

Order Reduction Method and Its Application to Singular Perturbed Delay Differential Equations

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Abstract

The present paper presents and analyzes the concept of order reduction method and its application to the delay differential equation. The principle idea is to reduce the 2nd order singular perturbed differential equation to pairs of reduced order ones. The obtained pairs are then solved, and the two common test problems are solved and the obtained results are compared with the available exact results. Both the test examples are solved twice for two different values of the parameter $\varepsilon = 2^{-8}$ & 2^{-10} and the computed results showed an excellent agreement with the exact results moreover, for the computed results became more accurate with $\varepsilon = 2^{-10}$.

Keywords: Perturbation techniques, singular perturbed delay differential equations, order reduction techniques

1. Introduction

Let us start the introduction by making a major difference between delay and ordinary differential equations that is the delay can be considered similarly to ordinary ones, but their evolutions involve past values state variable [1], [2]. Resolving backward differential equations does not only require knowledge of the current state, but requires knowledge of the case at a specific earlier time. We would like to point out that in some cases, mathematical models generally formulated are the first estimates of the physical models studied and interest in this system often arises when a replacement is made, conventional wise modeling assumptions with more realistic distributed assumptions [3]. Hence it is useful to remember that more realistic models should include some of past and future states of the system, therefore, real system in certain cases should be modeled by differential equations with delay or advance [4]. The manner in which the properties of delay differential equations (D.D.E.)

differ from ODE equation has been remains active area of research. Let us now mention the major difference between ODEs and DDEs is the initial or boundary data [5]. Regarding the solution of the DDEs can be determined by making use prior values at the points prior to the initial or boundary points. An equation in which the highest order derivative is multiplied by a tiny parameter and includes at least one delay term is known as the singular perturbed D.D.E., and this form of equation appears often in the mathematical modeling of numerous practical phenomena [6]–[8]. In what follows, we will mention some examples in which singular perturbed D.D.E., appears, for example, in modeling of the human pupil-light reflex [9], model of HIV infection [10], the study of bistable devices in digital electronics [11], variational problem in control theory [12], first exit time problem in modeling of activation of neuronal variability [13], Immune response [14], evolutionary biology [13], dynamics of networks of two identical amplifier [15], mathematical ecology [16], population dynamics [17], modeling of biological oscillator [18] and in a variety of models for physiological process [19].

2. The proposed method

This paper considers a singly perturbed delay differential equation (SPDDEq), which has the following form:

$$\{-\varepsilon + a(t)\} \frac{dg(t)}{dt} + b(t)g(t-1) = F(t), \quad \forall t \in [-1, 2], \quad (1)$$

$$0 < \varepsilon < 1,$$

with boundary conditions (BC):

$$g(t) = R(t), \forall t \in [-1, 0], \quad g(2) = \beta, \quad (2)$$

in equation (2), both $R(t)$ and β are known function and constant, respectively.

2.1. Assumptions taken throughout the full text

- In the present paper, it is assumed that $a(t)$, $b(t)$ and $F(t)$, sufficient continuous functions over the interval $[-1, 2]$.
- Over the prescribed interval $[-1, 2]$ there exists a positive constant M , such that the following condition is usually satisfied, that is $0 < M < a(t)$, based on this assumption, the neighborhood of $t = 2$ will indicate to the boundary layer.

2.2. Main steps of the proposed method

2.2.1. Major 1st step

- (1) Put $\varepsilon = 0$, on equation (1), to obtain the reduced problem and obtain an approximate solution, assuming that the approximate solution is $g_0(t)$.

$$a(t) \frac{dg_0(t)}{dt} + b(t)g_0(t-1) = F(t), \quad \forall t \in [0,1], \quad (3)$$

with BC,

$$g_0(t) = R(t), \quad \forall t \in [0,1], \quad (4)$$

- (2) Applying the classical 4th order Runge-Kutta method, leads to

$$g_0(t) = \gamma, \quad \forall t \in [0,1].$$

- (3) Now it is required to solve the equation (3), with BC (4), :

2.2.2. Major 2nd step

Perform two 1st O.D.E. equivalents to equation (1) as:

$$\frac{dz(t)}{dt} - \frac{da(t)}{dt} g(t) + b(t)g(t-1) = F(t), \quad \forall t \in [0,2], \quad (5)$$

and

$$\varepsilon \frac{dg(t)}{dt} + a(t)g(t) = z(t), \quad \forall t \in [0,2], \quad (6)$$

