Derivation on β_1 Near-Ring

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Abstract
In this paper, A particular type of Near-Ring called β_1 Near-Ring is taken and existence of derivation on it, is described. Moreover, some results regarding derivation on β_1 Near-Ring are discussed.

Indexterms:β_1 Near-Ring, Derivation, commutative, Zero-symmetric.

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I. INTRODUCTION

This present paper is inspired by work of G.Sugantha and R.Balakrishna who have introduced the notion of β_1Near-Ring. A non-void set N_r together with additive and multiplicative operation is called a Near-Ring if N_r is additive group, multiplicative semi group and possesses one sided distributive law [4]. A Right Near-Ring N_r is known as β_1 Near-Ring if any a,b in N_r it satisfies $aN_rb = N_rab$ [1]. A Near-Ring is zero-symmetric if $\forall\ x_1 \in N_r$, $0x_1 = 0$. An additive mapping d from N_r to N_r is a derivation if $\forall \alpha, \beta \in N_r$

$d(\alpha\beta) = \alpha d(\beta) + \beta d(\alpha)$
or $d(\alpha\beta) = d(\alpha)\beta + \alpha d(\beta)$

ForanyaandbinNear-Ring $N_r$thesymbol[p,q]represents the commutatorpq−qp,andthesymbolp∘qrepresents theskewcommutatorpq+qp.
A Derivation dis known as commuting if
[d(x), x]=0,∀x ∈N_r.A Derivation d iscalledskewcommutingifd(x) ∘ x = 0,∀ x ∈N_r[5] . This paper moves in direction of extending some results on β_1 Near-Ring admitting derivation.
Throughout this paper N_r represents a β_1 Near-Ring with identity, d is derivation and Z represents centre of N_r.

II. PRELIMINARIES

Definition 2.1→ A Near-Ring N_r is known as pseudo commutative Near-Ring if ∀ x_1,y_1,z_1∈ N, x_1,y_1,z_1 = z_1,y_1,x_1 [Definition 2.1 of (2)].

Proposition 2.2 →Every β_1 Near-Ring which has identity is always zero-symmetric Near-Ring. [Prop. 5.1 of (1)]

Proposition 2.3 →Every Near-Ring possesses derivation iff it is zero-symmetric Near-Ring. [Theorem 2.7 of (3)]

Proposition 2.4 →Every pseudo commutative Near-Ring which has identity is a β_1 Near-Ring. [Prop. 5.5 of (1)]

Lemma 2.5 →For a Near-Ring N_r and its ideal$X$, if x_1x_2∈ X ⟹ x_1 n x_2∈I ∀ x_1x_2∈Nrthen $N_r$has strong I.F.P. [Prop. 9.2 of (4)]

Lemma 2.6→For a Near-Ring N_r which admits derivation d, (a_1, a_2) is constant where a_1 in A is commuting or skew commuting elements and a_2 ∈ N_r. Here A is additive non-zero N_r -subgroup of N_r [Lemma 2.5 of (5)]

III. Examples on $\boldsymbol{\beta_1}$ Near-Ring

3.1 β_1Near-Ring which admits derivation:

Example 1 →(N_r = {0,2,4,6,8}, $+_{10}$) is a β_1Near-Ring which is defined on cyclic group of order 5, where second composition is defined according to Pilz [4], Scheme 'F', p.410 and example 10.

.	0	2	4	6	8
0	0	0	0	0	0
2	0	2	4	6	8
4	0	4	8	2	6
6	0	6	2	8	4
8	0	8	6	4	2

Example 2 →(N_r = {0,2,4,6,8,10,12},$+_{14}$) is a β_1 Near-Ring which is defined on cyclic group of order 7,where second operation is defined according to Pilz [4], Scheme 'I', p.411 and example 23.

.	0	2	4	6	8	10	12
0	0	0	0	0	0	0	0
2	0	2	4	6	8	10	12
4	0	4	8	12	2	6	10
6	0	6	12	4	10	2	8
8	0	8	2	10	4	12	6
10	0	10	6	2	12	8	4
12	0	12	10	8	6	4	2

3.2 Near-Ring which are not β_1 Near-Ring

Example 1 →(N_r = {0,2,4,6,8,10,12},$+_{14}$) is a Near-Ring defined on cyclic group of order 7,where second operation is defined according to Pilz [4] Scheme 'I', p.412 and example 19.

.	0	2	4	6	8	10	12
0	0	0	0	0	0	0	0
2	2	2	2	2	2	2	2
4	4	4	4	4	4	4	4

6	6	6	6	6	6	6	6
8	8	8	8	8	8	8	8
10	10	10	10	10	10	10	10
12	12	12	12	12	12	12	12

Example 2 → (N_r = {0,2,4,6,8,10,12,14},$+_{16}$) is a Near-Ring defined on cyclic group of order 8,where second operation is defined according to Pilz [4], Scheme 'J', p.414 and example 118.

