Radius and Diameter of F-Average Eccentric Graph of Graphs

T. Sathiyanandham[1], S. Arockiaraj[2]

[1]Department of Mathematics, Government Arts and Science College

Sivakasi, Tamilnadu, India. sathiyan.maths@gmail.com

[2]Department of Mathematics, Government Arts and Science College

Sivakasi, Tamilnadu, India. psarockiaraj@gmail.com

Article Info
Page Number: 4717-4726
Publication Issue:
Vol. 71 No. 4 (2022)

Article History
Article Received: 25 March 2022
Revised: 30 April 2022
Accepted: 15 June 2022
Publication: 19 August 2022

Abstract
The F-average eccentric graph $AE_F(G)$ of a graph G has the vertex set as in G and any two vertices u and v are adjacent in $AE_F(G)$ if either they are at a distance $\left\lfloor\frac{e(u)+e(v)}{2}\right\rfloor$ while G is connected or they belong to different components while G is disconnected. In this paper, the radius and diameter of $AE_F(G)$ have been analysed and find the solutions for $AE_F(G) = AE_F(\overline{G})$ and $AE_F(G) = \overline{AE_F(\overline{G})}$.

Keywords: F-average eccentric vertex, F-average eccentric graph.

1. Introduction

Throughout this paper, a graph means a non trivial simple graph. For other graph theoretic notation and terminology, we follow [7,8,9]. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. $d(v)$ denotes the degree of a vertex $v \in V(G)$, the order of G is $|V(G)|$ and the size is $|E(G)|$. The distance $d(u,v)$ between a pair of vertices u and v is the length of a shortest path joining them. The eccentricity $e(u)$ of a vertex u is the distance to a vertex farthest from u. The radius $r(G)$ of G is the minimum eccentricity among the eccentricities of the vertices of G and the diameter $d(G)$ of G is the maximum eccentricity among the eccentricities of the vertices of G. A vertex v is called a peripheral vertex of G if $e(v) = d(G)$. A vertex v is called an eccentric vertex of a vertex u if $d(u,v) = e(u)$. A vertex v of G is called an eccentric vertex of G if it is the eccentric vertex of some vertex of G. Let $S_i(G)$ denote a subset of the vertex set of G such that $e(u) = i$ for all $u \in V(G)$. The concept of antipodal graph was initially introduced by Singleton [1] and was further expanded by Aravamuthan and Rajendran [3,4]. The antipodal graph of a graph G, denoted by $A(G)$, is the graph on the same vertices as of G, two vertices being adjacent if the distance between them is equal to the diameter of G. The concept of eccentric graph was introduced by Akiyama et al. [2]. The eccentric graph based on G is denoted by G_e whose vertex set is $V(G)$ and two vertices u and v are adjacent in G_e if $d(u,v) = \min\{e(u), e(v)\}$. The concept of radial graph was introduced by Kathiresan and Marimuthu [5]. The radial graph $R(G)$ based on G has the vertex set as in G and two vertices are adjacent if the distance between them is equal to the radius of G while G is connected. If G is disconnected, then two vertices are adjacent in $R(G)$ if they belong to different components of G. Sathiyanandham and Arockiaraj introduced a new graph, called F-average eccentric graph [6]. Two vertices u and v of a graph are said to be F-average eccentric to each other if $d(u,v) = \left\lfloor \frac{e(u)+e(v)}{2} \right\rfloor$. The F-average eccentric graph of a graph G, denoted by $AE_F(G)$, has the vertex set as in G and any two vertices u and v are adjacent in $AE_F(G)$ if either they are at a distance $d(u,v) = \left\lfloor \frac{e(u)+e(v)}{2} \right\rfloor$ while G is connected or they belong to different components while G is disconnected. In this paper, the radius and diameter of $AE_F(G)$ has been analysed and find the solutions for $AE_F(\overline{G}) = AE_F(G)$ and $AE_F(\overline{G}) = \overline{AE_F(G)}$.

Let $F_{11}, F_{12}, F_{22}, F_{23}, F_{24}, F_3$ denote the set of all connected graphs G for which $r(G) = d(G) = 1$, $r(G) = 1$ and $d(G) = 2$, $r(G) = d(G) = 2$, $r(G) = 2$ and $d(G) = 3$, $r(G) = 2$ and $d(G) = 4$, $r(G) \geq 3$ respectively and F_4 denote the set of all disconnected graphs. Next we provide some results which will be used to prove some theorems in this paper.

Theorem A[8] If G is a simple graph with diameter at least 3, then $\overline{G}$ has diameter at most 3.

Theorem B[8] If G is a simple graph with diameter at least 4, then $\overline{G}$ has diameter at most 2.

Theorem C[8] If G is a simple graph with radius at least 3, then $\overline{G}$ has radius at most 2.

Theorem D[7] If G is a self centered graph with radius at least 3, then $\overline{G}$ is a self centered graph of radius 2.

