q\right)} \sigma(\partial_i\,(h_n)).2 = \vartheta_i(h_n)\,\vartheta'_j(L_g)$$

(b) $j = 3\ and\ g \in \{(123), 132)\}$.

$$\vartheta_{(i,j)}(h_{ng})$$
$$= \sum_{\sigma \in Gal\left(^{Q\partial_i(h_n)}/_Q\right)} \sigma(-\partial_i\,(h_n))$$
$$= -\sum_{\sigma \in Gal\left(^{Q\partial_i(h_n)}/_Q\right)} \sigma(\partial_i\,(h_n))$$
$$= \sum_{\sigma \in Gal\left(^{Q\partial_i(h_n)}/_Q\right)} \sigma(\partial_i\,(h_n)).-1 = \vartheta_i(h_n)\,\vartheta'_j(L_g)$$

(c) $j = 3\ and\ g \in \{(12)(3),(13)(2),(23)(1)\}$

$$\vartheta_{(i,j)}(h_{ng}) = \sum_{\sigma \in Gal\left(^{Q\partial_i(h_n)}/_Q\right)} \sigma(0.\partial_i\,(h_n)) = 0.\sum_{\sigma \in Gal\left(^{Q\partial_i(h_n)}/_Q\right)} \sigma(\partial_i\,(h_n)) = \sum_{\sigma \in Gal\left(^{Q\partial_i(h_n)}/_Q\right)} \sigma(\partial_i\,(h_n)).0 = 0 = \vartheta_i(h_n)\,\vartheta'_j(L_g)$$

From (1),(2)and (3) we have:

$\vartheta_{(i,j)} = \vartheta_i . \vartheta'_j$.

Hence $\equiv^* (C_{q^\delta} \times S_3) = \equiv^* (C_{q^\delta}) \otimes \equiv^* (S_3)$

Example(3.3):

To find the irreducible character of $C_{5^2} \times S_3$ by use theorem (3.2).

$\equiv^* (C_{5^2}) =$

Γ-classes	$[I]$	$[x^5]$	$[x]$
ϑ'_1	20	-5	0
ϑ'_2	4	4	-1
ϑ'_3	1	1	1

And

$\equiv^* (S_3) =$

Γ-classes	$[L_1]$	$[L_2]$	$[L_3]$
ϑ'_1	1	1	1
ϑ'_2	1	1	-1
ϑ'_3	2	-1	0

Then: $\equiv^* (\,C_{5^2} \times S_3) =$

Γ-classes	$[I, L_1]$	$[I, L_2$	$[I, L_3$	$[x^5, L$	$[x^5, L_2$	$[x^5, L_3]$	$[x, L_1]$	$[x, L_2]$	$[x, L_3$
$\vartheta_{(1,1)}$	20	20	20	-5	-5	-5	0	0	0
$\vartheta_{(1,2)}$	20	20	-20	-5	-5	5	0	0	0
$\vartheta_{(1,3)}$	40	-20	0	10	-5	0	0	0	0
$\vartheta_{(2,1)}$	4	4	4	4	4	4	-1	-1	-1
$\vartheta_{(2,2)}$	4	4	-4	4	4	-4	-1	-1	1

$\vartheta_{(2,3)}$	8	-4	0	8	-4	0	-2	1	0
$\vartheta_{(3,1)}$	1	1	1	1	1	1	1	1	1
$\vartheta_{(3,2)}$	1	1	-1	1	1	-1	1	1	-1
$\vartheta_{(3,3)}$	2	-1	0	2	-1	0	2	-1	0

Table(3.2)

Proposition(3.4):

If q is a prime number and δ is a positive integer, then:

$$M(C_{q^\delta} \times S_3) = \begin{bmatrix} \Re & \Re & \Re & \cdots & \Re \\ \Im & \Re & \Re & \cdots & \Re \\ \Im & \Im & \Re & \cdots & \Re \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Im & \Im & \Im & \Im & \Re \end{bmatrix}$$

and

$$W(C_{q^\delta} \times S_3) = \begin{bmatrix} B & -B & \Im & \Im & \Im & \cdots & \Im & \Im \\ \Im & B & -B & \Im & \Im & \cdots & \Im & \Im \\ \Im & \Im & B & -B & \Im & \cdots & \Im & \Im \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ \Im & \Im & \Im & \Im & \Im & \cdots & B & -B \\ \Im & \Im & \Im & \Im & \Im & \cdots & \Im & B \end{bmatrix}$$

which is of the size $3(\delta+1) \times 3(\delta+1)$, where $\Re = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $\Im = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$.