.	0	2	4	6	8	10	12	14
0	0	0	0	0	0	0	0	0
2	2	2	2	2	2	2	2	2
4	4	4	4	4	4	4	4	4
6	6	6	6	6	6	6	6	6
8	8	8	8	8	8	8	8	8
10	10	10	10	10	10	10	10	10
12	12	12	12	12	12	12	12	12
14	14	14	14	14	14	14	14	14

IV. MAIN RESULTS

Proposition 4.1→Every β_1 Near-Ring$N_r$with identity element '1' always admits derivation.

Proof: Let N_r be a β_1 Near-Ring which has identity element one then by proposition 2.2, it will be zero-symmetric also. Again, using proposition 2.3 it can be easily proven that it must admit derivation. Hence N_r always admits a derivation.

Theorem 4.2→For any pseudo commutative Near-Ring N_r with identity one, the following results are true:

(i) N_r is a β_1 Near-Ring.

(ii) N_r admits derivation.

(iii) N_r satisfies partial distributive law which is:$x(d(yz))=x(d(y)z+yd(z))= xd(y)z+xyd(z),\forall x,y,z\in N_r$.

(iv) N_r has strong I.F.P.

V. **Proof:**Let N_r be a pseudo commutative Near-Ring with identity one then,

(i) Proof is straight forward (by Prop.2.4)

(ii) Since every pseudo commutative Near-Ring is β_1 Near-Ring and every β_1 Near-Ring admits derivation (by Proposition 4.1). Combining both yield to the result that every pseudo commutative Near-Ring with identity one also admits derivation.

(iii)Take x,y,z$\in N_r$, then we have

d((xy)z) = xyd(z) + (xd(y) +d(x)y)z (1)

and also d(x(yz)) = xd(yz) + d(x)yz = x(yd(z) + d(y)z) + d(x)yz = xyd(z) + xd(y)z +d(x)yz(2)

by (1) and (2), we will get the required result.

(vi) By (i), every pseudo commutative Near-Ring with identity '1' is a β_1 Near-Ring ⇒ ∀ p,q ∈ N_r,

VI. $pN_rq= pN_rq$ (1)

Now since N_r is zero-symmetric, so for any ideal X of N_r, $N_r X \subseteq X$ (2)

Let x_1, $x_2 \in N_r$ and any r $\in N_r$

VII. $x_1rx_2 \in x_1N_r x_2 = N_r x_1 x_2$(by 1)

$\subseteq N_r$ x

$\subseteq I$ (by 2)

Now if we use Lemma 2.5, it is clear that N_r has strong I.F.P.

Corollary 4.3: Let N_r be a β_1 Near-Ring which has derivation d then N_r has strong I.F.P.

Proof: Since N_r has derivation ⇒ N_r zero-symmetric then by above it is obvious that N_r has strongI.F.P.

Lemma 4.4: Let N_r be a β_1 Near-Ring with 1 ∈ N_r then d(0) = 0 where d is derivation on N_r.

Proof: Since every β_1 Near-Ring is zero-symmetric. Hence, d(0) = d(00) = d(0)0 + 0d(0)⇒ d(0) = 0

Proposition 4.5 →The necessary and sufficient conditions for a β_1 Near-Ring N_r with non-zero derivation 'd' to be an abelian Near-Ring are-
(i) $d(x_1\ x_2) = d\ d(x_2\ x_1)\ x_1\ x_2 \in N_r$
(ii) For multiplicative commutators in N_r $\exists \alpha \in N_r$ s.t. $d(\alpha)$is not a left zero divisor.

