

Families of Weakly Compatible Mappings Satisfying $(E.A)$ and Common Limit Range Properties

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Abstract

The purpose of this paper is to prove common fixed point theorems for families of weakly compatible mappings satisfying $(E.A)$ property, common limit range property and a weak contraction condition involving cubic terms of distance function. Our results generalize and extend the results by Kumar *et al.* [11]. Results are supported with relevant example.

Keywords and Phrases: Common fixed point, weak contraction, weakly compatible mappings, $(E.A)$ property, common limit range property.

1. Introduction and Preliminaries

Banach's fixed point theorem [14] states that if P is a contraction mapping of a complete metric space (M, d) into itself then P has a unique fixed point in M . Several authors explored some new type contraction and proved various fixed point theorems in order to generalize the Banach fixed point theorem (see [2],[6],[9],[12],[13]). In 1976, for generalization of Banach's fixed point theorem, Jungck [3] used the notion of commuting maps to prove a common fixed point theorem.

In 1982, Sessa [15] generalized the notion of commutativity to weak commutativity and proved some common fixed point theorems for weakly commuting mappings. Further, in 1996, Jungck [5] extended the notion of compatible mappings to a larger class of mappings

known as weakly compatible. Infact weakly compatible mappings relax the condition of continuity of the mappings.

Definition 1.1. [5] Let P and Q be two mappings from a metric space (M, d) into itself. If P and Q commute at their coincidence point, i.e., if $Pt = Qt$ for some $t \in M$ implies $PQt = QPt$, then P and Q are called weakly compatible.

In the general setting, the notion of property $(E.A)$, which requires the closedness of the subspace, was introduced by Aamri and El- Moutawakil [7].

Definition 1.2. [7] A pair of self-mappings P and Q on a metric space (M, d) is said to satisfy property $(E.A)$ if there exists a sequence $\{u_n\} \in M$ such that $\lim_{n \rightarrow \infty} Pu_n = \lim_{n \rightarrow \infty} Qu_n = t$ for some $t \in M$.

Remark 1.3. [7] One can note that weakly compatibility and property $(E.A)$ are independent concepts.

In 2011, Sintunavarat and Kumam [17] introduced common limit range property (CLR_Q) as follows:

Definition 1.4. [17] A pair of self-mappings (P, Q) on a metric space (M, d) is said to satisfy common limit range property with respect to Q , denoted by (CLR_Q) , if there exists a sequence $\{u_n\} \in M$ such that $\lim_{n \rightarrow \infty} Pu_n = \lim_{n \rightarrow \infty} Qu_n = t$ for some $t \in Q(M)$.

Thus, one can note that a pair (P, Q) satisfying the property $(E.A)$ along with the closedness of the subspace $Q(M)$ always enjoys the (CLR_Q) property with respect to the mapping Q (see Examples 2.16-2.17 of [8]).

Imdad *et al.* [8] extend this notion of common limit range property for two pairs of mappings.

Definition 1.5. [8] Two pairs (P, S) and (Q, T) of a metric space (M, d) are said to satisfy the common limit range property with respect to the mappings S and T , denoted by (CLR_{ST}) , if there exists sequences $\{u_n\}$ and $\{v_n\}$ in M such that

$$\lim_{n \rightarrow \infty} Pu_n = \lim_{n \rightarrow \infty} Su_n = \lim_{n \rightarrow \infty} Qv_n = \lim_{n \rightarrow \infty} Tv_n = t, \text{ for some } t \in S(M) \cap T(M).$$

In 2017, a new type of common limit range property is introduced by Popa [16].

Definition 1.6. [16] Let P, S and T be self mappings of a metric space (M, d) . The pair (P, S) is said to satisfy a common limit range property with respect to T , denoted by, $(CLR)_{(P,S)T}$, if there exist a sequence $\{u_n\}$ in M such that

$$\lim_{n \rightarrow \infty} Pu_n = \lim_{n \rightarrow \infty} Su_n = u, \text{ for some } u \in S(M) \cap T(M).$$

In this paper, we prove common fixed point theorems for families of weakly compatible mappings enjoying $(E.A)$ and $(CLR)_{(P,S)T}$ properties.

2. Main Results

In 2021, Kumar *et al.* [11] introduced a new weak contraction condition that involves cubic terms of distance function and proved common fixed point theorems for compatible mappings and its variants.