2.2.3. Major 3rd step

Perform the initial conditions (IC), by making use of both $g_0(t)$ and $g_0(1)$, the solution of the reduced problem appears in equation (6), leads to:

$$z(0) = -\varepsilon \frac{dg_0(0)}{dt} + a(0)g_0(0), \quad (7)$$

In equation (7);

$$\frac{dg_0(0)}{dt} = \left(\frac{F(0) - b(0)R(-1)}{a(0)} \right),$$

and from the equation:

$$a(t) \frac{dg_0(t)}{dt} = F(t) - b(t)g_0(t-1),$$

we can obtain:

$$z(1) = -\varepsilon \frac{dg_0(1)}{dt} + a(1)g_0(1), \quad (8)$$

Now it is important to remain the following:

Equation (6), will be the IC of (5), while $g(2)$, will be IC for Eq.(6).

2.2.4. Major 4th step

Getting pairs of initial value problems from equation (6), through replacing:

$g(t)$ by $g_0(t)$ and $g_0(t)$ by $g_0(1)$, leads to:

$$-\varepsilon \frac{dg(t)}{dt} + a(t)g(t) = z(t), g(2) = \beta, \forall t \in [1,2], \tag{9}$$

and

$$-\varepsilon \frac{dg(t)}{dt} + a(t)g(t) = z(t), g(1) = \gamma, \forall t \in [0,1], \tag{10}$$

Thus, a pair of initial value problems (IVPs) have been substituted for the boundary value problem (BVP), given by equations (1), (2). In terms of integration, these IVPs go in the opposite direction, and the solution to the second problem (10) can only be determined by knowing the solution to the first one (9). Solving these IVPs provides us the solution over the interval [0,2].

When it comes to addressing initial value issues, there are a variety of efficient strategies that may be used. In our numerical experiments, we apply the conventional fourth order Runge-Kutta technique to solve the IVPs that we encounter. With consideration for the possibility of using any conventional analytical or numerical techniques.

3. Numerical results and discussions

3.1. Test example 1

In the present section, two common test problems are solved; each one is tested for two different values of the parameter $\varepsilon, \varepsilon = 2^{-8}$ and 2^{-10} , to see the direct effect on the obtained results. consider a singular perturbed delay (SPD), governing differential equation (DEq)

$$\begin{cases} \varepsilon \frac{d^2g(t)}{dt^2} + 3 \frac{dg(t)}{dt} - g(t-1) = 0, \forall t \in [-1,2], \\ g(t) = 1, \quad \forall t \in [-1,0] \text{ and } g(2) = 2, \end{cases} \tag{11}$$

The accurate solution of the 2nd ODE takes the following form:

$$g(t) = \begin{cases} 1 + C_1 \left(\exp\left(\frac{3t}{s}\right) - 1 \right) + \frac{t}{3}, & 0 \leq t \leq 1 \\ \left(C_2 + \frac{t}{3} + \frac{(t-1)^2}{18} + \frac{\varepsilon t}{27} - \frac{C_1 t}{3} - \frac{C_1 t}{3} \exp\left(\frac{3(t-1)}{s}\right) + \right. \\ \left. + \exp\left(\frac{3(t-1)}{s}\right) \left(\frac{23}{18} - \frac{2\varepsilon}{27} - C_2 + \frac{2C_1}{3} + \frac{2C_1}{3} \exp\left(\frac{3}{s}\right) \right) \right), & 1 \leq t \\ \leq 2, \end{cases} \tag{12}$$

In equation (12):

$$C_1 = \exp\left(\frac{-6}{\varepsilon}\right) \left\{ \frac{\left(\frac{4\varepsilon}{9} - \frac{\varepsilon^2}{27} - 3\right)}{\left\{3 - 4\exp\left(\frac{-6}{\varepsilon}\right) + \frac{2\varepsilon}{3}\left(\exp\left(\frac{-3}{\varepsilon}\right) - \exp\left(\frac{-6}{\varepsilon}\right)\right)\right\}} \right\}, \tag{13}$$

$$C_2 = \frac{\left(1 - \frac{22}{18}\exp\left(\frac{-3}{\varepsilon}\right) - \frac{\varepsilon}{27} - 3\right) + C_1\exp\left(\frac{3}{\varepsilon}\right)\left(1 - \exp\left(\frac{-3}{\varepsilon}\right) - \frac{2}{3}\exp\left(\frac{-6}{\varepsilon}\right)\right)}{\left\{1 - \exp\left(\frac{-3}{\varepsilon}\right)\right\}}, \tag{14}$$