Theorem E[6] Let G_i be a connected graph with r_i vertices, for $i = 1,2,\ldots,n$. If G is the union of $G_1, G_2, \ldots, G_n$, then $AE_F(G) = K_{r_1,r_2,\ldots,r_n}$.

Theorem F[6] Let G be a graph on n vertices. Then a vertex is a full degree vertex in $AE_F(G)$ if and only if either it is an isolated vertex or a full degree vertex or a non full degree vertex adjacent to the full degree vertices only in G.

Theorem G[6] Let G be a graph. Then $AE_F(G) = K_m + K_{r_1,r_2,\ldots,r_l}$ for any positive integers m, l, r_i and $1 \leq i \leq l$ if and only if any one of the following holds

(1) G is disconnected with exactly $l + m$ components and it has at least m isolated vertices

(2) G is connected and it has m full degree vertices so that the deletion of these full degree vertices in G forms a disconnected graph with l components in which each component is complete.

Theorem H[6] If $r(G) \geq 2$, then $AE_F(G) \subseteq \overline{G}$.

Theorem I[6] Let G be a graph of order n. Then $AE_F(G) = G$ if and only if $G \in F_{11}$.

Theorem J[6] Let G be a graph and let $S_3(G)$ be the set of all vertices of $V(G)$ whose eccentricities are 3. Then $AE_F(G) = \overline{G}$ if and only if any one of the following conditions hold

(1) $G \in F_{22}$

(2) $G \in F_{23}$ and there is no vertex adjacent to atleast two non adjacent vertices in $S_3(G)$

(3) G is disconnected in which each component is complete.

2. Radius and diameter of F-average eccentric graph of graphs

In this section, we discuss about the radius and diameter of the graphs obtained from the F-average eccentric operation.

Proposition 2.1. Let $G \in F_4$. Then $AE_F(G) \in F_{11}$ if and only if G is totally disconnected.

Proof. By Theorem G, the result follows. □

Proposition 2.2. Let $G \in F_4$. Then $AE_F(G) \in F_{22}$ if and only if G has no isolated vertex.

Proof. Suppose $AE_F(G) \in F_{22}$. If G has an isolated vertex, then by the definition, $AE_F(G)$ has a full degree vertex. Hence $AE_F(G) \in F_{11} \cup F_{12}$, a contradiction.

If G has no isolated vertex, then by Theorem E, $AE_F(G) \in F_{22}$. □

Theorem 2.3. Let $G \in F_4$. Then $AE_F(G) \in F_{12}$ if and only if G has at least one isolated vertex and a non trivial component.

Proof. Suppose G has at least one isolated vertex and a non trivial component. Then by the definition, $AE_F(G) \in F_{12}$.

Suppose $AE_F(G) \in F_{12}$. If G has no isolated vertex, then by Proposition 2.2, $AE_F(G) \in F_{22}$, a contradiction. If G is a totally disconnected graph, then by Proposition 2.1, $AE_F(G) \in F_{11}$, a contradiction. Thus G has at least one isolated vertex and a non trivial component.

□ **Proposition 2.4.** Let G be a connected graph on n vertices. Then $AE_F(G) \in F_{11}$ if and only if either $G \in F_{11}$ or $G \in F_{12}$ in which no two non full degree vertices are mutually adjacent.

Proof. By Theorem G, the result follows. □

Proposition 2.5. Let G be a connected graph on n vertices. Then $AE_F(G) \in F_{12}$ if and only if $G \in F_{12}$ and at least one pair of non full degree vertices in G are mutually adjacent.

Proof. By Theorem F and Theorem G, the result follows. □

Theorem 2.6. Let G be a connected graph with no full degree vertex and $\overline{G} \in F_4$. Then $AE_F(G) \in F_4$ if and only if $G \in F_{22}$.

Proof. Suppose $AE_F(G) \in F_4$. If $G \in F_{23}$, then by Theorem A, $\overline{G} \in F_{22} \cup F_{23}$, a contradiction. If $G \in F_{24} \cup F_3$, then by Theorem B and Theorem C, $\overline{G} \in F_{22}$, a contradiction. Hence $G \in F_{22}$.

Since G has no full degree vertex, $r(G) \geq 2$ and by Theorem H, $AE_F(G) \subseteq \overline{G}$. If $G \in F_{22}$, then by Theorem J, $AE_F(G) = \overline{G} \in F_4$. □

Theorem 2.7. Let $G \in F_{22}$. Then $AE_F(G) \in F_{22}$ (or F_{23} or F_{24} or F_3 or F_4) if and only if $\overline{G} \in F_{22}$ (or F_{23} or F_{24} or F_3 or F_4, respectively).

Proof. By Theorem J, the results follows. □

Theorem 2.8. Let $G \in F_{23}$ and has no vertex adjacent to atleast two non adjacent vertices in $S_3(G)$. Then $AE_F(G) \in F_{22}$ (or F_{23}) if and only if $\overline{G} \in F_{22}$ (or F_{23}, respectively).