Theorem 2.1. [11] Let f, g, S and T be four mappings of a complete metric space (M, d) into itself satisfying the following conditions:

$$(C1) f(M) \subset T(M), g(M) \subset S(M),$$

$$(C2) d^3(fx, gy) \leq p \max\left\{\frac{1}{2}[d^2(Sx, fx)d(Ty, gy) + d(Sx, fx)d^2(Ty, gy)],\right. \\ \left. d(Sx, fx)d(Sx, gy)d(Ty, fx), d(Sx, gy)d(Ty, fx)d(Ty, gy)\right\} \\ - \phi(m(Sx, Ty)),$$

for all $x, y \in M$, where

$$m(Sx, Ty) = \max\{d^2(Sx, Ty), d(Sx, fx)d(Ty, gy), d(Sx, gy)d(Ty, fx), \\ \frac{1}{2}[d(Sx, fx)d(Sx, gy) + d(Ty, fx)d(Ty, gy)]\},$$

where p is a real number satisfying $0 < p < 1$ and $\phi: [0, \infty) \rightarrow [0, \infty)$ is a continuous function with $\phi(t) = 0$ iff $t = 0$ and $\phi(t) > 0$ for each $t > 0$.

(C3) One of f, g, S and T is continuous.

Assume that the pairs (f, S) and (g, T) are compatible or compatible of type (A) or compatible of type (B) or compatible of type (C) or compatible of type (P) , then f, g, S and T have a unique common fixed point in M .

Now we prove our main results for families of weakly compatible mappings enjoying $(E.A)$ property.

Theorem 2.2. Let $Q_1, Q_2, \dots, Q_{2n}, P_0$ and P_1 be self mappings on a metric space (M, d) , satisfying the following conditions:

$$(C4) Q_2(Q_4 \dots Q_{2n}) = (Q_4 \dots Q_{2n})Q_2, Q_2Q_4(Q_6 \dots Q_{2n}) \\ = (Q_6 \dots Q_{2n})Q_2Q_4, \dots, Q_2 \dots Q_{2n-2}(Q_{2n}) = (Q_{2n})Q_2 \dots Q_{2n-2}; \\ P_1(Q_4 \dots Q_{2n}) = (Q_4 \dots Q_{2n})P_1, P_1(Q_6 \dots Q_{2n}) = (Q_6 \dots Q_{2n})P_1, \dots, P_1Q_{2n} = Q_{2n}P_1,$$

$$Q_1(Q_3 \dots Q_{2n-1}) = (Q_3 \dots Q_{2n-1})Q_1, \quad Q_1Q_3(Q_5 \dots Q_{2n-1}) = (Q_5 \dots Q_{2n-1})Q_1Q_3, \\ \dots, \quad Q_1 \dots Q_{2n-3}(Q_{2n-1}) = (Q_{2n-1})Q_1 \dots Q_{2n-3};$$

$$P_0(Q_3 \dots Q_{2n-1}) = (Q_3 \dots Q_{2n-1})P_0, \quad P_0(Q_5 \dots Q_{2n-1}) = (Q_5 \dots Q_{2n-1})P_0, \dots, P_0Q_{2n-1} \\ = Q_{2n-1}P_0,$$

(C5) $P_0(M) \subset Q_2 \dots Q_{2n}(M)$ and the pair $(P_0, Q_1 \dots Q_{2n-1})$ satisfies (E.A) property and $Q_1 \dots Q_{2n-1}(M)$ is a closed subset of M .

or

(C6) $P_1(M) \subset Q_1 \dots Q_{2n-1}(M)$ and the pair $(P_1, Q_2 \dots Q_{2n})$ satisfies (E.A) property and $Q_2 \dots Q_{2n}(M)$ is a closed subset of M .

$$(C7) \quad d^3(P_0x, P_1y)$$

$$\leq p \max\left\{\frac{1}{2} [d^2(Q_1 \dots Q_{2n-1}x, P_0x)d(Q_2 \dots Q_{2n}y, P_1y) \right. \\ \left. + d(Q_1 \dots Q_{2n-1}x, P_0x)d^2(Q_2 \dots Q_{2n}y, P_1y)], \right. \\ \left. d(Q_1 \dots Q_{2n-1}x, P_0x)d(Q_1 \dots Q_{2n-1}x, P_1y)d(Q_2 \dots Q_{2n}y, P_0x), \right. \\ \left. d(Q_1 \dots Q_{2n-1}x, P_1y)d(Q_2 \dots Q_{2n}y, P_0x)d(Q_2 \dots Q_{2n}y, P_1y)\right\} \\ - \phi(m(x, y)),$$

for all $x, y \in M$, where

$$m(x, y) = \max\{d^2(Q_1 \dots Q_{2n-1}x, Q_2 \dots Q_{2n}y), d(Q_1 \dots Q_{2n-1}x, P_0x)d(Q_2 \dots Q_{2n}y, P_1y), \\ d(Q_1 \dots Q_{2n-1}x, P_1y)d(Q_2 \dots Q_{2n}y, P_0x), \\ \frac{1}{2} [d(Q_1 \dots Q_{2n-1}x, P_0x)d(Q_1 \dots Q_{2n-1}x, P_1y) + \\ d(Q_2 \dots Q_{2n}y, P_0x)d(Q_2 \dots Q_{2n}y, P_1y)],$$

where p is a real number satisfying $0 < p < 1$ and $\phi: [0, \infty) \rightarrow [0, \infty)$ is a continuous function with $\phi(t) = 0$ iff $t = 0$ and $\phi(t) > 0$ for each $t > 0$.