Theorem(3.5):

Let q be aprime number and δ is a positive integer then :

$$K(C_{q^\delta} \times S_3) = \oplus \sum_{i=1}^{3\delta} (C_{q^\delta} \times S_3) = \bigoplus_{i=1}^{3\delta} K(C_{q^\delta}) \bigoplus_{i=1}^{\delta} K(C_6)\ .$$

Proof:

To prove the theorem ,by proposition(3.1) we obtain $\equiv^* (C_{q^\delta} \times S_3)$and by proposition (3.4) we obtain $M\left(C_{q^\delta} \times S_3\right)$ and $W\left(C_{q^\delta} \times S_3\right)$.

Now we use remark (2.6) and theorem (2.7) we obtain:

$$M\left(C_{q^\delta} \times S_3\right) . \equiv^* \left(C_{q^\delta} \times S_3\right) . W\left(C_{q^\delta} \times S_3\right) =$$

$\{6q^{\delta}, q^{\delta}, -q^{\delta}, 6q^{\delta-1}, q^{\delta-1}, -q^{\delta-1}, \cdots, 6q^2, q^2, -q^2, 6q, q, -q, 6, 1, -1\}$

$K\left(C_{q^{\delta}} \times S_3\right) = C_{6q^{\delta}} \oplus C_{q^{\delta}} \oplus C_{q^{\delta}} \oplus C_{6q^{\delta-1}} \oplus C_{q^{\delta-1}} \oplus C_{q^{\delta-1}} \oplus \cdots \oplus C_{6q^2} \oplus C_q \oplus C_q \oplus C_6$

$$= \oplus \sum_{i=1}^{3\delta} (C_{q^i}) \oplus \sum_{i=1}^{\delta} (C_6)$$

$$= \underset{=1}{\overset{3\delta}{\oplus}} K(C_{q^{\delta}}) \underset{i=1}{\overset{\delta}{\oplus}} K(C_6)$$

Theorem(3.6):

Let $n = \prod_{i=1}^{k} q_i^{\eta_i}$ where q_i are distinct primes and η_i are positive integers,where $i = 1,2,\cdots,k$,then:

$$K(C_n \times S_3) = \oplus \sum_{i=1}^{k} (\oplus \sum K\left(C_{q^{\eta_i}} \times S_3\right) \left[\prod_{\substack{i\neq 1\\ j=1}}^{k} (\eta_i + 1)\right] \text{time.}$$

Proof:

$$K(C_n \times S_3) = \underbrace{K\left(C_{q_1^{\eta_1}} \times s_3\right) \oplus \cdots \oplus K\left(C_{q_1^{\eta_k}} \times s_3\right)}_{(\eta_2+1)(\eta_3+1)\cdots(\eta_k+1)\text{time}} \oplus \underbrace{K\left(C_{q_2^{\eta_2}} \times s_3\right) \oplus \cdots \oplus K\left(C_{q_2^{\eta_k}} \times s_3\right)}_{(\eta_1+1)(\eta_3+1)\cdots(\eta_k+1)\text{time}}$$

$$\oplus \cdots \oplus \underbrace{K(C_{q_k^{\eta_1}} \times s_3) \oplus \cdots \oplus K(q_k^{\eta_{k-1}} \times s_3)}_{(\eta_1+1)(\eta_2+1)\cdots(\eta_{k-1}+1)\text{time}}$$

By theorem (2.12) we can find .

$$K(C_n \times S_3) = \underset{i=1}{\overset{3(\eta_2+1)(\eta_3+1)\cdots(\eta_k+1)}{\oplus}} K\left(C_{q_1^{\eta_1}}\right) \underset{i=1}{\overset{(\eta_2+1)(\eta_3+1)\cdots(\eta_k+1)}{\oplus}} K(C_6)$$

$$\oplus \cdots \oplus \underset{i=1}{\overset{3(\eta_1+1)(\eta_2+1)\cdots(\eta_{k-1}+1)}{\oplus}} k\left(C_{q_k^{\eta_k}}\right) \underset{i=1}{\overset{(\eta_1+1)(\eta_2+1)\cdots(\eta_{k-1}+1)}{\oplus}} k(C_6) .$$

$$= \underset{i=1}{\overset{3(\eta_2+1)(\eta_3+1)\cdots(\eta_k+1)}{\oplus}} K\left(C_{q_i^{\eta_i}}\right) \underset{i=1}{\overset{(\eta_2+1)(\eta_3+1)\cdots(\eta_k+1)}{\oplus}} K(C_6) .$$

Theorem (3.7):

Suppose $n = q_1^{\eta 1} \cdot q_2^{\eta 2} \ldots\ldots\ldots q_m^{\eta m}$,where $q_1, q_2, \ldots\ldots, q_m$ are distinct primes and η_i are positive integers , $i = 1,2,\cdots,m$ then :

$$K(C_n \times S_3) = \underset{i=1}{\overset{3}{\oplus}} K(C_n) \underset{i=1}{\overset{(\eta_2+1)(\eta_3+1)\cdots(\eta_k+1)}{\oplus}} K(C_6)$$

proof:

using theorem(3.2) and proposition(2.4) we obtain:

$$D(\equiv^* \left(C_{q^\delta} \times S_3\right)) = D(\equiv^* \left(C_{q^\delta}\right)) \otimes D(\equiv^* (S_3)) .$$