Proof. By Theorem J, the results follows. □

Proposition 2.9. If $G \in F_{23}$ in which there is a vertex u such that u is adjacent to atleast two non adjacent pairs of vertices w_1 and w_2 in $S_3(G)$, then $AE_F(G) \in F_{23} \cup F_3$.

Proof. If $AE_F(G) \in F_{11} \cup F_{12}$, then by Proposition 2.4 and Proposition 2.5, $G \in F_{11} \cup F_{12}$, a contradiction. Suppose $AE_F(G) \in F_{22} \cup F_{24} \cup F_4$. Since u is adjacent to both non adjacent pairs of vertices w_1 and w_2 in $S_3(G)$, both w_1 and w_2 are adjacent to its antipodal vertices w of G in $AE_F(G)$. Let x be the adjacent vertices of u in G. Then both w_1 and w_2 are adjacent to x and u is adjacent to w in $AE_F(G)$. So $d(u,x) = 3$ in $AE_F(G)$ and hence $e(u) = 3 = e(x)$ in $AE_F(G)$, a contradiction to $AE_F(G) \in F_{22}$. For every pair of antipodal vertex and its eccentric vertex, there exists a shortest path P_4 in G and $AE_F(P_4) = \overline{P_4} \in F_{23}$. So the eccentricities of the adjacent vertices of the vertex v in $S_2(G)$ are 2 or 3 and $e(v) = 3$ in $AE_F(G)$. Also the eccentricities of the adjacent vertices of the antipodal vertex w are 2 or 3 and $e(w) = 2$ or 3 in $AE_F(G)$. Hence $AE_F(G) \in F_{23}$, a contradiction to $AE_F(G) \in F_{22} \cup F_{24} \cup F_4$. Thus $AE_F(G) \in F_{23} \cup F_3$. □

Lemma 2.10. If $AE_F(G) \in F_{24}$, then $G \in F_{22}$.

Proof. Let $u_2^{(l)}$ and $u_4^{(l)}$ be two adjacent vertices of the peripheral vertex $u_1^{(l)}$ and its eccentric vertex $u_5^{(l)}$ in $AE_F(G)$ respectively, where l is the positive integer. Then there exists a shortest path $P_5^{(l)}$ between $u_1^{(l)}$ and $u_5^{(l)}$ in $AE_F(G)$. Let $P_5^{(l)}: u_1^{(l)} u_2^{(l)} u_3^{(l)} u_4^{(l)} u_5^{(l)}$ be a shortest path between $u_1^{(l)}$ and $u_5^{(l)}$. Then $e(u_1^{(l)}) = 4 = e(u_5^{(l)})$, $e(u_2^{(l)}) = 3 = e(u_4^{(l)})$ and $e(u_3^{(l)}) = 2$ in $AE_F(G)$. Since $AE_F(\overline{P_5^{(l)}}) = P_5^{(l)}$, $\overline{P_5^{(l)}}$ in G. So $e(u_i^{(l)}) = 2$ for all $u_i^{(l)} \in \overline{P_5^{(l)}}$, $1 \le i \le 5$. For each vertex u in $AE_F(G) - \{u_1^{(l)}, u_2^{(l)}, u_3^{(l)}, u_4^{(l)}, u_5^{(l)}\}$ and any one of its peripheral vertices in $AE_F(G)$, there is a shortest path $P_h^{(m)}$ in $AE_F(G)$ where m is the positive integer and $h = 3,4,5$. Let $P_h^{(m)}: u_1^{(m)} u_2^{(m)} \ldots u_h^{(m)}$ be a shortest path between $u_1^{(m)}$ and $u_h^{(m)}$ in $AE_F(G)$ of length $h-1$. Since $AE_F(\overline{P_h^{(m)}}) = P_h^{(m)}$ for $h = 3,4,5$, $\overline{P_h^{(m)}}$ in G for $h = 3,4,5$. Also $u_i^{(l)} u_j^{(m)} \in G$ and $u_i^{(l)} \neq u_j^{(m)}$ whenever $u_i^{(l)} u_j^{(m)} \not\subseteq AE_F(G)$ for $1 \le i \le 5$, $1 \le j \le h$ and $l \neq m$. If $u_i^{(l)}$ and $u_j^{(m)}$ are adjacent in $AE_F(G)$, then $u_i^{(l)} u_k^{(m)} \not\subseteq G$ for $k \neq j$, $1 \le j,k \le h$, $1 \le i \le 5$ and $l \neq m$. This implies that $e(v) = 2$ for all $v \in V(G)$. Thus $G \in F_{22}$.

Proposition 2.11. If $G \in F_3$, then $AE_F(G) \in F_3 \cup F_4$.