Now, by applying the solution procedure, as follows:

From the 1st step, one can get the reduced form:

$$3 \frac{d\varphi_0(t)}{dt} - \varphi_0(t - 1) = 0, \quad \varphi_0(0) = 1,$$

$$\frac{d\varphi_0(t)}{dt} = \frac{\varphi_0(t - 1)}{3}, \quad \forall t \in [0,1], \quad \varphi_0(0) = 1,$$

Then, the solution takes the form:

$$\varphi_0(t) = \frac{t}{3} + 1,$$

$$3 \frac{d\varphi_0(t)}{dt} = \varphi_0(t - 1),$$

$$\frac{d\varphi_0(t)}{dt} = \frac{\left(\frac{t-1}{3} + 1\right)}{3}, \forall t \in [1,2], \quad \varphi_0(1) = \frac{4}{3},$$

Then the solution will be:

$$\varphi_0(t) = \frac{t^2}{18} + \frac{2t}{9} + \frac{19}{18}$$

Now,

$$\varphi_0(t) = \frac{t}{3} + 1, \forall t \in [1,2],$$

$$\varphi_0(t) = \frac{t^2}{18} + \frac{2t}{9} + \frac{19}{18}, \forall t \in [1,2].$$

The two 1st order equations of the 2nd step; which are equivalent to equation (11) are:

$$-\varepsilon \frac{d\varphi(t)}{dt} + 3\varphi(t) = z(t),$$

$$\frac{dz(t)}{dt} = -\varepsilon \frac{d^2\varphi(t)}{dt^2} + 3 \frac{d\varphi(t)}{dt}, \quad \frac{dz(t)}{dt} = \varphi(t - 1)$$

From the 3rd step, one can get:

$$z(0) = -\varepsilon \frac{d\varphi_0(t)}{dt} + 3\varphi_0(t)$$

$$z(0) = 3 - \frac{\varepsilon}{3}$$

Its solution is:

$$z(t) = t + 3 - \frac{\varepsilon}{3}, \quad \forall t \in [0,1]$$

$$z(1) = -\varepsilon \frac{d\varphi_0(1)}{dt} + 3\varphi_0(1), \quad z(1) = 4 - \frac{\varepsilon}{3}$$

Therefore, the solution takes the form:

$$z(t) = \frac{t^2}{6} - \frac{t}{3} + t + \frac{19}{6} - \frac{\varepsilon}{3}, \forall t \in [1,2]$$

Hence, the pair of IVPs, is:

$$-\varepsilon \frac{d\varphi(t)}{dt} + 3\varphi(t) = z(t), \quad \varphi(2) = 2, \forall t \in [1,2]$$

$$-\varepsilon \frac{d\varphi(t)}{dt} + 3\varphi(t) = z(t), \quad \varphi(1) = \frac{4}{3}, \forall t \in [0,1]$$

3.1.1. Results and analysis of test problem 1, case 1

The first case results for $\varepsilon = 2^{-8}$ are shown in Table 1, where a comparison between the exact and the computed results due to the proposed method.

Table 1. Test problem 1, case I- comparison between the exact and present

t	$\varphi(t)$	Exact solution	Present solution	Abs. Error	Min & Max
0	1.0000	1.0000	1.0000	0.0000	
0.01	1.0033	1.0034	1.0035	0.0001	0.0001
0.02	1.0067	1.0077	1.0078	0.0001	
0.03	1.0100	1.0102	1.0104	0.0002	
0.04	1.0133	1.0136	1.0139	0.0003	
0.05	1.0177	1.0182	1.0187	0.0005	
0.06	1.6886	1.6897	1.6906	0.0009	
0.07	1.6941	1.6932	1.6841	0.0009	
0.08	1.7044	1.7066	1.7076	0.0010	
0.09	1.7053	1.7076	1.7088	0.0012	
1.00	1.7069	1.7099	1.7115	0.0016	0.0016