Then $Q_1, Q_2, \dots, Q_{2n}, P_0$ and P_1 have a unique common fixed point in M given that the pairs $(P_0, Q_1 \dots Q_{2n-1})$ and $(P_1, Q_2 \dots Q_{2n-2})$ are weakly compatible.

Proof. Let $Q'_1 = Q_1Q_3 \dots Q_{2n-1}$ and $Q'_2 = Q_2Q_4 \dots Q_{2n}$. First we assume that the pair $(P_0, Q_1 \dots Q_{2n-1})$ satisfies (E.A) property, then there exists a sequence $\{u_n\}$ in M such that $\lim_{n \rightarrow \infty} P_0 u_n = \lim_{n \rightarrow \infty} Q_1 \dots Q_{2n-1} u_n = u$, for some $u \in M$.

Since $P_0(M) \subset Q_2 \dots Q_{2n}(M)$, there exists a sequence $\{v_n\}$ in M such that $Q_2 \dots Q_{2n}(v_n) = P_0 u_n$. Hence $\lim_{n \rightarrow \infty} Q_2 \dots Q_{2n} v_n = u$.

Step (i): First we show that $\lim_{n \rightarrow \infty} P_1 v_n = u$. On putting $x = u_n$ and $y = v_n$ in (C7), we have

$$d^3(P_0 u_n, P_1 v_n) \leq p \max\left\{\frac{1}{2} [d^2(Q_1 \dots Q_{2n-1} u_n, P_0 u_n) d(Q_2 \dots Q_{2n} v_n, P_1 v_n) + d(Q_1 \dots Q_{2n-1} u_n, P_0 u_n) d^2(Q_2 \dots Q_{2n} v_n, P_1 v_n)], d(Q_1 \dots Q_{2n-1} u_n, P_0 u_n) d(Q_1 \dots Q_{2n-1} u_n, P_1 v_n) d(Q_2 \dots Q_{2n} v_n, P_0 u_n), d(Q_1 \dots Q_{2n-1} u_n, P_1 v_n) d(Q_2 \dots Q_{2n} v_n, P_0 u_n), d(Q_1 \dots Q_{2n-1} u_n, P_1 v_n) d(Q_2 \dots Q_{2n} v_n, P_1 v_n)\right\} - \phi(m(u_n, v_n)),$$

where

$$m(u_n, v_n) = \max\{d^2(Q_1 \dots Q_{2n-1} u_n, Q_2 \dots Q_{2n} v_n), d(Q_1 \dots Q_{2n-1} u_n, P_0 u_n) d(Q_2 \dots Q_{2n} v_n, P_1 v_n), d(Q_1 \dots Q_{2n-1} u_n, P_1 v_n) d(Q_2 \dots Q_{2n} v_n, P_0 u_n), \frac{1}{2} [d(Q_1 \dots Q_{2n-1} u_n, P_0 u_n) d(Q_1 \dots Q_{2n-1} u_n, P_1 v_n) + d(Q_2 \dots Q_{2n} v_n, P_0 u_n) d(Q_2 \dots Q_{2n} v_n, P_1 v_n)]\}.$$

Taking limits as $n \rightarrow \infty$, we have

$$d^3(u, P_1 v_n) \leq p \max\left\{\frac{1}{2} [d^2(u, u) d(u, P_1 v_n) + d(u, u) d^2(u, P_1 v_n)], d(u, u) d(u, P_1 v_n) d(u, u), d(u, P_1 v_n) d(u, u) d(u, P_1 v_n)\right\} - \phi(m(u_n, v_n)),$$

where

$$m(u_n, v_n) = \{d^2(u, u), d(u, u) d(u, P_1 v_n), d(u, P_1 v_n) d(u, u), \frac{1}{2} [d(u, u) d(u, P_1 v_n) + d(u, u) d(u, P_1 v_n)]\}.$$

which implies that $\lim_{n \rightarrow \infty} d^3(u, P_1 v_n) = 0$. Hence

$$\lim_{n \rightarrow \infty} P_0 u_n = \lim_{n \rightarrow \infty} Q_1 \dots Q_{2n-1} u_n = \lim_{n \rightarrow \infty} P_1 v_n = \lim_{n \rightarrow \infty} Q_2 \dots Q_{2n} v_n = u.$$

Now suppose that $Q_1 \dots Q_{2n-1}(M)$ is closed subset of M , then exists $v \in M$ such that $Q_1 \dots Q_{2n-1} v = u$.