Proof. If $AE_F(G) \in F_{11} \cup F_{12}$, then by Proposition 2.4 and Proposition 2.5, $G \in F_{11} \cup F_{12}$, a contradiction. If $AE_F(G) \in F_{24}$, then by Lemma 2.10, $G \in F_{22}$, a contradiciton.

Case 1. $AE_F(G) \in F_{22}$. Then $AE_F(G)$ has no pendant vertex.

Case 1.1. Suppose $AE_F(G)$ has a triangle. Let uvw be a triangle in $AE_F(G)$. Since $uv \in E(AE_F(G))$, there is a shortest path between u and v of length $\left\lfloor\frac{e(u)+e(v)}{2}\right\rfloor$ in G.

Case 1.1.1. Suppose $e(u)+e(v)$ is even and $\left\lfloor\frac{e(u)+e(v)}{2}\right\rfloor = M$. If $e(u) \neq e(v)$, then $d_G(u,v) < M$. So $e(u) = M = e(v)$. Let $P_1: ux_1x_2 \ldots x_{m-1}v$ be a shortest path between u and v in G of length M. Since $vw \in E(AE_F(G))$, $w \neq u$ and $w \neq x_{m-1}$. So $w = x_1$ and $uw \in E(G)$. Since $uw \in E(AE_F(G))$, either u or w is a full degree vertex in G, a contradiction.

Case 1.1.2. Suppose $e(u)+e(v)$ is odd and $\left\lfloor\frac{e(u)+e(v)}{2}\right\rfloor = M-1$. In this case, the eccentricity of any one of u, v is $M-1$. Let $e(u) = M-1$. Then $e(v) = M$ and $e(w) = M$. So $e(v)+e(w)$ is even. By Case 1.1.1, a contradiction arises.

Case 1.2. Suppose $AE_F(G)$ has no triangle. Then every vertex in $AE_F(G)$ is adjacent to non adjacent pair of vertices in $AE_F(G)$. Let v_2 be a vertex in $AE_F(G)$ such that v_2 is adjacent to non adjacent pair of vertices v_1 and v_3 in $AE_F(G)$. Since $d(v_3) \geq 2$ in $AE_F(G)$, there exists a vertex v_4 such that v_4 is adjacent to v_3 in $AE_F(G)$. Since $v_2v_3 \in E(AE_F(G))$, ther exists a shortest path between v_2 and v_3 in G. let $P_3: v_2y_1y_2 \ldots y_{m-1}v_3$ be a shortest path between v_2 and v_3 in G of length M. Since $v_1v_2 \in E(AE_F(G))$, $v_1 \neq v_3$ and $v_1 \neq y_1$. So $v_1 = y_{m-1}$ and $e(y_{m-1}) = M-1$ in G. Since $v_3v_4 \in E(AE_F(G))$, $v_4 \neq v_2$ and $v_4 \neq y_{m-1}$. So $v_4 = y_1$ and $e(y_1) = M-1$ in G. Therefore y_i is not adjacent to v_j for $2 \leq i \leq M-2$ and $1 \leq j \leq 4$. Hence G has at least two components, a contradiction to $G \in F_3$. If $M-1 = 2$, then $v_4v_2v_3v_1$ is a path in G and hence $G \in F_{23}$, a contradiction.

Case 2. $AE_F(G) \in F_{23}$. Then there is a path $P'_1: u_1u_2u_3u_4$ in $AE_F(G)$ of length 3 so that their eccentricities are 3,2,2,3 respectively. Since $AE_F(\overline{P'_1}) = P'_1$, $\overline{P'_1}: u_2u_4u_1u_3$ is in G. So $e_G(u_2) = 3 = e_G(u_3)$ and $e_G(u_4) = 2 = e_G(u_1)$. For a peripheral vertex and its eccentric vertex in $AE_F(G)$, there is a shortest path P''_1 in $AE_F(G)$ of length 3. Let $P''_1: u'_1u'_2u'_3u'_4$ be a shortest path between u'_1 and u'_4 in $AE_F(G)$ of length 3. Since $AE_F(\overline{P''_1}) = P''_1$, $\overline{P''_1}: u'_2u'_4u'_1u'_3$ is in G. Also $u_iu'_j \in G$ and $u_i \neq u'_j$ whenever $u_iu'_j \not\in AE_F(G)$ for $1 \leq i,j \leq 4$. If u_i and u'_j are adjacent in $AE_F(G)$, $u_iu'_k \in G$ for $k \neq j, 1 \leq i,j \leq 4$. This implies that $e(v) = 2$ in G for each peripheral vertex v in $AE_F(G)$, a contradiction. □

Proposition 2.12. Let G be a graph such that $d(G) - r(G) \geq 2$. Then $AE_F(G) \in F_4$.