1.20	1.7110	1.7120	1.7130	0.0010	
1.30	1.7154	1.7156	1.7158	0.0002	
1.40	1.7261	1.7265	1.7269	0.0004	
1.50	1.7282	1.7285	1.7289	0.0004	
1.60	1.7429	1.7439	1.7449	0.0010	
1.70	1.7629	1.7635	1.7641	0.0006	
1.80	1.8001	1.8005	1.8009	0.0004	
1.90	1.9708	1.9710	1.9712	0.0002	
2.00	2.0000	2.0000	2.0000	0.0000	

As it is clear from Table 1, the absolute error ranges from 0.0001 to maximum value of 0.0016, also we should remember that the values at the extremes remain fixed as boundary conditions. The error variations versus horizontal axis for the present case are plotted in Figure 1.

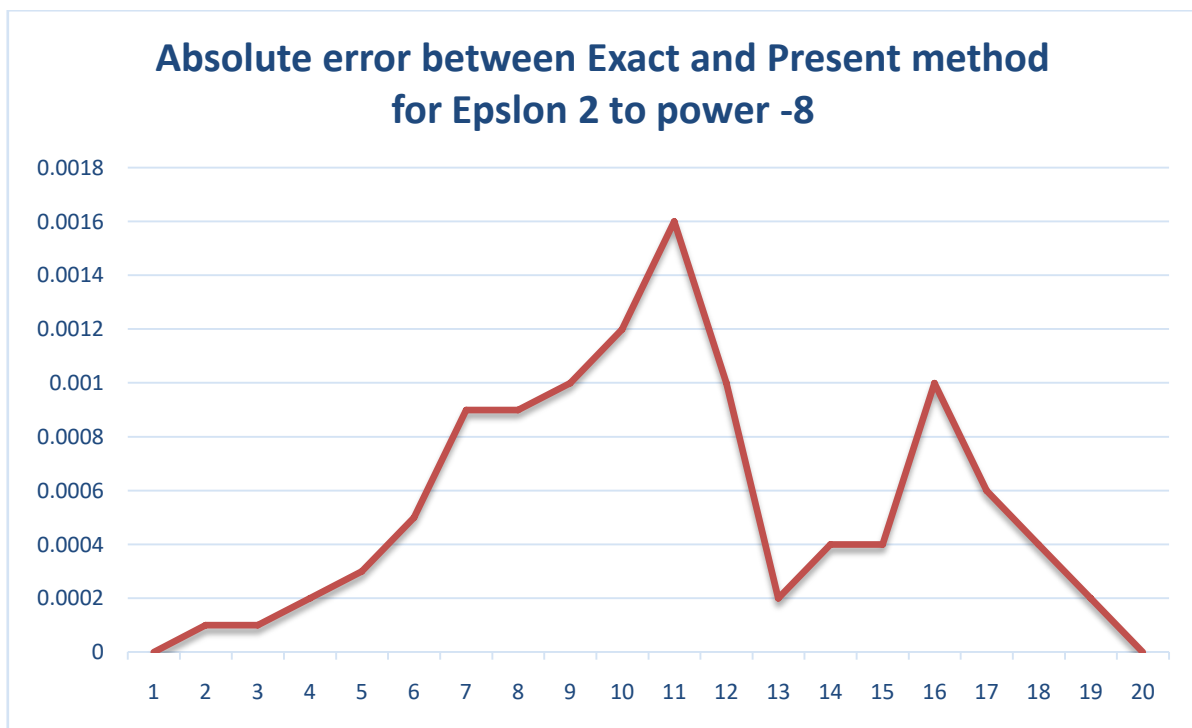


Figure 1. Error variations for test problem 1, case 1.

3.1.2. Results and analysis of test problem 1, case 2.

The first case results for $\varepsilon = 2^{-10}$ are shown in Table 2, where a comparison between the exact and the computed results due to the proposed method.