Step (ii): Now we show that $P_0 v = u$. Using (C7) with $x = v$ and $y = v_n$, we get

$$d^3(P_0 v, P_1 v_n) \leq p \max\left\{\frac{1}{2} [d^2(Q_1 \dots Q_{2n-1} v, P_0 v) d(Q_2 \dots Q_{2n} v_n, P_1 v_n) + d(Q_1 \dots Q_{2n-1} v, P_0 v) d^2(Q_2 \dots Q_{2n} v_n, P_1 v_n)], d(Q_1 \dots Q_{2n-1} v, P_0 v) d(Q_1 \dots Q_{2n-1} v, P_1 v_n) d(Q_2 \dots Q_{2n} v_n, P_0 v), d(Q_1 \dots Q_{2n-1} v, P_1 v_n) d(Q_2 \dots Q_{2n} v_n, P_0 v) d(Q_2 \dots Q_{2n} v_n, P_1 v_n)\right\}$$

$$-\phi(m(v, v_n)),$$

where

$$m(v, v_n) = \max\{d^2(Q_1 \dots Q_{2n-1}v, Q_2 \dots Q_{2n}v_n), d(Q_1 \dots Q_{2n-1}v, P_0v) \\ \cdot d(Q_2 \dots Q_{2n}v_n, P_1v_n), d(Q_1 \dots Q_{2n-1}v, P_1v_n)d(Q_2 \dots Q_{2n}v_n, P_0v), \\ \frac{1}{2}[d(Q_1 \dots Q_{2n-1}v, P_0v)d(Q_1 \dots Q_{2n-1}v, P_1v_n) + \\ dQ_2\dots Q_{2n}v_n, P_0vdQ_2\dots Q_{2n}v_n, P_1v_n]\}.$$

Approaching limits as $n \rightarrow \infty$, we get $d^3(P_0v, u) \leq 0$, i.e., $P_0v = u$. Hence $P_0v = Q_1Q_3 \dots Q_{2n-1}v = u$. Since $P_0(M) \subset Q_2 \dots Q_{2n}(M)$, there exists $w \in M$ such that $P_0v = Q_2 \dots Q_{2n}w = u$.

Step (iii): Next we show that $Q_2Q_4 \dots Q_{2n}w = u$. On using (C7) with $x = v$ and $y = w$, we get

$$d^3(P_0v, P_1w) \leq p \max\{\frac{1}{2}[d^2(Q_1 \dots Q_{2n-1}v, P_0v)d(Q_2 \dots Q_{2n}w, P_1w) \\ + d(Q_1 \dots Q_{2n-1}v, P_0v)d^2(Q_2 \dots Q_{2n}w, P_1w)], \\ d(Q_1 \dots Q_{2n-1}v, P_0v)d(Q_1 \dots Q_{2n-1}v, P_1w)d(Q_2 \dots Q_{2n}w, P_0v), \\ d(Q_1 \dots Q_{2n-1}v, P_1w)d(Q_2 \dots Q_{2n}w, P_0v)d(Q_2 \dots Q_{2n}w, P_1w)\} \\ -\phi(m(v, w)),$$

where

$$m(v, w) = \max\{d^2(Q_1 \dots Q_{2n-1}v, Q_2 \dots Q_{2n}w), d(Q_1 \dots Q_{2n-1}v, P_0v) \\ \cdot d(Q_2 \dots Q_{2n}w, P_1w), d(Q_1 \dots Q_{2n-1}v, P_1w)d(Q_2 \dots Q_{2n}w, P_0v), \\ \frac{1}{2}[d(Q_1 \dots Q_{2n-1}v, P_0v)d(Q_1 \dots Q_{2n-1}v, P_1w) + \\ dQ_2\dots Q_{2n}w, P_0vdQ_2\dots Q_{2n}w, P_1w]\}.$$

which implies that $d^3(u, P_1w) \leq 0$, i.e., $P_1w = u$ and hence $P_0v = Q_1Q_3 \dots Q_{2n-1}v = P_1w = Q_2Q_4 \dots Q_{2n}w = u$. Since the pairs $(P_0, Q_1Q_3 \dots Q_{2n-1})$ and $(P_1, Q_2Q_4 \dots Q_{2n})$ are weakly compatible, we have

$$Q_1Q_3 \dots Q_{2n-1}u = Q_1Q_3 \dots Q_{2n-1}(P_0v) = P_0(Q_1Q_3 \dots Q_{2n-1}v) = P_0u$$

and

$$Q_2Q_4 \dots Q_{2n}u = Q_2Q_4 \dots Q_{2n}(P_1w) = P_1(Q_2Q_4 \dots Q_{2n}w) = P_1u.$$