Proof. Suppose $G \in F_{24}$. Then the non adjacent vertices of $S_2(G)$ are adjacent between them in $AE_F(G)$. Also by the definition, no vertices of $S_2(G)$ are adjacent to the remaining vertices of G in $AE_F(G)$. Hence $AE_F(G) \in F_4$. Suppose $G \in F_3$ and $d(G) - r(G) \geq 2$. Then the peripheral vertices and its F-average eccentric vertices are adjacent only in $AE_F(G)$ and the remaining vertices of G are not adjacent to those vertices in $AE_F(G)$. Hence $AE_F(G) \in F_4$. □

Theorem 2.13. Let G be a graph such that $d(G) - r(G) \geq 2$. Then $AE_F(G) \in F_4$ if and only if $G \in F_{24} \cup F_3$.

Proof. Suppose $AE_F(G) \in F_4$. If $G \not\in F_{24} \cup F_3$, then $G \in F_{11} \cup F_{12} \cup F_{22} \cup F_{23}$. This implies that $d(G) - r(G) < 2$, a contradiction. Hence $G \in F_{24} \cup F_3$.

Suppose $G \in F_{24} \cup F_3$. Then by Proposition 2.12, $AE_F(G) \in F_4$. □

Proposition 2.14. Let G be a connected graph on n vertices. Then $r(AE_F(G)) = 1$ if and only if $r(G) = 1$.

Proof. The result follows from Proposition 2.4 and Proposition 2.5. □

Theorem 2.15. Let G be a graph on n vertices. Then $AE_F(G) \in F_{22}$ if and only if any one of the following conditions hold

(1) $G \in F_{22}$ and $\overline{G} \in F_{22}$

(2) $G \in F_{23}$ in which there is no vertex adjacent to at least two non adjacent vertices in $S_3(G)$ and $\overline{G} \in F_{22}$.

(3) $G \in F_4$ and it has no isolated vertex.

Proof. If $G \in F_4$ and it has no isolated vertex, then by Proposition 2.2, $AE_F(G) \in F_{22}$. If G has at least one isolated vertex, then by Proposition 2.1 and Theorem 2.3, $AE_F(G) \in F_{11} \cup F_{12}$ and hence $AE_F(G) \not\in F_{22}$. So $r(G) \geq 1$. By Proposition 2.14, $r(G) = 1$ if and only if $r(AE_F(G)) = 1$. In this case, $AE_F(G) \not\in F_{22}$. If $G \in F_{22}$ and $\overline{G} \in F_{22}$, then by Theorem 2.7, $AE_F(G) \in F_{22}$. Also by Theorem 2.7, $AE_F(G) \not\in F_{22}$ if $\overline{G} \not\in F_{22}$. If $G \in F_{23}$ in which there is no vertex adjacent to at least two non adjacent vertices in $S_3(G)$ and $\overline{G} \in F_{22}$, then by Theorem 2.8, $AE_F(G) \in F_{22}$. Also by Theorem 2.8, $AE_F(G) \not\in F_{22}$ if $\overline{G} \not\in F_{22}$. If $G \in F_{23}$ in which there is a vertex u such that u is adjacent to atleast two non adjacent pairs of vertices w_1 and w_2 in $S_3(G)$, then by Proposition 2.9, $AE_F(G) \in F_{23} \cup F_3$. So $AE_F(G) \not\in F_{22}$. If $G \in F_{24}$, then by Proposition 2.12, $AE_F(G) \in F_4$ and hence $AE_F(G) \not\in F_{22}$. If $G \in F_3$, then by Proposition 2.11, $AE_F(G) \in F_3 \cup F_4$. So $AE_F(G) \not\in F_{22}$. By considering all the cases, the result follows. □

Theorem 2.16 Let G be a graph on n vertices. Then $AE_F(G) \in F_{23}$ if and only if any one of the following conditions hold

(1) $G \in F_{22}$ and $\overline{G} \in F_{23}$

(2) $G \in F_{23}$ in which there is no vertex adjacent to at least two non adjacent vertices in $S_3(G)$ and $\overline{G} \in F_{23}$.

Proof. If $G \in F_4$, then by Proposition 2.1, Proposition 2.2 and Theorem 2.3, $AE_F(G) \in F_{11} \cup F_{12} \cup F_{22}$ and hence $AE_F(G) \not\in F_{23}$. So $r(G) \geq 1$. By Proposition 2.14, $r(G) = 1$ if and only if $r(AE_F(G)) = 1$. In this case, $AE_F(G) \not\in F_{23}$. If $G \in F_{22}$ and $\overline{G} \in F_{23}$, then by Theorem 2.7, $AE_F(G) \in F_{23}$. Also by Theorem 2.7, $AE_F(G) \not\in F_{23}$ if $\overline{G} \not\in F_{23}$. If $G \in F_{23}$ in which there is no vertex adjacent to at least two non adjacent vertices in $S_3(G)$ and $\overline{G} \in F_{23}$, then by Theorem 2.8, $AE_F(G) \in F_{23}$. Also by Theorem 2.8, $AE_F(G) \not\in F_{23}$ if $\overline{G} \not\in F_{23}$. If $G \in F_{24}$, then by Proposition 2.12, $AE_F(G) \in F_4$. So $AE_F(G) \not\in F_{23}$. If $G \in F_3$, then by Proposition 2.11, $AE_F(G) \in F_3 \cup F_4$ and hence $AE_F(G) \not\in F_{23}$. By considering all the cases, the result follows. □

Theorem 2.17. Let G be a graph on n vertices. Then $AE_F(G) \in F_{24}$ if and only if $G \in F_{22}$ and $\overline{G} \in F_{24}$.