By proposition(2.11) we obtain:

$(D \equiv^* (C_n))$,then:

$$D(\equiv^* (C_n \times S_3)) = [D(\equiv^* (C_n \times S_3))] \otimes \begin{bmatrix} 6 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 6D(\equiv^* (C_n) & & 0 \\ & D(\equiv^* (C_n)) & \\ 0 & & -D(\equiv^* (C_n)) \end{bmatrix}$$

$$= \{6d_1, 6d_2, \ldots, 6d_{(\eta_1+1)(\eta_2+1)\ldots(\eta_m+1)}, d_1, d_2, \ldots, d_{(\eta_1+1)(\eta_2+1)\ldots(\eta_m+1)},$$
$$-d_1, -d_2, \ldots, -d_{(\eta_1+1)(\eta_2+1)\ldots(\eta_m+1)}\}$$

Where d_i is the invariant factor of $\equiv^* (C_n)$;then by using theorem (2.12) we have:

$$K(C_n \times S_3) = \bigoplus_{i=1}^{(\eta_1+1)(\eta_2+1)(\eta_3+1)\cdots(\eta_m+1)} C_{6d_i} \bigoplus_{i=1}^{(\eta_1+1)(\eta_2+1)(\eta_3+1)\cdots(\eta_m+1)} C_{d_i}$$

$$\bigoplus_{i=1}^{(\eta_1+1)(\eta_2+1)(\eta_3+1)\cdots(\eta_m+1)} C_{d_i}$$

$$= \bigoplus_{i=1}^{(\eta_1+1)(\eta_2+1)(\eta_3+1)\cdots(\eta_m+1)} C_{6d_i} \bigoplus_{i=1}^{2(\eta_1+1)(\eta_2+1)(\eta_3+1)\cdots(\eta_m+1)} C_{d_i}$$

$$= \bigoplus_{i=1}^{(\eta_1+1)(\eta_2+1)(\eta_3+1)\cdots(\eta_m+1)} C_{d_i} \bigoplus_{i=1}^{(\eta_1+1)(\eta_2+1)\cdots(\eta_m+1)} (\eta_3+1)C_6$$
$$\bigoplus_{i=1}^{2(\eta_1+1)(\eta_2+1)(\eta_3+1)\cdots(\eta_m+1)} C_{d_i}$$

$$= \bigoplus_{i=1}^{3(\eta_1+1)(\eta_2+1)(\eta_3+1)\cdots(\eta_m+1)} C_{d_i} \bigoplus_{i=1}^{(\eta_1+1)(\eta_2+1)(\eta_3+1)\cdots(\eta_m+1)} C_6$$

By theorems (3.5)and (3.6),we obtain:

$$K(C_n \times S_3) = \mathrm{i} \bigoplus_{i=1}^{3} K(C_n) \bigoplus_{i=1}^{(\eta_1+1)(\eta_2+1)(\eta_3+1)\cdots(\eta_m+1)} C_6$$

Example (3.8):

To find the cyclic decomposition $(C_{25} \times S_3)$, $K(C_{1125} \times S_3)$ and $K(C_{1157625} \times S_3)$

By Theorem (3.7) :

$$K(C_{25} \times S_3) = K(C_{5^2} \times S_3) = \bigoplus_{i=1}^{3} K(C_{5^2}) \bigoplus_{i=1}^{(2+1)} C_6$$

$$= \bigoplus_{i=1}^{3} K(C_{5^2}) \bigoplus_{i=1}^{3} C_6 .$$

$$K(C_{1125} \times S_3) = K(C_{3^2.5^5} \times S_3) = \bigoplus_{i=1}^{3} K(C_{3^2.5^5}) \bigoplus_{i=1}^{(2+1)(5+1)} C_6$$

$$= \bigoplus_{i=1}^{3} K(C_{3^2.5^5}) \bigoplus_{i=1}^{18} C_6 .$$

$$K(C_{1157625} \times S_3) = K(C_{3^3.5^3.7^3} \times S_3)$$

$$= \bigoplus_{i=1}^{3} K(C_{3^3.5^3.7^3}) \bigoplus_{i=1}^{(3+1)(3+1)(3+1)} C_6$$

$$= \bigoplus_{i=1}^{3} K(C_{3^3.5^3.7^3}) \bigoplus_{i=1}^{64} C_6 .$$

Conclusion:

According to this paper we have found a new method companied with a new results for the cyclic decomposition of the factor group $K(C_n \times S_3)$,for that we can extend this paper in future work .

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