Step (iv): In this step, we prove that $P_0v = P_1u$. Suppose that $P_0v \neq P_1u$. On putting $x = v$ and $y = u$ in (C7), we obtain

$$d^3(P_0v, P_1u) \leq p \max\left\{\frac{1}{2} [d^2(Q_1 \dots Q_{2n-1}v, P_0v)d(Q_2 \dots Q_{2n}u, P_1u) + d(Q_1 \dots Q_{2n-1}v, P_0v)d^2(Q_2 \dots Q_{2n}u, P_1u)], d(Q_1 \dots Q_{2n-1}v, P_0v)d(Q_1 \dots Q_{2n-1}v, P_1u)d(Q_2 \dots Q_{2n}u, P_0v), d(Q_1 \dots Q_{2n-1}v, P_1u)d(Q_2 \dots Q_{2n}u, P_0v)d(Q_2 \dots Q_{2n}u, P_1u)\right\} - \phi(m(v, u)),$$

where

$$m(v, u) = \max\{d^2(Q_1 \dots Q_{2n-1}v, Q_2 \dots Q_{2n}u), d(Q_1 \dots Q_{2n-1}v, P_0v) \cdot d(Q_2 \dots Q_{2n}u, P_1u), d(Q_1 \dots Q_{2n-1}v, P_1u)d(Q_2 \dots Q_{2n}u, P_0v), \frac{1}{2} [d(Q_1 \dots Q_{2n-1}v, P_0v)d(Q_1 \dots Q_{2n-1}v, P_1u) + dQ_2\dots Q_{2nu}, P_0vdQ_2\dots Q_{2nu}, P_1u]\}.$$

On simplification, we have $d^3(P_0v, P_1u) \leq -\phi(d^2(P_0v, P_1u))$, a contradiction. Thus, we have $P_0v = P_1u = u$, i.e., $P_1u = Q_2Q_4 \dots Q_{2n}u = u$.

Further on putting $x = y = u$ in (C7), we have $P_0u = u$ hence $P_0u = Q_1Q_3 \dots Q_{2n-1}u = u$.

Step (v): On putting $x = u$ and $y = Q_4 \dots Q_{2n}u$ in (C7) and using condition (C4) and $Q'_1 = Q_1Q_3 \dots Q_{2n-1}$ and $Q'_2 = Q_2Q_4 \dots Q_{2n}$, we have

$$d^3(P_0u, P_1Q_4 \dots Q_{2n}u) \leq p \max\left\{\frac{1}{2} [d^2(Q'_1u, P_0u)d(Q'_2Q_4 \dots Q_{2n}u, P_1Q_4 \dots Q_{2n}u) + d(Q'_1u, P_0u)d^2(Q'_2Q_4 \dots Q_{2n}u, P_1Q_4 \dots Q_{2n}u)], d(Q'_1u, P_0u)d(Q'_1u, P_1Q_4 \dots Q_{2n}u)d(Q'_2Q_4 \dots Q_{2n}u, P_0u), d(Q'_1u, P_1Q_4 \dots Q_{2n}u)d(Q'_2Q_4 \dots Q_{2n}u, P_0Q_4 \dots Q_{2n}u) \cdot d(Q'_2Q_4 \dots Q_{2n}u, P_1u)\right\} - \phi(m(u, Q_4 \dots Q_{2n}u)),$$

where

$$m(u, Q_4 \dots Q_{2n}u) = \max\{d^2(Q'_1u, Q'_2Q_4 \dots Q_{2n}u), d(Q'_1u, P_0u) \cdot d(Q'_2Q_4 \dots Q_{2n}u, P_1u), d(Q'_1u, P_1Q_4 \dots Q_{2n}u)d(Q'_2Q_4 \dots Q_{2n}u, P_0u), \frac{1}{2} [d(Q'_1u, P_0u)d(Q'_1u, P_1Q_4 \dots Q_{2n}u) + dQ_2'Q_4\dots Q_{2nu}, P_0udQ_2'Q_4\dots Q_{2nu}, P_1Q_4\dots Q_{2nu}]\}.$$

On simplification, we get $d^3(u, Q_4 \dots Q_{2n}u) \leq -\phi(d^2(u, Q_4 \dots Q_{2n}u))$, i.e., $Q_4 \dots Q_{2n}u = u$.

Hence $Q_2Q_4 \dots Q_{2n}u = Q_2u = u$. Continuing like this, we have

$$P_0u = Q_2u = Q_4u = \dots = Q_{2n}u = u.$$

Therefore,

$$P_0 u = P_1 u = Q_1 u = Q_2 u = \dots = Q_{2n-1} u = Q_{2n} u = u.$$

Hence u is a common fixed point of the given mappings. Uniqueness follows easily from condition (C7). Similarly, the proof holds for the condition (C6). This completes the proof.

Now we prove the following theorem for families of mappings which is a slight generalization of Theorem 2.2.