Table 2. Test problem 1, case II- comparison between the exact and present

t	$g(t)$	Exact solution	Present solution	Abs. Error	Min & Max
0	1.0000	1.0000	1.00000	0.00000	
0.01	1.0033	1.0034	1.00341	0.00001	0.00001
0.02	1.0067	1.0077	1.00771	0.00001	
0.03	1.0100	1.0102	1.01022	0.00002	
0.04	1.0133	1.0136	1.01363	0.00003	
0.05	1.0177	1.0182	1.01825	0.00005	
0.06	1.6886	1.6897	1.68979	0.00009	0.00009
0.07	1.6941	1.6932	1.69329	0.00009	
0.08	1.7044	1.7066	1.70661	0.00001	
0.09	1.7053	1.7076	1.70762	0.00002	
1.00	1.7069	1.7099	1.70996	0.00006	
1.20	1.7110	1.7120	1.71201	0.00001	
1.30	1.7154	1.7156	1.71562	0.00002	
1.40	1.7261	1.7265	1.72654	0.00004	
1.50	1.7282	1.7285	1.72854	0.00004	
1.60	1.7429	1.7439	1.74393	0.00003	
1.70	1.7629	1.7635	1.76352	0.00002	
1.80	1.8001	1.8005	1.80052	0.00002	
1.90	1.9708	1.9710	1.97102	0.00002	
2.00	2.0000	2.0000	2.00000	0.00000	

According to Table 2, the absolute error ranges from 0.00001 to maximum value of 0.00009, also we should remember that the values at the extremes remain fixed as boundary conditions. The error variations versus horizontal axis for the present case are plotted in Figure 2. As we see the error bounds decreased significantly due to the apparent decrease of the parameter ε .

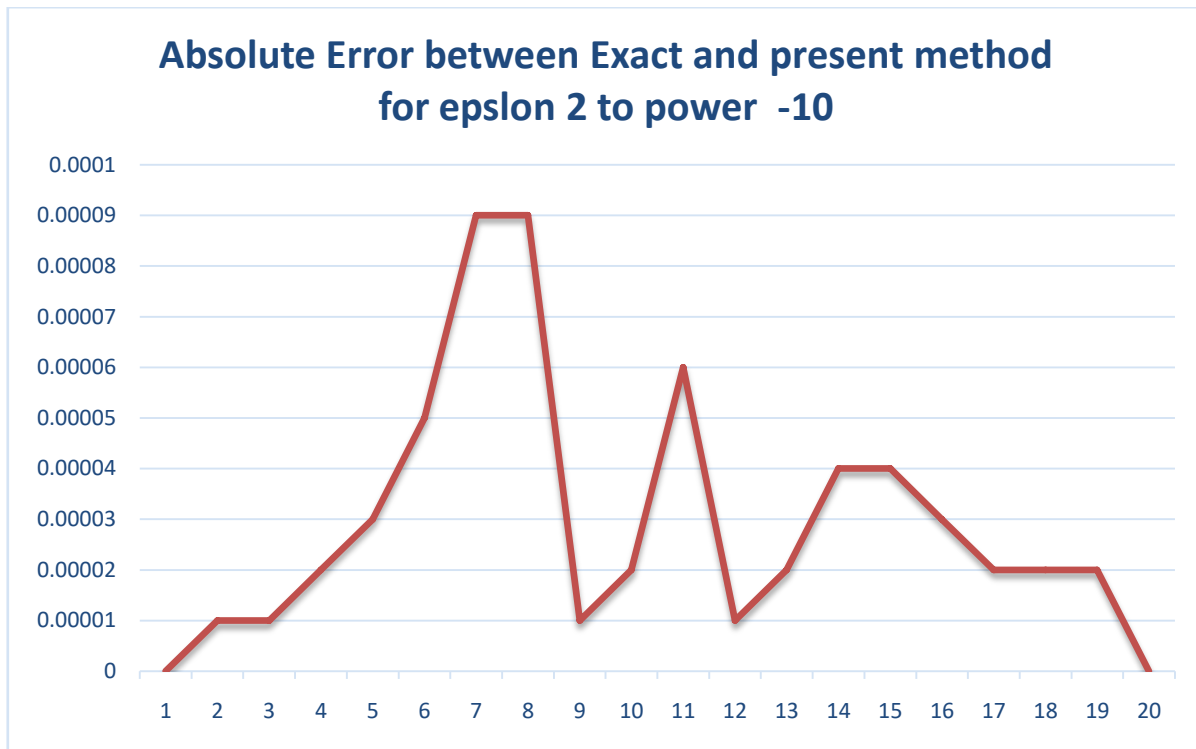


Figure 2. Error variations for test problem 1, case 2.