Proof: Let N_r be a β_1 Near-Ring and suppose the following condition holds.
Now $d(x_1\ x_2) = d(x_2\ x_1)$ (given)
put $x_1 = x_2\, x_1 \Rightarrow d(x_2\, x_1\, x_2) = d(x_2\, x_2 x_1)$
Hence $d(x_2\ x_1\ x_2) - d(x_2\ x_1\ x_2) = 0$
$\Rightarrow d(x_2\,(x_1\, x_2 - x_2\, x_1)) = 0$
$\Rightarrow x_2 d(x_1\, x_2 - x_2\, x_1) + d(x_2)(x_1\, x_2 - x_2\, x_1) = 0$
$\Rightarrow d(x_2)(x_1\ x_2 - x_2\ x_1) = 0\ \forall x_1\ x_2 \in N_r$
[As $d(x_1\ x_2) = d(x_2\ x_1)$]
Also $d(\alpha)$ is not a left zero divisor $\forall [x_1\, x_2]$
$\Rightarrow d(\alpha)(x_1\ \alpha - \alpha x_1) = 0 \Rightarrow (x_1\ \alpha - \alpha x_1) = 0,\ \forall x_1 \in N_r$
Hence $\alpha \in Z(N_r)$.
Again $d(\alpha(x_1\, x_2)) = d((\alpha\, x_1)\, x_2) = d(x_2)(\alpha\, x_1) = d((x_2 d(x_2 \alpha) x_1) = d((\alpha\, x_2) x_1) = d(\alpha\, (x_2\, x_1))$
[As$d(x_1\ x_2) = d\ (x_2\ x_1)$]
$\Rightarrow d(\alpha(x_1\ x_2)) - d(\alpha(x_2\ x_1)) = d(\alpha(x_1\ x_2 - x_2\ x_1)) = 0$
$\Rightarrow \alpha\ d(x_1\ x_2 - x_2\ x_1) + d(\alpha)(x_1\ x_2 - x_2\ x_1) = d(\alpha)(x_1\ x_2 - x_2\ x_1) = 0$
Hence $x_1\ x_2 - x_2 x_1 = 0,\ \forall x_1 x_2 \in N_r$
$N_r$is commutative.
Conversely if N_r is commutative then clearly
$d(x_1\ x_2) = d(x_2\ x_1)$ and $[x_1,\ x_2] = 0\ \forall x_1\ x_2 \in N_r$.
which leads to the result.

Corollary 4.6: For any β_1 Near-Ring N_r which admits derivation s.t.$d(N_r) \subseteq Z(N_r)$, then N_r will be a commutative ring if $\exists\ \alpha N_r$s.t. $d(\alpha)$ is not a left zero divisor in N_r.

Theorem 4.7: For a β_1 Near-Ring N_r admitting a derivation d s.t.$d(d(r)s) = d(s\ d(r))$, $\forall r,s \in N_r$, and if $d^2(\alpha)$ is not a left zero divisor in β_1 Near-Ring $N_r$for some $\alpha \in N_r$ then N_r is an abelian Ring.

Proof:$\because d(d(r)s) = d(sd(r))$
Replace $s = d(r)s$ then
$d(d(r))\ (d(r)s - s\ d(r)) = 0$
$\Rightarrow d^2(r)(d(r)s\text{-}s\ d(r)) = 0\ \forall r,s\in N_r$ (1)
$\Rightarrow d(\alpha)s = s\ d(\alpha)\ \forall s\in N_r[\because d^2\ (\alpha)$ is not a left zero divisor].
Now $\forall r, s\in N_r$
$d(d(\alpha)(r\ d(s))) = d((d(\alpha)r)d(s)) = d(d(s)\ (d\ (\alpha)\ r))$
$= d((ds)d(\alpha))r) = d((d(\alpha)d(s))r) = d(d(\alpha)\ (d(s)r))$
$\Rightarrow d(\alpha)\ d(r\ d(s) - d(s)r) + d^2\ (\alpha)(rds) - d\ (s)r) = 0$
$\Rightarrow d^2\ (\alpha)(r\ d\ (s)) - d(s)\ r) = 0$
$\Rightarrow r\ d(s) - d(s)\ r = 0\ \forall r, s\in N_r$
$\Rightarrow d(N_r) \subseteq Z\ (N_r)$.
Replacing $b = d(\alpha) \Rightarrow d(b) \neq 0$ is not a left zero divisor in N_r.$\Rightarrow$ byCorollary 4.6 N_r is an abelian Ring.

Theorem4.8→For a β_1Near-Ring N_r which has zero element is the only divisor of zero and also has a non-trivial commuting or skew commuting derivation d then N_r is a commutative Near-Ring.

Proof: Let N_r be aβ_1 Near-Ring and c_1 be any additive commutator in N_r.
$\Rightarrow c_1$ is constant (by Lemma 2.6 and Prop. 4.1) also c_1x_1 will also be an additive commutator$\forall x_1\in N_r$
$\Rightarrow c_1x_1$ is also constant.
Hence $d(c_1\ x_1) = d(c_1)x_1 + c_1\ d(x_1) = 0$
and$c_1\ d(x_1) = 0$.
Now $\exists\ y_1\in N_r$s.t.$d(y_1) \neq 0$ (d is non-zero)
and by above $c_1\ d\ (x_1) = 0$.
$\because N_r$ has no non-zero divisors of zero $\Rightarrow c_1 = 0$
Hence N_r is abelian.

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