Theorem 2.3. Let $\{P_\alpha\}_{\alpha \in J}$ and $\{Q_i\}_{i=1}^{2p}$ be two families of self-mappings on a metric space (M, d) . Suppose that there exists a fixed $\beta \in J$ such that:

$$(C8) Q_2(Q_4 \dots Q_{2n}) = (Q_4 \dots Q_{2n})Q_2, \quad Q_2 Q_4(Q_6 \dots Q_{2n})$$

$$= (Q_6 \dots Q_{2n})Q_2 Q_4,$$

$$\dots, \quad Q_2 \dots Q_{2n-2}(Q_{2n}) = (Q_{2n})Q_2 \dots Q_{2n-2};$$

$$P_\beta(Q_4 \dots Q_{2n}) = (Q_4 \dots Q_{2n})P_\beta, \quad P_\beta(Q_6 \dots Q_{2n}) = (Q_6 \dots Q_{2n})P_\beta, \dots, \quad P_\beta Q_{2n} = Q_{2n} P_\beta,$$

$$Q_1(Q_3 \dots Q_{2n-1}) = (Q_3 \dots Q_{2n-1})Q_1, \quad Q_1 Q_3(Q_5 \dots Q_{2n-1}) = (Q_5 \dots Q_{2n-1})Q_1 Q_3,$$

$$\dots, \quad Q_1 \dots Q_{2n-3}(Q_{2n-1}) = (Q_{2n-1})Q_1 \dots Q_{2n-3};$$

$$P_\alpha(Q_3 \dots Q_{2n-1}) = (Q_3 \dots Q_{2n-1})P_\alpha, \quad P_\alpha(Q_5 \dots Q_{2n-1}) = (Q_5 \dots Q_{2n-1})P_\alpha, \dots, \quad P_\alpha Q_{2n-1}$$

$$= Q_{2n-1} P_\alpha,$$

(C9) $P_\alpha(M) \subset Q_2 \dots Q_{2n}(M)$ and the pair $(P_\alpha, Q_1 \dots Q_{2n-1})$ satisfies (E.A) property and $Q_1 \dots Q_{2n-1}(M)$ is a closed subset of M .

or

(C10) $P_\beta(M) \subset Q_1 \dots Q_{2n-1}(M)$ and the pair $(P_\beta, Q_2 \dots Q_{2n})$ satisfies (E.A) property and $Q_2 \dots Q_{2n}(M)$ is a closed subset of M .

$$(C11) d^3(P_\alpha x, P_\beta y)$$

$$\leq p \max\left\{\frac{1}{2} [d^2(Q_1 \dots Q_{2n-1} x, P_\alpha x) d(Q_2 \dots Q_{2n} y, P_\beta y) \right.$$

$$+ d(Q_1 \dots Q_{2n-1} x, P_\alpha x) d^2(Q_2 \dots Q_{2n} y, P_\beta y)],$$

$$d(Q_1 \dots Q_{2n-1} x, P_\alpha x) d(Q_1 \dots Q_{2n-1} x, P_\beta y) d(Q_2 \dots Q_{2n} y, P_\alpha x),$$

$$d(Q_1 \dots Q_{2n-1} x, P_\beta y) d(Q_2 \dots Q_{2n} y, P_\alpha x) d(Q_2 \dots Q_{2n} y, P_\beta y)\}$$

$$- \phi(m(x, y)),$$

for all $x, y \in M$, where

$$m(x, y) = \max\{d^2(Q_1 \dots Q_{2n-1} x, Q_2 \dots Q_{2n} y), d(Q_1 \dots Q_{2n-1} x, P_\alpha x) d(Q_2 \dots Q_{2n} y, P_\beta y),$$

$$d(Q_1 \dots Q_{2n-1}x, P_\beta y)d(Q_2 \dots Q_{2n}y, P_\alpha x),$$

$$\frac{1}{2} [d(Q_1 \dots Q_{2n-1}x, P_\alpha x)d(Q_1 \dots Q_{2n-1}x, P_\beta y) +$$

$$d(Q_2 \dots Q_{2n}y, P_\alpha x)d(Q_2 \dots Q_{2n}y, P_\beta y)],$$

where p is a real number satisfying $0 < p < 1$ and $\phi: [0, \infty) \rightarrow [0, \infty)$ is a continuous function with $\phi(t) = 0$ iff $t = 0$ and $\phi(t) > 0$ for each $t > 0$.

Then all Q_i and P_α have a unique common fixed point in M given that the pairs $(P_\alpha, Q_1 \dots Q_{2n-1})$ and $(P_\beta, Q_2 \dots Q_{2n-2})$ are weakly compatible.