3.2. Test example 2

Consider a singular perturbed delay (SPD) governing differential equation (DEq) of the following form:

$$\varepsilon \frac{d^2 g(t)}{dt^2} + 3 \frac{dg(t)}{dt} - 5g(t - 1) = 0, \forall t \in [-1, 2], \text{ with } g(t) = 1, \forall t \in [-1, 0] \text{ and } g(2) = 2, \tag{15}$$

By applying the solution procedure, as follows:

From the 1st step, one can get the reduced form:

$$2 \frac{dg_0(t)}{dt} - 5g_0(t - 1) = 0, \quad g_0(0) = 1,$$

$$\frac{dg_0(t)}{dt} = \frac{5g_0(t - 1)}{2}, \forall t \in [0, 1], g_0(0) = 1$$

The solution takes the following form:

$$g_0(t) = \frac{5t}{2} + 1$$

$$2 \frac{dg_0(t)}{dt} = 5g_0(t - 1)$$

$$\frac{dg_0(t)}{dt} = \frac{25(t - 1)}{4} + \frac{5}{2}, \quad \forall t \in [1, 2], g_0(1) = \frac{7}{2}$$

Then the solution will be:

$$g_0(t) = \frac{25}{4} \left(\frac{t^2}{2} - t \right) + \frac{5t}{2} + \frac{33}{8}$$

Now,

$$g_0(t) = \frac{5t}{2} + 1, \forall t \in [0,1]$$

$$g_0(t) = \frac{25}{4} \left(\frac{t^2}{2} - t \right) + \frac{5t}{2} + \frac{33}{8}, \forall t \in [1,2]$$

From the 2nd step; the two 1st order equations equivalent to equation (15) are:

$$-\varepsilon \frac{dg(t)}{dt} + 2g(t) = z(t),$$

$$\frac{dz(t)}{dt} = -\varepsilon \frac{d^2g(t)}{dt^2} + 2 \frac{dg(t)}{dt}, \quad \frac{dz(t)}{dt} = 5g(t - 1),$$

From the 3rd step, one can get:

$$= -\varepsilon \frac{dg_0(0)}{dt} + 2g_0(0),$$

$$z(0) = \frac{4 - 5\varepsilon}{2},$$

Its solution is:

$$z(t) = 5t + \frac{4 - 5\varepsilon}{2},$$

3.2.1. Results and analysis of test problem 2, case 1

The first case results for $\varepsilon = 2^{-8}$ are shown in Table 3, where a comparison between the exact and the computed results due to the proposed method.

Table 3. Test problem 2, case I- comparison between the exact and present

t	$g(t)$	Exact solution	Present solution	Absolute Error	Min & Max
0	1.000	1.0000	1.0000	0.0000	
0.01	1.0049	1.0050	1.0051	0.0001	0.0001
0.02	1.0099	1.0100	1.0101	0.0001	
0.03	1.0399	1.0496	1.0503	0.0007	
0.04	1.0799	1.0810	1.0811	0.0001	
0.05	1.2049	1.2052	1.2055	0.0003	
0.06	1.3249	1.3259	1.3265	0.0006	

0.07	1.3889	1.3898	1.3907	0.0009	
0.08	1.4299	1.4315	1.4324	0.0009	
0.09	1.4645	1.4648	1.4650	0.0002	
1.00	1.4749	1.4752	1.4755	0.0003	
1.20	7.7020	7.7025	7.7030	0.0005	
1.30	7.8880	7.8890	7.8890	0.0009	
1.40	7.9665	7.9668	7.9671	0.0003	
1.50	8.2707	8.2709	8.2711	0.0002	
1.60	8.4675	8.4685	8.4694	0.0009	
1.70	8.6674	8.6685	8.6695	0.0010	0.0010
1.80	8.7355	8.7359	8.7363	0.0004	
1.90	9.0241	9.0243	9.0245	0.0002	
2.00	2.0000	2.0000	2.0000	0.0000	

In Table 3, shows that the absolute error ranges from 0.0001 to maximum value of 0.0010, also we should remember that the values at the extremes remain fixed as boundary conditions. The error variations versus horizontal axis for the present case are plotted in Figure 2. As we see the error bounds decreased significantly due to the apparent decrease of the parameter ε .