Proof. If $G \in F_4$, then by Proposition 2.1, Proposition 2.2 and Theorem 2.3, $AE_F(G) \in F_{11} \cup F_{12} \cup F_{22}$ and hence $AE_F(G) \not\in F_{24}$. So $r(G) \geq 1$. By Proposition 2.14, $r(G) = 1$ if and only if $r(AE_F(G)) = 1$. In this case, $AE_F(G) \not\in F_{24}$. If $G \in F_{22}$ and $\overline{G} \in F_{24}$, then by Theorem 2.7, $AE_F(G) \in F_{24}$. Also by Theorem 2.7, $AE_F(G) \not\in F_{24}$ if $\overline{G} \not\in F_{24}$. If $G \in F_{23}$, then by Theorem 2.8 and Proposition 2.9, $AE_F(G) \in F_{22} \cup F_{23} \cup F_3$ and hence $AE_F(G) \not\in F_{24}$. If $G \in F_{24}$, then by Proposition 2.12, $AE_F(G) \in F_4$. So $AE_F(G) \not\in F_{24}$. If $G \in F_3$, then by Proposition 2.11, $AE_F(G) \in F_3 \cup F_4$ and hence $AE_F(G) \not\in F_{24}$. By considering all the cases, the result follows.

Theorem 2.18. Let G be a graph on n vertices. Then $AE_F(G) \in F_3 \cup F_4$ if and only if any one of the following conditions hold

(1) $G \in F_{22}$ and $\overline{G} \in F_3 \cup F_4$

(2) $G \in F_{24}$

(3) $G \in F_3$.

Proof. If $G \in F_4$, then by Proposition 2.1, Proposition 2.2 and Theorem 2.3, $AE_F(G) \in F_{11} \cup F_{12} \cup F_{22}$ and hence $AE_F(G) \not\in F_3 \cup F_4$. So $r(G) \geq 1$. By Proposition 2.14, $r(G) = 1$ if and only if $r(AE_F(G)) = 1$. In this case, $AE_F(G) \not\in F_3 \cup F_4$. If $G \in F_{22}$ and $\overline{G} \in F_3 \cup F_4$, then by Theorem 2.7, $AE_F(G) \in F_3 \cup F_4$. Also by Theorem 2.7, $AE_F(G) \not\in F_3 \cup F_4$ if $\overline{G} \not\in F_3 \cup F_4$. If $G \in F_{24}$, then by Proposition 2.12, $AE_F(G) \in F_4$. If $G \in F_3$, then by Proposition 2.11, $AE_F(G) \in F_3 \cup F_4$. By considering all the cases, the result follows. □

3. The graph G and its complement $\overline{G}$ satisfying $AE_F(G) = AE_F(\overline{G})$ and $AE_F(G) = \overline{AE_F(\overline{G})}$

In this section, we analyse the graphs in which the F-average eccentric graph of graph and its complement are one and the same. Also we analyse the graph G so that the complement of F-average eccentric graph of $\overline{G}$ is same as F-average eccentric graph of graph G.

Theorem 3.1. Let G be a graph on n vertices. Then $AE_F(G) = AE_F(\overline{G})$ if and only if $G \in F_{11}$.

Proof. If $G \in F_{11}$, then by Theorem I, $AE_F(G) = G = K_n$. Also $\overline{G}$ is totally disconnected and hence $AE_F(\overline{G}) = K_n = AE_F(G)$.

Suppose $AE_F(G) = AE_F(\overline{G})$. Let G be a graph with $r(G) \geq 2$. Then $AE_F(G) \subseteq \overline{G}$. Therefore $AE_F(\overline{G}) \subseteq \overline{G}$. Since $r(G) \geq 2$, $r(\overline{G}) \geq 2$. Therefore there exists a pair of antipodal vertices u and v in $\overline{G}$ such that $uv \not\in E(\overline{G})$ and $d_{\overline{G}}(u,v) = d(\overline{G}) = \left\lfloor \frac{e(u)+e(v)}{2} \right\rfloor$. Hence $uv \in AE_F(\overline{G})$ but $uv \not\in E(\overline{G})$, a contradiction. Hence $r(G) = 1$. So either $G \in F_{11}$ or $G \in F_{12}$. Suppose $G \in F_{12}$. Let v be a full degree vertex and u, w be two non adjacent vertices in G. Then by the definition uvw is a triangle in $AE_F(G)$. Since v is an isolated vertex in $\overline{G}$, v is a full degree vertex in $AE_F(\overline{G})$. Since $uw \not\in E(G)$, $uw \in E(\overline{G})$ and hence u and w are in the same component of the disconnected graph $\overline{G}$. So $uw \not\in AE_F(\overline{G})$, a contradiction. □