Proof. Let P_{α_0} be a fixed element in $\{P_\alpha\}_{\alpha \in J}$. By Theorem 2.2 with $P_\alpha = P_{\alpha_0}$ and $P_1 = P_\beta$ it follows that there exists some $u \in M$ such that

$$P_\beta u = P_{\alpha_0} u = Q_2 Q_4 \dots Q_{2n} u = Q_1 Q_3 \dots Q_{2n-1} u = u.$$

Let $\alpha \in J$ be arbitrary. Then from condition (C11), we have

$$d^3(P_\alpha u, P_\beta u) \leq p \max\left\{\frac{1}{2} [d^2(Q_1 \dots Q_{2n-1}u, P_\alpha u)d(Q_2 \dots Q_{2n}u, P_\beta u)\right.$$

$$+ d(Q_1 \dots Q_{2n-1}u, P_\alpha u)d^2(Q_2 \dots Q_{2n}u, P_\beta u)],$$

$$d(Q_1 \dots Q_{2n-1}u, P_\alpha u)d(Q_1 \dots Q_{2n-1}u, P_\beta u)d(Q_2 \dots Q_{2n}u, P_\alpha u),$$

$$d(Q_1 \dots Q_{2n-1}u, P_\beta u)d(Q_2 \dots Q_{2n}u, P_\alpha u)d(Q_2 \dots Q_{2n}u, P_\beta u)\}$$

$$- \phi(m(u, u)),$$

where

$$m(u, u) = \max\{d^2(Q_1 \dots Q_{2n-1}u, Q_2 \dots Q_{2n}u), d(Q_1 \dots Q_{2n-1}u, P_\alpha u)d(Q_2 \dots Q_{2n}u, P_\beta u),$$

$$d(Q_1 \dots Q_{2n-1}u, P_\beta u)d(Q_2 \dots Q_{2n}u, P_\alpha u),$$

$$\frac{1}{2} [d(Q_1 \dots Q_{2n-1}u, P_\alpha u)d(Q_1 \dots Q_{2n-1}u, P_\beta u)$$

$$+ d(Q_2 \dots Q_{2n}u, P_\alpha u)d(Q_2 \dots Q_{2n}u, P_\beta u)]\}$$

and hence

$$d^3(P_\alpha u, u) \leq p \max\left\{\frac{1}{2} [d^2(u, P_\alpha u)d(u, u) + d(u, P_\alpha u)d^2(u, u)],$$

$$d(u, P_\alpha u)d(u, u)d(u, P_\alpha u),$$

$$d(u, u)d(u, P_\alpha u)d(u, u)\} - \phi(d^2(u, u)),$$

which implies that $d^3(P_\alpha u, u) \leq 0$, i.e., $P_\alpha u = u$ for each $\alpha \in J$. Uniqueness follows easily.

Next, we prove a theorem for any even number of weakly compatible mappings satisfying common limit range property.

Theorem 2.4. Let $Q_1, Q_2, \dots, Q_{2n}, P_0$ and P_1 be self mappings on a metric space (M, d) , satisfying the conditions (C4), (C7) and

(C12) the pair (P_0, Q'_1) and Q'_2 satisfy $(CLR)_{(P_0, Q'_1)Q'_2}$ property.

Then $Q_1, Q_2, \dots, Q_{2n}, P_0$ and P_1 have a unique common fixed point in M given that the pairs (P_0, Q'_1) and (P_1, Q'_2) are weakly compatible, where $Q'_1 = Q_1 Q_3 \dots Q_{2n-1}$ and $Q'_2 = Q_2 Q_4 \dots Q_{2n}$.

Proof. Since the pair (P_0, Q'_1) and Q'_2 satisfy $(CLR)_{(P_0, Q'_1)Q'_2}$ property, there exists a sequence $\{u_n\}$ in M such that

$$\lim_{n \rightarrow \infty} P_0 u_n = \lim_{n \rightarrow \infty} Q'_1 u_n = \lim_{n \rightarrow \infty} Q_1 Q_3 \dots Q_{2n-1} u_n = u,$$

for some $u \in Q'_1 \cap Q'_2 = Q_2 Q_4 \dots Q_{2n}(M) \cap Q_1 Q_3 \dots Q_{2n-1}(M)$.

Since $u \in Q_2 Q_4 \dots Q_{2n}(M)$, this gives $u = Q_2 Q_4 \dots Q_{2n} v$, for some v in M .

We show that $u = P_1 v$. On putting $x = u_n$ and $y = v$ in (C7), we have

$$\begin{aligned} d^3(P_0 u_n, P_1 v) \leq & p \max\left\{\frac{1}{2} [d^2(Q_1 \dots Q_{2n-1} u_n, P_0 u_n) d(Q_2 \dots Q_{2n} v, P_1 v) \right. \\ & + d(Q_1 \dots Q_{2n-1} u_n, P_0 u_n) d^2(Q_2 \dots Q_{2n} v, P_1 v)], \\ & d(Q_1 \dots Q_{2n-1} u_n, P_0 u_n) d(Q_1 \dots Q_{2n-1} u_n, P_1 v) d(Q_2 \dots Q_{2n} v, P_0 u_n), \\ & d(Q_1 \dots Q_{2n-1} u_n, P_1 v) d(Q_2 \dots Q_{2n} v, P_0 u_n) d(Q_2 \dots Q_{2n} v, P_1 v)\} \\ & - \phi(m(u_n, v)), \end{aligned}$$

where

$$\begin{aligned} m(u_n, v) = & \max\{d^2(Q_1 \dots Q_{2n-1} u_n, Q_2 \dots Q_{2n} v), d(Q_1 \dots Q_{2n-1} u_n, P_0 u_n) \\ & \cdot d(Q_2 \dots Q_{2n} v, P_1 v), d(Q_1 \dots Q_{2n-1} u_n, P_1 v) d(Q_2 \dots Q_{2n} v, P_0 u_n), \\ & \frac{1}{2} [d(Q_1 \dots Q_{2n-1} u_n, P_0 u_n) d(Q_1 \dots Q_{2n-1} u_n, P_1 v) + \\ & d(Q_2 \dots Q_{2n} v, P_0 u_n) d(Q_2 \dots Q_{2n} v, P_1 v)]\}. \end{aligned}$$