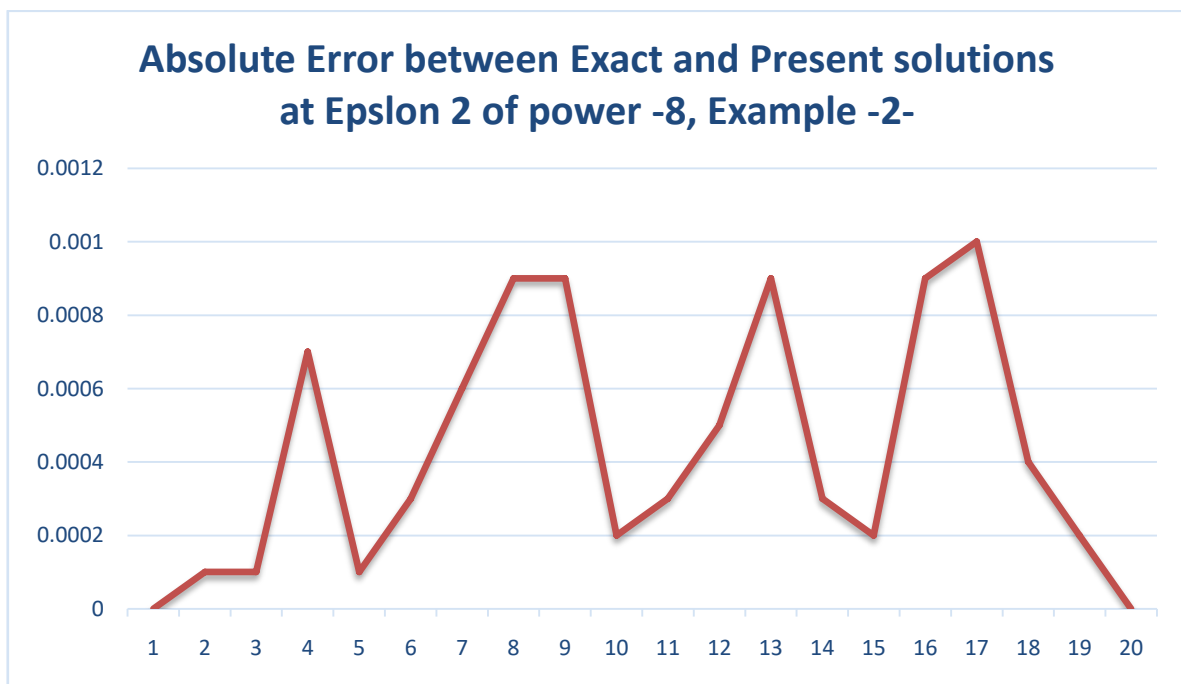


Figure 3. Error variations for test problem 2, case 1

3.2.2. Results and analysis of test problem 2, case 2

The first case results for $\varepsilon = 2^{-10}$ are shown in Table 2, where a comparison between the exact and the computed results due to the proposed method.

Table 4. Test problem 2, case II- comparison between the exact and present

t	$g(t)$	Exact solution	Present solution	Absolute Error	Min & Max
0	1.000	1.000	1.00000	0.00000	
0.01	1.0049	1.0050	1.00501	0.00001	0.00001
0.02	1.0099	1.0100	1.01001	0.00001	
0.03	1.0399	1.0496	1.04967	0.00007	
0.04	1.0799	1.0810	1.08109	0.00009	0.00009
0.05	1.2049	1.2052	1.20523	0.00003	
0.06	1.3249	1.3259	1.32595	0.00005	
0.07	1.3889	1.3898	1.38983	0.00003	
0.08	1.4299	1.4315	1.43153	0.00003	
0.09	1.4645	1.4648	1.46482	0.00002	
1.00	1.4749	1.4752	1.47523	0.00003	
1.20	7.7020	7.7025	7.70255	0.00005	
1.30	7.8880	7.8890	7.88901	0.00001	
1.40	7.9665	7.9668	7.96683	0.00003	
1.50	8.2707	8.2709	8.27092	0.00002	
1.60	8.4675	8.4685	8.46851	0.00001	
1.70	8.6674	8.6685	8.66851	0.00001	
1.80	8.7355	8.7359	8.73594	0.00004	
1.90	9.0241	9.0243	9.02432	0.00002	
2.00	2.0000	2.0000	2.00000	0.00000	

Based on Table 4, it is clear that the absolute error ranges from 0.00001 to maximum value of 0.00009, also we should remember that the values at the extremes remain fixed as boundary conditions. The error variations versus horizontal axis for the present case are plotted in Figure 2. As we see the error bounds decreased significantly due to the apparent decrease of the parameter ε .