Theorem 3.2. Let G be a graph on n vertices. Then $AE_F(G) = \overline{AE_F(\overline{G})}$ if and only if any one of the following will hold

(1) $G \in F_{22}$ and $\overline{G} \in F_{22}$

(2) $G \in F_{22}$ and $\overline{G} \in F_{23}$ in which there is no vertex adjacent to at least two non adjacent vertices in $S_3(\overline{G})$

(3) $G \in F_{22}$ and $\overline{G}$ is disconnected in which each component is complete

(4) $G \in F_{23}$ in which there is no vertex adjacent to at least two non adjacent vertices in $S_3(G)$ and $\overline{G} \in F_{22}$

(5) $G \in F_{23}$ in which there is no vertex adjacent to at least two non adjacent vertices in $S_3(G)$ and $\overline{G} \in F_{23}$ in which there is no vertex adjacent to at least two non adjacent vertices in $S_3(\overline{G})$

(6) $G \in F_4$ in which each component is complete and $\overline{G} \in F_{22}$

Proof. Suppose G is connected. Then by Theorem 3.1, $G \in F_{11}$ if and only if $AE_F(G) = AE_F(\overline{G})$. In this case, $AE_F(G) \neq \overline{AE_F(\overline{G})}$. If $G \in F_{12}$, then G has a full degree vertex say v. Let u and w be two non adjacent vertices in G. Then by the definition, uvw is a triangle in $AE_F(G)$. Since v is an isolated vertex in $\overline{G}$ and uw is an edge in $\overline{G}$, vu and $vw \in E(AE_F(\overline{G}))$ but $uw \not\in E(AE_F(\overline{G}))$. So $AE_F(G) \neq \overline{AE_F(\overline{G})}$.

Case 1. $G \in F_{22}$. Then $\overline{G} \in F_{22} \cup F_{23} \cup F_{24} \cup F_3 \cup F_4$. If $\overline{G} \in F_{22}$, then by Theorem J, $AE_F(\overline{G}) = G$ and $AE_F(G) = \overline{G}$. Hence $AE_F(G) = \overline{AE_F(\overline{G})}$. Suppose $\overline{G} \in F_{23}$. If $\overline{G}$ has a vertex w such that w is adjacent to at least two non adjacent pairs of vertices u and v in

$S_3(\overline{G})$, then by Theorem J, $AE_F(\overline{G}) \neq G$. Hence $AE_F(G) \neq \overline{AE_F(\overline{G})}$. If $\overline{G}$ has no vertex adjacent to at least two non adjacent vertices in $S_3(\overline{G})$, then by Theorem J, $AE_F(\overline{G}) = G$. Also $AE_F(G) = \overline{G}$ since $G \in F_{22}$. Hence $AE_F(G) = \overline{AE_F(\overline{G})}$. Suppose $\overline{G} \in F_{24} \cup F_3$. Let $u \in V(\overline{G})$. Then u is adjacent to all the vertices v in G such that $d_{\overline{G}}(u,v) \geq 2$. since $r(\overline{G}) \geq 2$, the vertex u and its non F-average eccentric vertices w in $\overline{G}$ are adjacent in G. Then $uw \notin E(AE_F(G))$ and $uw \notin E(AE_F(\overline{G}))$. Hence $AE_F(G) \neq \overline{AE_F(\overline{G})}$. Suppose $\overline{G} \in F_4$. If $\overline{G}$ has at least one non complete component, then by Theorem J, $AE_F(\overline{G}) \neq G$. Since $G \in F_{22}$, $AE_F(G) = \overline{G}$ and hence $AE_F(G) \neq \overline{AE_F(\overline{G})}$. If $\overline{G}$ has no non complete component, then by Theorem J, $AE_F(G) = \overline{G}$ and hence $AE_F(G) = \overline{AE_F(\overline{G})}$.

Case 2. $G \in F_{23}$. Then $\overline{G} \in F_{22} \cup F_{23}$. Suppose $G \in F_{23}$ in which there is no vertex adjacent to at least two non adjacent vertices in $S_3(G)$. Then by Theorem J, $AE_F(G) = \overline{G}$. If $\overline{G} \in F_{22}$, then by Theorem J, $AE_F(\overline{G}) = G$. So $AE_F(G) = \overline{AE_F(\overline{G})}$. If $\overline{G} \in F_{23}$ in which there is no vertex adjacent to at least two non adjacent vertices in $S_3(\overline{G})$, then by Theorem J, $AE_F(\overline{G}) = G$ and hence $AE_F(G) = \overline{AE_F(\overline{G})}$. If $\overline{G} \in F_{23}$ in which there is a vertex w' such that w' is adjacent to at least two non pairs of vertices u' and v' in $S_3(\overline{G})$, then by Theorem J, $AE_F(\overline{G}) \neq G$. Hence $AE_F(G) \neq \overline{AE_F(\overline{G})}$.