Taking limits as $n \rightarrow \infty$ and on simplification, we have $d^3(u, P_1 v) \leq 0$, i.e., $P_1 v = u$ and hence $u = P_1 v = Q_2 Q_4 \dots Q_{2n} v$.

Also $u \in Q_1 Q_3 \dots Q_{2n-1}(M)$ which gives $u = Q_1 Q_3 \dots Q_{2n-1} w$, for some $w \in M$.

If we put $x = w$ and $y = v$ in condition (C7), on simplification we get, $P_0 w = u$.

Hence $P_0 w = Q_1 Q_3 \dots Q_{2n-1} w = P_1 v = Q_2 Q_4 \dots Q_{2n} v = u$.

Due to weakly compatibility of the given mappings, we obtain

$$P_0u = P_0Q_1Q_3 \dots Q_{2n-1}w = Q_1Q_3 \dots Q_{2n-1}P_0w = Q_1Q_3 \dots Q_{2n-1}u$$

and

$$P_1u = P_1Q_2Q_4 \dots Q_{2n}v = Q_2Q_4 \dots Q_{2n}P_1v = Q_2Q_4 \dots Q_{2n}u.$$

Further using steps (iv) and (v) of Theorem 2.2, we get u is a unique common fixed of the given mappings.

The following theorem is a slight generalization of Theorem 2.4.

Theorem 2.5. Let $\{P_\alpha\}_{\alpha \in J}$ and $\{Q_i\}_{i=1}^{2p}$ be two families of self-mappings on a metric space (M, d) . Suppose that there exists a fixed $\beta \in J$ such that conditions (C8) and (C11) are satisfied. Further,

(C13) the pair $(P_\alpha, Q_1Q_3 \dots Q_{2n-1})$ and $Q_2Q_4 \dots Q_{2n}$ satisfy $(CLR)_{(P_\alpha, Q_1Q_3 \dots Q_{2n-1})Q_2Q_4 \dots Q_{2n}}$ property.

Then all Q_i and P_α have a unique common fixed point in M given that the pairs $(P_\alpha, Q_1 \dots Q_{2n-1})$ and $(P_\beta, Q_2 \dots Q_{2n-2})$ are weakly compatible.

Proof. Proof of the Theorem 2.5 follows from the proof of Theorem 2.3.

Corollary 2.6. Let f, g, S and T are self-mappings on a metric space (M, d) satisfying (C2) and the following conditions

(C14) $f(M) \subset T(M)$ and the pair (f, S) satisfies (E.A) property and $S(M)$ is a closed subset of M .

or

(C15) $g(M) \subset S(M)$ and the pair (g, T) satisfies (E.A) property and $T(M)$ is a closed subset of M .

Then f, g, S and T have a unique common fixed point in M provided that the pairs (f, S) and (g, T) are weakly compatible.

Proof. If we take $P_0 = f, P_1 = g, Q_1 = S, Q_2 = T$ and $Q_3 = Q_4 = \dots = Q_{2n} = I$ (Identity Map) in Theorem 2.2, we get required result.

Remark 2.7. Corollary 2.6 is a generalization of Theorem 2.1 in the sense that conditions of completeness of space and continuity of the mappings are relaxed. Theorems 2.2-2.5 generalize and extend Theorem 2.1 for families of weakly compatible mappings.

Now we give an example in support of our theorems.

Example 2.8. Let $M = [0, 1]$ and d be usual metric on M . Define

$$P_\alpha(x) = \frac{x^3}{1+x^3} \text{ for each } \alpha \in J \text{ and all } x \in M,$$

$$Q_i(x) = x^{\sqrt[n]{3}} \text{ for each } i \in \{1, 2, \dots, 2n\} \text{ and all } x \in M.$$

$$\text{Then } Q_2 Q_4 \dots Q_{2n} x = x^3, Q_1 Q_3 \dots Q_{2n-1} x = x^3.$$

Let $\phi: [0, \infty) \rightarrow [0, \infty)$ be a function defined by $\phi(t) = \frac{t}{30}$, for $t \geq 0$. Then all the conditions of Theorems 2.2-2.5 are satisfied for $p = \frac{9}{10}$ and 0 is the unique common fixed point of the mappings.

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