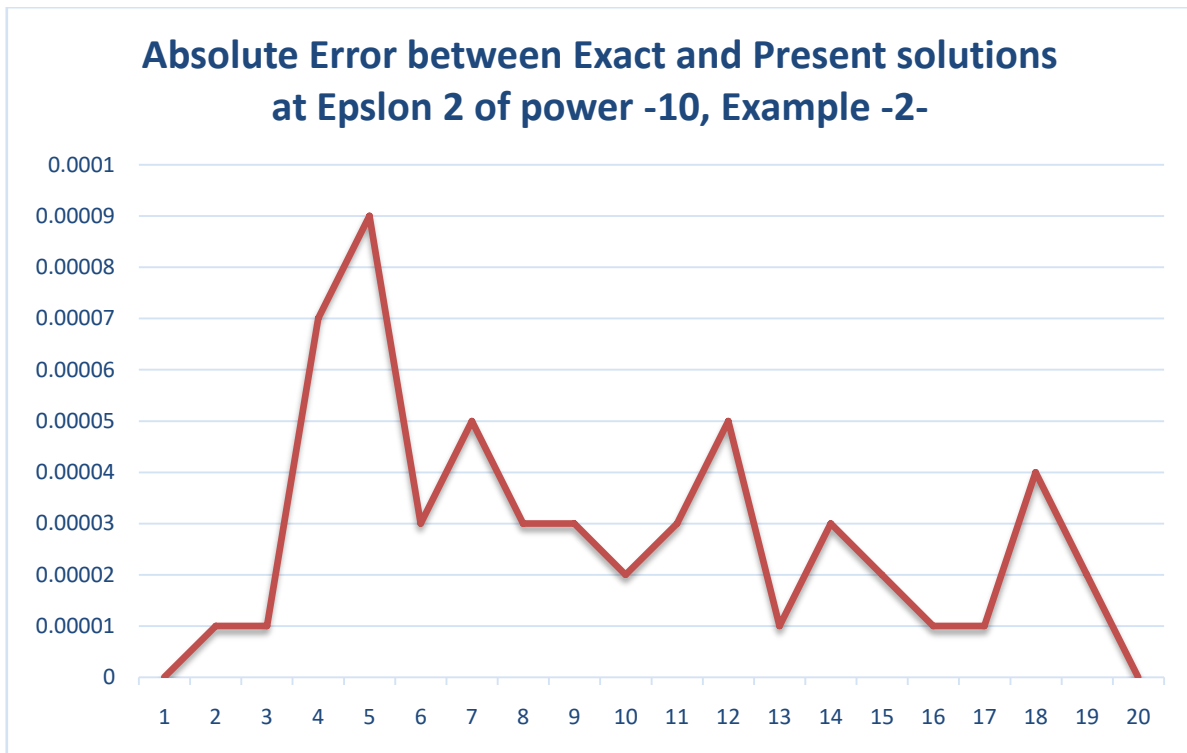


Figure 4. Error variations for test problem 2, case 2.

Table 5. Overall Max and Min errors

Error type	Test problem (1)		Test problem (2)	
	Case (1)	Case (2)	Case (1)	Case (2)
Minimum error	0.0001	0.00001	0.0001	0.00001
Maximum error	0.0016	0.00009	0.0010	0.00009

From Table 5, it is clear that the parameter ϵ has direct effect on decreasing the error significantly, meanwhile, even $\epsilon = 2^{-8}$ gave a good approximation, and it can be used well.

4. Conclusion

In this paper, the reduction of order method has presented and illustrated for solving singular perturbed D.D.E. The computed results are computed numerically by solving a pair of initial value problems, which are deduced from the original problem. The paper also has implemented the present method on two examples. The classical fourth order Runge-Kuttamethod have used to solve the IVPs. Taking into account that we can use any standard analytical or numerical method. It can be observed from the tables that present method agrees very well with exact solution, which shows the method's effectiveness.

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