Suppose $G \in F_{23}$ in which there is a vertex w such that w is adjacent to at least two non adjacent pairs of vertices u and v in $S_3(G)$. Then by Theorem J, $AE_F(G) \neq \overline{G}$. If $\overline{G} \in F_{22}$, then by Theorem J, $AE_F(\overline{G}) = G$. So $AE_F(G) \neq \overline{AE_F(\overline{G})}$. If $\overline{G} \in F_{23}$ in which there is no vertex adjacent to at least two non adjacent vertices in $S_3(\overline{G})$, then by Theorem J, $AE_F(\overline{G}) = G$ and hence $AE_F(G) \neq \overline{AE_F(\overline{G})}$. Suppose $\overline{G} \in F_{23}$ in which there is a vertex w' such that w' is adjacent to at least two non pairs of vertices u' and v' in $S_3(\overline{G})$. Since $G \in F_{23}$ in which there is a vertex w such that w is adjacent to at least two non adjacent pairs of vertices u and v in $S_3(G)$, $uv \in E(\overline{G})$ and $uv \notin E(AE_F(G))$. So $uv \notin E(AE_F(\overline{G}))$. Hence $AE_F(G) \neq \overline{AE_F(\overline{G})}$.

Case 3. $G \in F_{24} \cup F_3$. Then $\overline{G} \in F_{22}$. Let $u \in V(G)$. Then u is adjacent to all the vertices v in $\overline{G}$ such that $d_G(u,v) \geq 2$. Since $r(G) \geq 2$, the vertex u and its non F-average eccentric vertices w in G are adjacent in $\overline{G}$. Then $uw \notin E(AE_F(G))$ and $uw \notin E(AE_F(\overline{G}))$. Hence $AE_F(G) \neq \overline{AE_F(\overline{G})}$.

Suppose G is disconnected. If G has a non complete component and $\overline{G} \in F_{22}$, then by Theorem J, $AE_F(G) \neq \overline{G}$ and $AE_F(\overline{G}) = G$. Hence $AE_F(G) \neq \overline{AE_F(\overline{G})}$. If each component of G is complete and $\overline{G} \in F_{22}$, then by Theorem J, $AE_F(\overline{G}) = G$ and $AE_F(G) = \overline{G}$. Hence $AE_F(G) = \overline{AE_F(\overline{G})}$. If G is totally disconnected, then $\overline{G} \in F_{11}$ and by Theorem 3.1,

$AE_F(\overline{G}) = AE_F(G)$ and hence $AE_F(G) \neq \overline{AE_F(\overline{G})}$. If G is disconnected with at least one isolated vertex and has a non trivial component, then by the definition, $\overline{G} \in F_{12}$. Let v' be a full degree vertex and u', w' be two non adjacent vertices in $\overline{G}$. Then by the definition, $u'v'w'$ is a triangle in $AE_F(\overline{G})$. Since u' and w' are adjacent and v' is an isolated vertex in G, by the definition, $u'v'$ and $w'v' \in E(AE_F(G))$ but $u'w' \not\in E(AE_F(G))$. Hence $AE_F(G) \neq \overline{AE_F(\overline{G})}$. By considering all the cases, the result follows.

References

[1] R.R. Singleton, "There is no irregular moore graph," Amer. Math. Monthly, 75 (1968), 42 - 43.

[2] J. Akiyama, K. Ando, D. Avis, "Eccentric graphs," Discrete math., 16 (1976), 187 - 195.

[3] R. Aravamuthan and B. Rajendran, "Graph equations involving antipodal graphs," Presented at the seminar on combinatories and applications held at ISI, Culcutta during 14 - 17 December, (1982), 40 - 43.

[4] R. Aravamuthan and B. Rajendran, "On antipodal graphs," Discrete math., 49 (1984), 193 - 195.

[5] KM. Kathiresan and G. Marimuthu, "A study on radial graphs," Ars Combin., 96 (2010), 353 - 360.

[6] T. Sathiyanandham and S. Arockiaraj, "F-average eccentric graphs," J. Algebraic stat., 13.3 (2022), 1239 - 1247.

[7] F. Buckley and F. Harary, "Distance in graphs," Addition-wesley, Reading (1990).

[8] D.B. West, "Introduction to graph theory," Prentice - Hall of india, New Delhi (2003).

[9] G. Chartrand and P. Zhang, "A first course in graph theory," Dover publications, inc., New York (2012).